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Inégalités intégrales fractionnaires via l'intégrale fractionnaire de Hadamard.

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Abstract

In this memoir, we present some fractional integral inequalities as Chebyshev-Gruss type using the Hadamard fractional integral. Also, we study new integral inequalities are obtained by using Young and weighted AM-GM inequalities.

Keywords :Hadamard fractional integral , Young and weighted AM-GM inequalities.

Résumé

Dans ce mémoire, nous présentons quelques inégalités intégrales fractionnaires comme type de Chebyshev-Gruss pour l'intégrale fractionnaire de Hadamard. Aussi, on étudie quelques nouvelles inégalitées intégrales qui btenu par l'inégalité de Young et l'inégalité ponderée de MA-MG.

Mots clés : Intégrale fractionnaire de Hadamard, inégalités de Young et MA-MG ponderée.

خلاصة

فى هذه المذكرة، قدمنا بعض المتباينات التكاملية الكسرية كمتباينة تشبشاف و متباينة غروس ،باستعمال التكامل الكسري لهادامر. اضافة لذالك قمنا بدراسة بعض المتباينات الحديثة الناتجة باستعمال متباينة يونغ ومتباينة المتوسط الحسابي- المتوسط الهندسي .

الكلمات المفتاحية:التكامل الكسرى لهادامار,. متباينة يونغ ومتباينة المتوسط الحسابي- المتوسط الهندسي.

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Dedicate

...je dédie ce travail :

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MES CHERS PARENTS pour tous leurs sacrifices, leur amour,
leur tendresse, leur soutien
et leurs prières tout au long de mes ètudes

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Introduction

In mathematical analysis, the fractional calculus is a very helpful tool to perform differentiation and integration with the real number or complex number powers of the differential or integral operators.

This subject has earned the attention of many researchers and mathematicians during lest few decades(see[[1, 8, 14, 17, 18]).

There is a large number of the fractional integral operators discussed in literature but because of their applications in many fields of sciences.

Another kind of fractional devirative that appears in the literature is the fractional devirative due to Hadamard introduced in 1892 (see[13, 14]), which differs from the Riemann-Liouville and Caputo deviratives in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional devirative and integral can be found in(see[3, 6, 15, 16]).

Recently in the literature, there appeared some results on fractional integral inequalities using Hadamard fractional integral.

The main attention in this memoir, was focused on the fractional integral inequalities using the Hadamarad fractional integral.Several new integral inequalities are obtained including a Gruss type Hadamard fractional integral inequality(see[12]).

This memoir consists of three chapters. **The first chapter,** contains the definitions of fractional analysis as functional spaces of Lebesgue measurable functions,absolutely functions, continuous functions and their weighted. Also some properties of fractional integrals as the Riemann-Liouville and Hadamard.

In the second chapter, we use Hadamard fractional integral to establish some integral inequalities of Chebyshev-Gruss type, by using one or two parameters. **In the last chapter,** we present some new fractional integral inequalities using the Hadamard fractional, are obtained by using Young and weighted AM-GM inequalities.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we present definitions of spaces as p-integrable (Lebesgue measurable functions), absolutely continuous, continuous functions and their weighted. We also give some properties of the Euler gamma function and the Hadamard fractional integral(see[16, 19]).

1.1.1 The spaces L_p, $L_p(\omega)$ and $X_c^p(\Omega)$

Definition 1.1. *Let $\Omega = [a, b], -\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real. We denote by, $L_{L_p} = L_{L_p}(\Omega)(1 \leq p \leq \infty)$ the set of all Lebesgue measurable functions $f(x)$, on Ω for which $\| f \|_{L_p} < \infty$, where*

$$\| f \|_{L_p} = \left(\int_\Omega | f(t) |^p \, dt \right)^{\frac{1}{p}} (1 \leq p < \infty),$$

and

$$\| f \|_{L_\infty} = ess \sup_{a \leq x \leq b} | f(x) | .$$

There $ess \sup | f(x) |$ is the eesential maximum of the function $| f(x) |$.

Definition 1.2. *Let $\omega(x)$ be a non-negative function. We denote by $L_p(\Omega) = L_p(\Omega, \omega)$ the space of functions $f(x)$, measurable on Ω for* *which*

$$\| f \|_{L_p(\omega)} = \left(\int_\Omega \omega(t) \mid f(t) \mid^p dt \right)^{\frac{1}{p}} < \infty.$$

Definition 1.3. *We denote by* $X_c^p(\Omega)(c \in R; 1 \leq p \leq \infty)$ *consists of all Lebesgue measurable functions* $f(x)$*, on* Ω*,* $\| f \|_{X_c^p} < \infty$*, with*

$$\| f \|_{X_c^p} = \left(\int_\Omega \mid t^c f(t) \mid^p dt \right)^{\frac{1}{p}} \quad (1 \leq p < \infty),$$

and

$$\| f \|_{X_c^\infty} = ess \sup_{x \in \Omega} \left[x^c \mid f(x) \mid \right].$$

In particuler, when $c = \frac{1}{p}$*, the space* X_c^p *coincides whith the* L_p*.*

1.1.2 The space $AC(\Omega)$

Definition 1.4. *A function* $f(x)$ *is called absolutely continuous on an interval* Ω*, if for any* $\varepsilon > 0$*, there exists a* $\delta > 0$*, such that for any finite set pairwise non intersecting intervals* $[a_k, b_k] \subset \Omega, k = 1, 2, ..., n$ *such that* $\sum_{k=1}^n (b_k - a_k) < \delta,$ *the inequalty* $\sum_{k=1}^n \mid f(b_k) - f(a_k) \mid < \varepsilon$ *holds.*
The space denoted by $AC(\Omega)$*.*

Remark 1.1. *It is known that* $AC(\Omega)$ *coincides with the space of primitives of Lebesgue summable functions*

$$f(x) \in AC(\Omega) \Longleftrightarrow f(x) = c + \int_a^x \varphi(t) dt, \qquad \varphi \in L^1(\Omega),$$

where, $\varphi(t) = f'(t), \; c = f(a)$*.*

1.1.3 Some properties

The Young inequality

Let $a, b \geq 0$ *and* $1 \leq \theta_1, \theta_2 \leq \infty$, be two conjugate exposents, alors

$$ab \leq \frac{a^{\theta_1}}{\theta_1} + \frac{b^{\theta_2}}{\theta_2}, \;\; \frac{1}{\theta_1} + \frac{1}{\theta_2}.$$

Example 1.1. *For* $\theta_1 = \theta_2 = 2$, *then* $\frac{a^2+b^2}{2} \geq ab$.

Property 1.1. *The inequality*

$$\theta_1 a + \theta_2 b \geq a^{\theta_1} b^{\theta_2},\ \theta_1 + \theta_2 = 1,$$

is well-known weighted AM-GM inequality.

The Minkowsky inequality

If $f, g \in L_p,\ p, p' \geq 1$ then $f + g \in L_p$ and

$$\| f + g \|_{L_p} \leq \| f \|_{L_p} + \| g \|_{L_p} .$$

The Hölder inequality

If $f \in L_p, g \in L_{p'}$, then $fg \in L_1(\Omega)$ and

$$\int_\Omega | f(x)g(x) | dx \leq \| f \|_{L_p} \| g \|_{L_{p'}}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

The Chebyshev-Grüss type inequality

$f, g : \Omega \to R$ are absolutely continuous functions, suth that
$\varphi \leq f(x) \leq \Phi,\ \psi \leq g(x) \leq \Psi$, for all $x \in [a, b]$. We have

$$| T(f, g) | \leq \frac{1}{4}(\Phi - \varphi)(\Psi - \psi), \quad \varphi, \Phi, \psi, \Psi \in R,$$

where,

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \right).$$

Chebyshev functional[2].

1.2 Some special functions

The gamma-function $\Gamma(x)$

The Euler integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \ \ x > 0,$$

is called the gamma-function.

Property 1.2.

- $\Gamma(x+1) = x\Gamma(x), x > 0.$
- $\Gamma(1) = 1.$

The beta-function $B(x, y)$

The Euler integral

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \ \ x, y > 0.$$

is called the beta-function.

Property 1.3.

- $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $\bullet B(x, y) = B(y, x)$.

1.3 Description of fractional calculus

We will give the notation and basic definitions.

1.3.1 Riemann-Liouville fractional integral

Definition 1.5. *[11] The Reimann-Liouville fractional integral of order $\alpha \in R$ ($\alpha > 0$), for a function $f \in L^1([a, b])$ is defined as*

$$\begin{aligned} I^\alpha f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt; \quad \alpha > 0, \, t > 0, \qquad (1.1) \\ I^0 f(x) &= f(x), \end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$

Property 1.4.

$$\begin{aligned} \bullet I^\alpha I^\beta f(x) &= I^{(\alpha+\beta)} f(x) \qquad (semigroupe). \\ \bullet I^\alpha I^\beta f(x) &= I^\beta I^\alpha f(x) \quad (commutative). \end{aligned}$$

1.3.2 Hadamard fractional integral

Definition 1.6. *Let (a,b)* $(0 \leq a < b \leq \infty)$ *and* $\alpha \in R$ $(\alpha > 0)$. *The Hadamard fractional integral of order* α *of function* $f(x) \in L^1([a,b])$, *for all* $x > 1$ *is defined as,*

$$_HJ^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln(\frac{x}{t})\right)^{\alpha-1} f(t) \frac{dt}{t}. \tag{1.2}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$

Property 1.5.

$$\begin{aligned} \bullet {}_HJ_H^\alpha J^\beta f(x) &= {}_HJ^{(\alpha+\beta)} f(x) \qquad (semigroupe). \\ \bullet {}_HJ_H^\alpha J^\beta f(x) &= {}_HJ_H^\beta J^\alpha f(x) \quad (commutative). \end{aligned} \tag{1.3}$$

• *Linearity verified.*

Property 1.6.

• If $f(x) = (\ln x)^{\beta-1}$.

$$_HJ^\alpha (\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln x)^{\beta+\alpha-1} \tag{1.4}$$

Example 1.2.

Let $\alpha > 0, \beta > 0, x > a$ and $f(x) = (\ln(\frac{x}{a}))^{\beta-1}$.
We obtain

$$_HJ^\alpha (\ln(\frac{x}{a}))^{\beta-1} = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln\frac{x}{t})^{\alpha-1} \ln(\frac{t}{a})^{\beta-1} \frac{dt}{t}, \tag{1.5}$$

suppose

$$u = \frac{\ln(\frac{x}{t})}{\ln(\frac{x}{a})} \tag{1.6}$$

then

$$\begin{aligned}
{}_HJ^\alpha(\ln(\frac{x}{a}))^{\beta-1} &= \frac{\ln(x)^{\alpha+\beta-1}}{\Gamma(\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du \\
&= \frac{B(\beta,\alpha)}{\Gamma(\alpha)}(\ln(\frac{x}{a}))^{\beta+\alpha-1} \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln(\frac{x}{a}))^{\beta+\alpha-1}.
\end{aligned}$$

If $a = 1$, we obtain inequality (1.3).

Example 1.3.

If $\beta = 1, f(x) = 1.$ in (1.4)

$${}_HJ^\alpha 1 = \frac{(\ln(x))^\alpha}{\Gamma(\alpha+1)}. \tag{1.7}$$

Hypothes 1.1. *Let f and g be two integrable functions on $[1,\infty)$. Assume the following$(H_1), (H_2)$ There exist real constants m, M, n, N such that*

$$(H_1) \quad m \le f(x) \le M \tag{1.8}$$
$$(H_2) \quad n \le g(x) \le N; \tag{1.9}$$

Hypothes 1.2. *Let f and g be two integrable functions on $[1,\infty)$. Assume the following$(H_1'), (H_2')$ There exist $\varphi_1, \varphi_2, \psi_1$ and ψ_2 integrable functions such that*

$$(H_1') \quad \varphi_1 \le f(x) \le \varphi_2 \tag{1.10}$$
$$(H_2') \quad \psi_1 \le g(x) \le \psi_2. \tag{1.11}$$

Chapter 2

Some integral inequalities for Hadamard fractional integral

2.1 Introduction

In this chapter, we use Hadamard fractional integral, to establish certain integral inequalities, as Chebyshev-Gruss type by using one or two parameters(see[7, 8, 9, 10]).

2.2 Some integral inequalities for ${}_HJ$ integral

Theorem 1. *Let f be an integrable function on $[1,\infty)$ satisfying the condition* (1.7). *Then for $t>1$ and $\alpha,\beta>0$, one has*

$$\begin{aligned} & m\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t)+M\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t) \\ \geq\ & mM\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+{}_HJ^{\alpha}f(t){}_HJ^{\beta}f(t). \end{aligned} \tag{2.1}$$

Proof

From (1.7), for all $\tau \geq 1, \rho \geq 1$, we have

$$(M - f(\tau))(f(\rho) - m) \geq 0. \tag{2.2}$$

Therefore,

$$Mf(\rho) + mf(\tau) \geq mM + f(\tau)f(\rho). \tag{2.3}$$

Mutiplying both sides of (2.3) by $\frac{(\ln \frac{t}{\tau})^{\alpha-1}}{\tau\Gamma(\alpha)}$, $\tau \in (1, t)$, we get

$$\begin{aligned} & Mf(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} + m\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau) \\ \geq \; & mM\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} + f(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau). \end{aligned} \tag{2.4}$$

Integrating both sides of (2.4) with respect to τ on (1,t), we obtain

$$\begin{aligned} & Mf(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\ + \; & m\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau} \\ \geq \; & mM\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\ + \; & f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}, \end{aligned} \tag{2.5}$$

which yeilds

$$Mf(\rho)_H J^\alpha 1 + m_H J^\alpha f(t) \geq mM_H J^\alpha 1 + f(\rho)_H J^\alpha f(t). \tag{2.6}$$

Multiplying both sides of (2.6) by $\frac{(\ln \frac{t}{\rho})^{\beta-1}}{\rho\Gamma(\beta)}, \rho \in (1, t)$, we have

$$\begin{aligned} & M_H J^\alpha \frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} f(\rho) \\ + \; & m_H J^\alpha f(t)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} \end{aligned}$$

$$\begin{aligned} &\geq \; mM_H J^{\alpha}\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} \\ &+ \; {}_H J^{\alpha} f(t)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} f(\rho). \end{aligned} \tag{2.7}$$

Integrating both sides of (2.7) with respect to ρ on (1,t), we get

$$\begin{aligned} & M_H J^{\alpha} 1 \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln\frac{t}{\rho}\right)^{\beta-1} f(\rho)\frac{d\rho}{\rho} \\ + \; & m_H J^{\alpha} f(t)\frac{1}{\Gamma(\beta)} \int_1^t \left(\ln\frac{t}{\rho}\right)^{\beta-1} \frac{d\rho}{\rho} \\ \geq \; & mM_H J^{\alpha} 1 \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln\frac{t}{\rho}\right)^{\beta-1} \frac{d\rho}{\rho} \\ + \; & {}_H J^{\alpha} f(t)\frac{1}{\Gamma(\beta)} \int_1^t \left(\ln\frac{t}{\rho}\right)^{\beta-1} f(\rho)\frac{d\rho}{\rho}. \end{aligned} \tag{2.8}$$

From(1.6), we deduce inequality (2.1). □

Theorem 2. *Let f be an integrable function on $[1,\infty)$ satisfying the condition* (1.7)*. Then for $t > 1$ and $\alpha, \beta > 0$, one has*

$$\begin{aligned} & (m+M)^2 \frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} +_H J^{\alpha} f^2(t)\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)} \\ + \; & {}_H J^{\beta} f^2(t)\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} + 2_H J^{\alpha} f(t)_H J^{\beta} f(t) \\ \geq \; & 2(m+M)\left({}_H J^{\alpha} f(t)\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)} +_H J^{\beta} f(t)\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\right). \end{aligned} \tag{2.9}$$

Proof

By inequality (2.2), we setting

$$a = M - f(\tau), \; b = f(\rho) - m.$$

Now, according to the Young's inequality and for, $p = p' = 2$, we have

$$(M-f(\tau))^2+(f(\rho)-m)^2 \geq 2(M-f(\tau))(f(\rho)-m). \qquad (2.10)$$

Therefore

$$\begin{aligned} & (M+m)^2+f^2(\tau)+f^2(\rho)+2f(\rho)f(\tau) \\ \geq\ & 2(M+m)(f(\tau)+f(\rho)). \end{aligned} \qquad (2.11)$$

Multiplying both sides of (2.11) by

$$(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta),\ \tau,\ \rho\in(1,t),$$

we get

$$\begin{aligned} & (M+m)^2(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta)) \\ +\ & (\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta))(f(\tau)+f(\rho))^2 \\ \geq\ & 2(M+m) \\ \times\ & (\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta))f(\tau)+f(\rho)). \end{aligned} \qquad (2.12)$$

Then integrating(2.12) over $(1,t)$, we obtain

$$\begin{aligned} & (M+m)^2_H J^\alpha 1_H J^\beta 1 +_H J^\beta 1_H J^\alpha f^2(t) \\ +\ & {}_H J^\alpha 1_H J^\beta f^2(t) + 2_H J^\alpha f(t)_H J^\beta f(t) \\ \geq\ & 2(m+M)\left({}_H J^\alpha 1_H J^\beta f(t)_H J^\beta 1_H J^\alpha f(t)\right). \end{aligned} \qquad (2.13)$$

Hence

$$\begin{aligned} & (m+M)^2\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} +_H J^\alpha f^2(t)\frac{(\ln t)^\beta}{\Gamma(\beta+1)} \\ +\ & {}_H J^\beta f^2(t)\frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} + 2_H J^\alpha f(t)_H J^\beta f(t) \\ \geq\ & 2(m+M)\left({}_H J^\alpha f(t)\frac{(\ln t)^\beta}{\Gamma(\beta+1)} +_H J^\beta f(t)\frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}\right). \qquad \square \end{aligned}$$

Theorem 3. *Let f be an integrable function on $[1,\infty)$ satisfying the condition* (1.7). *Then for $t>1$ and $\alpha,\beta>0$, one has*

$$
\begin{aligned}
& M\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\beta} f(t) \\
\geq\ & m\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\ln t)^{\beta}}{\Gamma(\alpha+1)}{}_H J^{\alpha} f(t) \\
+\ & 2{}_H J^{\alpha}(M-f)^{1/2}(t){}_H J^{\beta}(f-m)^{1/2}(t). \qquad (2.14)
\end{aligned}
$$

Proof

From property 1.1 and $\theta_1=\theta_2=\frac{1}{2}$,
by setting

$$a=M-f(\tau),\ b=f(\rho)-m,\ \tau,\rho>1,$$

we have

$$\frac{(M-f(\tau))+(f(\rho)-m)}{2}\geq\sqrt{(M-f(\tau))}\sqrt{(f(\rho)-m)} \qquad (2.15)$$

Multiplying both sides of (2.15) by,

$$(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta),\ \tau,\rho\in(1,t),$$

we get

$$
\begin{aligned}
& \frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(M-f(\tau)) \\
+\ & \frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(f(\rho)-m) \\
\geq\ & 2\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\sqrt{(M-f(\tau))}\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\sqrt{(f(\rho)-m)}, \qquad (2.16)
\end{aligned}
$$

where

$$M\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}+\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}f(\rho)$$

$$
\begin{aligned}
&\geq\ m\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}+\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}f(\tau)\\
&+\ 2\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\sqrt{(M-f(\tau))}\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\sqrt{(f(\rho)-m)},\quad (2.17)
\end{aligned}
$$

then integrating (2.17), over $(1,t)$, we obtain

$$
\begin{aligned}
&M_H J^{\alpha}1_H J^{\beta}1+_H J^{\beta}1_H J^{\beta}f(t)\\
\geq\ &m_H J^{\alpha}1_H J^{\beta}1+_H J^{\beta}1_H J^{\alpha}f(t)\\
+\ &2_H J^{\alpha}(M-f)^{\frac{1}{2}}(t)_H J^{\beta}(f-m)^{\frac{1}{2}}(t). \quad (2.18)
\end{aligned}
$$

The inequalty (2.14), *is proved.* □

Theorem 4. *Let f and g be two integrable functions on $[1,\infty)$ satisfying the condition* (1.7) *and* (1.8). *Then for $t>1$ and $\alpha,\beta>0$, one has*

$$
\begin{aligned}
&\frac{n(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_H J^{\alpha}f(t)+\frac{M(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\beta}g(t)\\
\geq\ &\frac{nM(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+_H J^{\alpha}f(t)_H J^{\beta}g(t), \quad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{m(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_H J^{\alpha}g(t)+\frac{N(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\beta}f(t)\\
\geq\ &\frac{mN(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+_H J^{\beta}f(t)_H J^{\alpha}g(t), \quad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{MN(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+_H J^{\alpha}f(t)_H J^{\beta}g(t)\\
\geq\ &\frac{M(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\beta}g(t)+\frac{N(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_H J^{\alpha}f(t), \quad (2.21)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{mn(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+_H J^{\alpha}f(t)_H J^{\beta}g(t)\\
\geq\ &\frac{m(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\beta}g(t)+\frac{n(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_H J^{\alpha}f(t). \quad (2.22)
\end{aligned}
$$

Proof

To prove inequality (2.19), from hypothes 1.1, we have for $t \in [1, \infty)$ that

$$(M - f(\tau))(g(\rho) - n) \geq 0.$$

Therfore,

$$Mg(\rho) + nf(\tau) \geq nM + f(\tau)g(\rho). \tag{2.23}$$

Multiplying both sides of (2.23) by $(ln(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha),\ \tau \in (1, t)$, we get

$$\begin{aligned} & g(\rho)\frac{M(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} + n\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau) \\ \geq\ & nM\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} + g(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau). \end{aligned} \tag{2.24}$$

Integrating both sides of (2.24) with respect to τ on (1,t), we obtain

$$\begin{aligned} & g(\rho)\frac{M}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1}\frac{d\tau}{\tau} \\ + \ & \frac{n}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1} f(\tau)\frac{d\tau}{\tau} \\ \geq\ & \frac{nM}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1}\frac{d\tau}{\tau} \\ + \ & g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1} f(\tau)\frac{d\tau}{\tau}, \end{aligned} \tag{2.25}$$

where

$$Mg(\rho)_H J^\alpha 1 + n_H J^\alpha f(t) \geq nM_H J^\alpha 1 + g(\rho)_H J^\alpha f(t). \tag{2.26}$$

Multiplying both sides of (2.26) by $(\ln(t/\rho))^{\beta-1}/\rho\Gamma(\beta),\ \rho \in (1, t)$, we have

$$
\begin{aligned}
& M_H J^{\alpha} 1 \frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} g(\rho) \\
+ & \ {}_H J^{\alpha} f(t) \frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} n \\
\geq & \ n M_H J^{\alpha} 1 \frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} \\
+ & \ {}_H J^{\alpha} f(t) \frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)} g(\rho).
\end{aligned} \tag{2.27}
$$

Integrating both sides of (2.27) with respect to ρ on $(1, t)$, we get the inequality (2.19).

Finally, to prove (2.20) − (2.22), we use similar arguments as in the proof of inequality (2.19).

Remark 2.1. *We use the following inequalities.*

(2.20) $(N - g(\tau))(f(\rho) - m) \geq 0,$

(2.21) $(M - f(\tau))(g(\rho) - N) \leq 0,$

(2.22) $(m - f(\tau))(g(\rho) - m) \leq 0.$ □

Lemma 2.1. *Let f be an integrable functions on $[1, \infty)$ satisfying* (1.7). *Then for all $t > 1, \alpha > 0$, one has*

$$
\begin{aligned}
& \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\alpha} f^2(t) - ({}_H J^{\alpha} f(t))^2 \\
= & \left(M \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} -_H J^{\alpha} f(t) \right) \left({}_H J^{\alpha} f(t) - m \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} \right) \\
- & \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\alpha}((M - f(t))(f(t) - m)).
\end{aligned} \tag{2.28}
$$

Proof

To prove (see[8]). □

Theorem 5. *Let f and g be an integrable function on $[1, \infty)$ satisfying the condition (1.7) and (1.8). Then for all $t > 1, \alpha > 0$, we have*

$$\begin{aligned} & \left| \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\alpha} fg(t) -_H J^{\alpha} f(t)_H J^{\alpha} g(t) \right| \\ \leq & \left(\frac{1(\ln t)^{\alpha}}{2\Gamma(\alpha+1)} \right)^2 (M-m)(N-n). \end{aligned} \quad (2.29)$$

Proof

To prove Theorem 5 we need the preceding equality (2.28), in Lemma 2.1. More details, one can consult [3]. □

Chapter 3

New generalisations of fractional integral inequalities using Hadamard fractional integral

3.1 Introduction

In this chapter, we present some new fractional integral inequalities, using the Hadamard fractional integral, are obtained by using Young and weighted AM-GM inequalities(see[4, 5, 20]).

3.2 Main results on fractional integral inequalities

Theorem 6. *Let f be an integrable function on $[1,\infty)$ satisfying the condition (1.9). Then for all $t>1, \alpha>0, \beta>0$, we have*

$$\begin{aligned} & {}_HJ^{\beta}\varphi_1(t)\,{}_HJ^{\alpha}f(t) + {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}f(t) \\ \geq\ & {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}\varphi_1(t) + {}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}f(t). \end{aligned} \tag{3.1}$$

Proof

From (1.9), for all $\tau \geq 1$, $\rho \geq 1$, we have

$$(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho) \geq 0. \tag{3.2}$$

Therefore

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{3.3}$$

Multiplying both sides of (3.3) by $\frac{\left(\ln(\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}$, $\tau \in (1, t)$, we get

$$\begin{aligned} & f(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + \varphi_1(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau) \\ \geq \;& \varphi_1(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + f(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau). \end{aligned} \tag{3.4}$$

Integrating both sides of (3.4) with respect to τ on $(1, t)$, we obtain

$$\begin{aligned} & f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} \\ + \;& \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau} \\ \geq \;& \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} \\ + \;& f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}, \end{aligned} \tag{3.5}$$

which yeilds

$$f(\rho)_H J^\alpha \varphi_2(\tau) + \varphi_1(\rho)_H J^\alpha f(t) \geq \varphi_1(\rho)_H J^\alpha \varphi_2(\tau) + f(\rho)_H J^\alpha f(t). \tag{3.6}$$

Multiplying both sides of (3.6) by $(\ln(t/\tau))^{\beta-1}/\rho\Gamma(\beta)$, $\rho \in (1, t)$, we have

$${}_H J^\alpha \varphi_2(\tau)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}f(\rho)$$

$$
\begin{aligned}
&+ \ {}_HJ^{\alpha}f(t)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\varphi_2(\tau)\\
&\geq \ {}_HJ^{\alpha}\varphi_2(\tau)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\varphi_1(\rho)\\
&+ \ {}_HJ^{\alpha}f(t)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}f(\rho). \qquad (3.7)
\end{aligned}
$$

Integrating both sides of (3.7) with respect to ρ on $(1, t)$, we get

$$
\begin{aligned}
&{}_HJ^{\alpha}\varphi_2(\tau)\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho}\\
&+ \ {}_HJ^{\alpha}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{\rho}\right)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho}\\
&\geq \ {}_HJ^{\alpha}\varphi_2(\tau)\frac{1}{\Gamma(\beta)}\int_1^t(\ln t\rho)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho}\\
&+ \ {}_HJ^{\alpha}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho}. \qquad (3.8)
\end{aligned}
$$

Hence, we deduce inequality (3.1). □

Example 3.1. *Let f be an integrable function on $[1,\infty)$ satisfying the condition*

$$\ln t \leq f(t) \leq 1 + \ln t,$$

for all $t \geq 1$. Then for $t > 1$ and $\alpha = \beta > 0$, we deduce

$$
\begin{aligned}
&\left(\frac{2(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\right)_H J^{\alpha}f(t)\\
&\geq \left(\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\\
&+ \ ({}_HJ^{\alpha}f(t))^2.
\end{aligned}
$$

Remark 3.1. *Applying Theorem 6 for $\varphi_1 = m$ and $\varphi_2 = M$, we obtain Theorem 1.*

Theorem 7. *Let f be an integrable function on $[1,\infty)$ satisfying the condition $1/\theta_1 + 1/\theta_2 = 1$. Suppose that (1.9) holds. Then for all $t > 1, \alpha > 0, \beta > 0$, we have*

$$\begin{aligned} & \frac{1}{\theta_1}\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}((\varphi_2 - f)^{\theta_1})(t) \\ + & \frac{1}{\theta_2}\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}((f - \varphi_1)^{\theta_2})(t) \\ + & {}_HJ^{\alpha}\varphi_2(t){}_HJ^{\beta}\varphi_1(t) +_H J^{\alpha}f(t){}_HJ^{\beta}f(t) \\ \geq & {}_HJ^{\alpha}\varphi_2(t){}_HJ^{\beta}f(t) +_H J^{\alpha}f(t){}_HJ^{\beta}\varphi_1(t). \end{aligned} \tag{3.9}$$

Proof

Putting

$$a = \varphi_2(\tau) - f(\tau),\ \tau > 1$$

and

$$b = f(\rho) - \varphi_1(\rho),\ \rho > 1.$$

By the Young's inequality, we have

$$\frac{1}{\theta_1}(\varphi_2(\tau) - f(\tau))^{\theta_1} + \frac{1}{\theta_2}(f(\rho) - \varphi_1(\rho)^{\theta_2} \geq (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho). \tag{3.10}$$

Multiplying both sides of $(3.10) by (\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta),\ \tau,\ \rho \in (1,t)$, we get

$$\begin{aligned} & \frac{1}{\theta_1}\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\alpha)}(\varphi_2(\tau) - f(\tau))^{\theta_1} \\ + & \frac{1}{\theta_2}\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\alpha)}(f(\rho) - \varphi_2(\tau))^{\theta_2} \\ \geq & \frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau) - f(\tau))\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}(f(\rho) - \varphi_1(\rho)(\end{aligned} \tag{3.11}$$

Then, integrating (3.11) with respect to τ and ρ from 1 to t, we have

$$\frac{1}{\theta_1}{}_HJ^{\beta}(1)(t){}_HJ^{\alpha}(\varphi_2(\tau) - f)^{\theta_1}(t)$$

$$
\begin{aligned}
&+ \quad \frac{1}{\theta_2}{}_H J^{\alpha}(1)(t)_H J^{\beta}(f-\varphi_1(\rho))^{\theta_2}(t) \\
&\geq \quad {}_H J^{\alpha}(\varphi_2(\tau)-f)(t)_H J^{\beta}(f-\varphi_1(\rho)(t). \qquad (3.12)
\end{aligned}
$$

Which implies (3.9). □

Example 3.2. *Let f be an integrable function on $[1,\infty)$ satisfying the condition*

$$\ln t \leq f(t) \leq 1+\ln t,$$

for all $t \geq 1$. Then for $t>1$ $\alpha=\beta>0$, and $\theta_1=\theta_2=\frac{1}{2}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{2(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{4(\ln t)^{\alpha+2}}{\Gamma(\alpha+3)}+2_H J^{\alpha}f^2(t)\right) \\
+ \quad &\left(\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}+({}_H J^{\alpha}f(t))^2 \\
\geq \quad &\left(\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}\right){}_H J^{\alpha}f(t) \\
+ \quad &\frac{2(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\alpha}(f\ln t)(t).
\end{aligned}
$$

Remark 3.2. *Applying Theorem 7 for $\varphi_1=m$, $\varphi_2=M$ and $\theta_1=\theta_2=2$, we obtain inequality (2.9) in Theorem 2.*

Theorem 8. *Let f be an integrable function on $[1,\infty)$ satisfying the condition $\theta_1+\theta_2=1$. Suppose that (1.9) holds. Then for all $t>1,\alpha>0,\beta>0$, we have*

$$
\begin{aligned}
&\theta_1\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_H J^{\alpha}\varphi_2(t)+\theta_2\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\beta}f(t) \\
\geq \quad &{}_H J^{\alpha}(\varphi_2-f)^{\theta_1}(t)_H J^{\beta}(f-\varphi_1)^{\theta_2}(t) \\
+ \quad &\theta_1\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}{}_H J^{\alpha}f(t)+\theta_2\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_H J^{\beta}\varphi_1(t). \qquad (3.13)
\end{aligned}
$$

Proof

Putting

$$a=\varphi_2(\tau)-f(\tau),\ \tau>1$$

and

$$b=f(\rho)-\varphi_1(\rho),\ \ \rho>1.$$

By the weighted AM-GM inequality, we get

$$\theta_1(\varphi_2(\tau)-f(\tau))+\theta_2(f(\rho)-\varphi_1(\rho))\geq(\varphi_2(\tau)-f(\tau))^{\theta_1}(f(\rho)-\varphi_1(\rho))^{\theta_2}. \tag{3.14}$$

Multiplying both sides of (3.14) *by* $(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}/\tau\rho\Gamma(\alpha)\Gamma(\beta),\ \ \tau,\rho\in(1,t)$,we get

$$\begin{aligned}
&\theta_1\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau)-f(\tau))\\
+\ \ &\theta_2\frac{(\ln(t/\tau))^{\alpha-1}(\ln(t/\rho))^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(f(\rho)-\varphi_2(\tau))\\
\geq\ \ &\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau)-f(\tau))^{\theta_1}\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}(f(\rho)-\varphi_1(\rho))^{\theta_2}
\end{aligned}\tag{3.15}$$

Then, integrating (3.15) with respect to τ and ρ from 1 to t, we have

$$\begin{aligned}
&\theta_{1H}J^{\beta}(1)(t)_HJ^{\alpha}(\varphi_2(\tau)-f)(t)+\theta_{2H}J^{\alpha}(1)(t)_HJ^{\beta}(f-\varphi_1(\rho))(t)\\
\geq\ \ &{}_HJ^{\alpha}(\varphi_2(\tau)-f)^{\theta_1}(t)_HJ^{\beta}(f-\varphi_1(\rho))^{\theta_2}(t).
\end{aligned}\tag{3.16}$$

Then using the inequalty (3.14), hence inequalty(3.13). □

Example 3.3. *Let f be an integrable function on $[1,\infty)$ satisfying the condition*

$$\ln t\leq f(t)\leq 1+\ln t,$$

for all $t\geq 1$*. Then for* $t>1$ *and* $\alpha=\beta>0$*, we have*

$$\begin{aligned}
&\frac{(\ln t)^{2\alpha}}{\Gamma^2(\alpha+1)}\\
\geq\ \ &2_HJ^{\alpha}\left(\sqrt{1+\ln t-f}\right)(t)_HJ^{\alpha}\left(\sqrt{-\ln t+f}\right)(t).
\end{aligned}$$

Remark 3.3. *Applying Theorem 8 for* $\varphi_1=m,\ \varphi_2=M$ *and* $\theta_1=\theta_2=\frac{1}{2}$*, we obtain inequality* (2.14) *in Theorem 3.*

Theorem 9. *Let f and g be two integrable functions on $[1,\infty)$ satisfying the condition* (1.9) *and* (1.10). *Then for $t>1$ and $\alpha,\beta>0$, one has*

$$\begin{aligned} & {}_HJ^\beta\psi_1(t)_HJ^\alpha f(t) +_H J^\alpha\varphi_2(t)_HJ^\beta g(t) \\ \geq\; & HJ^\beta\psi_1(t)_HJ^\alpha\varphi_2(t) +_H J^\alpha f(t)_HJ^\beta g(t). \end{aligned} \tag{3.17}$$

$$\begin{aligned} & {}_HJ^\beta\varphi_1(t)_HJ^\alpha g(t) +_H J^\alpha\psi_2(t)_HJ^\beta f(t) \\ \geq\; & HJ^\beta\varphi_1(t)_HJ^\alpha\psi_2(t) +_H J^\beta f(t)_HJ^\alpha g(t). \end{aligned} \tag{3.18}$$

$$\begin{aligned} & {}_HJ^\beta\psi_2(t)_HJ^\alpha\varphi_2(t) +_H J^\alpha f(t)_HJ^\beta g(t) \\ \geq\; & HJ^\alpha\varphi_2(t)_HJ^\beta g(t) +_H J^\beta\psi_2(t)_HJ^\alpha f(t). \end{aligned} \tag{3.19}$$

$$\begin{aligned} & {}_HJ^\alpha\varphi_1(t)_HJ^\beta\psi_1(t) +_H J^\alpha f(t)_HJ^\beta g(t) \\ \geq\; & HJ^\alpha\varphi_1(t)_HJ^\beta g(t) +_H J^\beta\psi_1(t)_HJ^\alpha f(t). \end{aligned} \tag{3.20}$$

Proof

To prove (3.17), *From* (1.9) *and* (1.10), *we have for $t\in[1,\infty)$ that* $(\varphi_2(\tau)-f(\tau))(g(\rho)-\psi_1(\rho))\geq 0$.

Therfore,

$$\varphi_2(\tau)g(\rho)+\psi_1(\rho)f(\tau)\geq\psi_1(\rho)\varphi_2(\tau)+f(\tau)g(\rho). \tag{3.21}$$

Multiplying both sides of (3.21) *by* $(\ln(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau\in(1,t)$, *we get*

$$\begin{aligned} & g(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau)+\psi_1(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau) \\ \geq\; & \psi_1(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau)+g(\rho)\frac{(\ln(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau). \end{aligned} \tag{3.22}$$

Integrating both sides of (3.22) *with respect to* τ *on* $(1,t)$, *we obtain*

$$\begin{aligned} & g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} + \psi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1}f(\tau)\frac{d\tau}{\tau} \\ \geq\ & \psi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} \\ +\ & g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t (\ln(t/\tau))^{\alpha-1}f(\tau)\frac{d\tau}{\tau}. \end{aligned} \tag{3.23}$$

Then we get

$$g(\rho)_H J^\alpha \varphi_2(t) + \psi_1(\rho)_H J^\alpha f(t) \geq \psi_1(\rho)_H J^\alpha \varphi_2(t) + g(\rho)_H J^\alpha f(t). \tag{3.24}$$

Multiplying both sides of (3.24) *by* $(\log(t/\rho))^{\beta-1}/\rho\Gamma(\beta),\ \rho\in(1,t)$, *we have*

$$\begin{aligned} & {}_H J^\alpha \varphi_2(t)\frac{(\log(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}g(\rho) +_H J^\alpha f(t)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\psi_1(\rho) \\ \geq\ & {}_H J^\alpha \varphi_2(t)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}\psi_1(\rho) +_H J^\alpha f(t)\frac{(\ln(t/\rho))^{\beta-1}}{\rho\Gamma(\beta)}g(\rho). \end{aligned} \tag{3.25}$$

Integrating both sides of (3.25) *with respect to* ρ *on* $(1,t)$, *we get the inequality* (3.17).

Finally, to prove (3.18) $-$ (3.20), *we use similar arguments as in the proof of inequality* (3.25).

Remark 3.4. *We use the following inequalities.*

(3.18) $(N-g(\tau))(f(\rho)-m)\geq 0$,

(3.19) $(M-f(\tau))(g(\rho)-N)\leq 0$,

(3.20) $(m-f(\tau))(g(\rho)-m)\leq 0$. $\square$

Corollary 3.1. *Applying Theorem 9 for* $\varphi_1=m, \varphi_2,=M, \psi_1=n$ *and* $\psi_2=N$, *ew obtain the inequalities* (2.19) $-$ (2.22) *in Theorem 4.*

Theorem 10. *Let f and g be two integrable functions on $[1,\infty)$ satisfying the condition $1/\theta_1 + 1/\theta_2 = 1$. Suppose that (1.9) and (1.10) hold. Then for all $t > 1, \alpha > 0, \beta > 0$, we have*

$$\begin{aligned} & \frac{1}{\theta_1}\frac{(\ln t)^\beta}{\Gamma(\beta+1)}{}_H J^\alpha(\varphi_2 - f)^{\theta_1}(t) \\ + & \frac{1}{\theta_2}\frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\beta(\psi_2 - g)^{\theta_2}(t) \\ \geq & {}_H J^\alpha(\varphi_2 - f)(t)\, {}_H J^\beta(\psi_2 - g)(t), \end{aligned} \tag{3.26}$$

$$\begin{aligned} & \frac{1}{\theta_1}{}_H J^\alpha(\varphi_2 - f)^{\theta_1}(t)\, {}_H J^\beta(\psi_2 - g)^{\theta_1}(t) \\ + & \frac{1}{\theta_2}{}_H J^\alpha(\psi_2 - g)^{\theta_2}(t)\, {}_H J^\beta(\varphi_2 - f)^{\theta_2}(t) \\ \geq & {}_H J^\alpha(\varphi_2 - f)(\psi_2 - g)(t)\, {}_H J^\beta(\varphi_2 - f)(\psi_2 - g)(t), \end{aligned} \tag{3.27}$$

$$\begin{aligned} & \frac{1}{\theta_1}\frac{(\ln t)^\beta}{\Gamma(\beta+1)}{}_H J^\alpha(f - \varphi_1)^{\theta_1}(t) \\ + & \frac{1}{\theta_2}\frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\beta(g - \psi_1)^{\theta_2}(t) \\ \geq & {}_H J^\alpha(f - \varphi_1)(t)\, {}_H J^\beta(g - \psi_1)(t), \end{aligned} \tag{3.28}$$

$$\begin{aligned} & \frac{1}{\theta_1}{}_H J^\alpha(f - \varphi_1)^{\theta_1}(t)\, {}_H J^\beta(g - \psi_1)^{\theta_1}(t) \\ + & \frac{1}{\theta_2}{}_H J^\alpha(g - \psi_1)^{\theta_2}(t)\, {}_H J^\beta(f - \varphi_1)^{\theta_2}(t) \\ \geq & {}_H J^\alpha(f - \varphi_1)(g - \psi_1)(t)\, {}_H J^\beta(f - \varphi_1)(g - \psi_1)(t). \end{aligned} \tag{3.29}$$

Proof

The inequalities (3.26) – (3.29) can be proved by choosing of the parameters in the Young inequality(Theorem7)

$$(3.26)\quad \begin{aligned} a &= \varphi(\tau) - f(\tau),\\ b &= \psi_2(\rho) - g(\rho). \end{aligned}$$

$$(3.27)\quad \begin{aligned} a &= (\varphi(\tau) - f(\tau))(\psi_2(\rho) - g(\rho)),\\ b &= (\varphi(\tau) - g(\tau))(\psi_2(\rho) - f(\rho)). \end{aligned}$$

$$(3.28)\quad \begin{aligned} a &= f(\tau) - \varphi_1(\tau),\\ b &= g(\rho) - \psi_1(\rho). \end{aligned}$$

$$(3.29)\quad \begin{aligned} a &= (f(\tau) - \varphi_1(\tau))(g(\rho) - \psi_1(\rho)),\\ b &= (g(\tau) - \psi_1(\tau))(f(\rho) - \varphi_1(\rho)). \qquad \square \end{aligned}$$

Theorem 11. *Let f and g be two integrable functions on $[1,\infty)$ satisfying the condition $\theta_1 + \theta_2 = 1$. Suppose that (1.9) and (1.10) hold. Then for all $t > 1, \alpha > 0, \beta > 0$, we have*

$$\begin{aligned} & \theta_1 \frac{(\ln t)^\beta}{\Gamma(\beta+1)}{}_H J^\alpha \varphi_2(t) + \theta_2 \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\beta \psi_2(t)\\ \geq\ & {}_H J^\alpha (\varphi_2 - f)^{\theta_1}(t)\, {}_H J^\beta (\psi_2 - g)^{\theta_2}(t)\\ +\ & \theta_1 \frac{(\ln t)^\beta}{\Gamma(\beta+1)}{}_H J^\alpha f(t) + \theta_2 \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\beta g(t), \end{aligned} \tag{3.30}$$

$$\begin{aligned} & \theta_{1H} J^\alpha \varphi_2(t)\, {}_H J^\beta \psi_2(t) + \theta_{1H} J^\alpha f(t)\, {}_H J^\beta g(t)\\ +\ & \theta_{2H} J^\alpha \psi_2(t)\, {}_H J^\beta \varphi_2(t) + \theta_{2H} J^\alpha g(t)\, {}_H J^\beta f(t)\\ \geq\ & {}_H J^\alpha (\varphi_2 - f)^{\theta_1} (\psi_2 - g)^{\theta_2}(t)\, {}_H J^\beta (\psi_2 - g)^{\theta_1}\\ \times\ & (\varphi_2 - f)^{\theta_2}(t)\\ +\ & \theta_{1H} J^\beta g(t)\, {}_H J^\alpha \varphi_2(t) + \theta_{1H} J^\alpha f(t)\, {}_H J^\beta \psi_2(t)\\ +\ & \theta_{2H} J^\beta f(t)\, {}_H J^\alpha \psi_2(t) + \theta_{2H} J^\alpha g(t)\, {}_H J^\beta \varphi_2(t), \end{aligned} \tag{3.31}$$

$$\begin{aligned} & \theta_1 \frac{(\ln t)^\beta}{\Gamma(\beta+1)}{}_H J^\alpha f(t) + \theta_2 \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\beta g(t)\\ \geq\ & {}_H J^\alpha (f - \varphi_1)^{\theta_1}(t)\, {}_H J^\beta (g - \psi)^{\theta_2}(t)\\ +\ & \theta_1 \frac{(\ln t)^\beta}{\Gamma(\beta+1)}{}_H J^\alpha \varphi_1(t) + \theta_2 \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\beta \psi_1(t), \end{aligned} \tag{3.32}$$

$$
\begin{aligned}
& \theta_{1H}J^{\alpha}f(t)_{H}J^{\beta}g(t)+\theta_{1H}J^{\alpha}\varphi_1(t)_{H}J^{\beta}\psi_1(t)\\
+ \; & \theta_{2H}J^{\alpha}g(t)_{H}J^{\beta}f(t)+\theta_{2H}J^{\alpha}\psi_1(t)_{H}J^{\beta}\varphi_1(t)\\
\geq \; & {}_{H}J^{\alpha}(f-\varphi_1)^{\theta_1}(g-\psi_1)^{\theta_2}(t)_{H}J^{\beta}(g-\psi_1)^{\theta_1}\\
\times \; & (f-\varphi_1)^{\theta_2}(t)\\
+ \; & \theta_{1H}J^{\alpha}f(t)_{H}J^{\beta}\psi_1(t)+\theta_{1H}J^{\alpha}\varphi_1(t)_{H}J^{\beta}g(t)\\
+ \; & \theta_{2H}J^{\beta}f(t)_{H}J^{\alpha}\psi_1(t)+\theta_{2H}J^{\beta}\varphi_1(t)_{H}J^{\alpha}g(t). \qquad (3.33)
\end{aligned}
$$

Proof

The inequalities (3.30) − (3.33) *can be proved by choosing of the parameters in the weighted AM-GM (Theorem8)*

(3.30) $a=\varphi(\tau)-f(\tau)$,
$b=\psi_2(\rho)-g(\rho)$.

(3.31) $a=(\varphi(\tau)-f(\tau))(\psi_2(\rho)-g(\rho))$,
$b=(\varphi(\tau)-g(\tau))(\psi_2(\rho)-f(\rho))$.

(3.32) $a=f(\tau)-\varphi_1(\tau)$,
$b=g(\rho)-\psi_1(\rho)$.

(3.33) $a=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho))$,
$b=(g(\tau)-\psi_1(\tau))(f(\rho)-\varphi_1(\rho))$. □

Lemma 3.1. *Let f be an integrable function on $[1,\infty)$ satisfying the condition* (1.9)*. Then for all $t>1,\alpha>0,$ we have*

$$
\begin{aligned}
& \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}f^2(t)-\left({}_{H}J^{\alpha}f(t)\right)^2\\
= \; & \left({}_{H}J^{\alpha}\varphi_2(t)-{}_{H}J^{\alpha}f(t)\right)\left({}_{H}J^{\alpha}f(t)-{}_{H}J^{\alpha}\varphi_1(t)\right)\\
- \; & \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}((\varphi_2-f)(f-\varphi_1)(t))\\
+ \; & \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}{}_{H}J^{\alpha}\varphi_1 f(t)-{}_{H}J^{\alpha}\varphi_1(t)_{H}J^{\alpha}f(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\alpha \varphi_2 f(t) -_H J^\alpha \varphi_2(t)_H J^\alpha f(t) \\
&+ \quad {}_H J^\alpha \varphi_1(t)_H J^\alpha \varphi_2(t) - \frac{(\ln t)^\alpha}{\Gamma(\alpha+1)}{}_H J^\alpha \varphi_1\varphi_2(t). \qquad (3.34)
\end{aligned}
$$

Proof

For any $\tau > 1, \rho > 1,$ *we have*

$$
\begin{aligned}
& \quad (\varphi_2(\rho) - f(\rho))(f(\tau) - \varphi_1(\tau)) \\
&+ \quad (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \\
&- \quad (\varphi_2(\tau) - f(\tau))(f(\tau) - \varphi_1(\tau)) \\
&- \quad (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) \\
&= \quad f^2(\tau) + f^2(\rho) - 2f(\tau)f(\rho) + \varphi_2(\rho)f(\tau) \\
&+ \quad \varphi_1(\tau)f(\rho) - \varphi_1(\tau)\varphi_2(\rho) \\
&+ \quad \varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) - \varphi_1(\rho)\varphi_2(\tau) \\
&- \quad \varphi_2(\tau)f(\tau) + \varphi_1(\tau)\varphi_2(\tau) \\
&- \quad \varphi_1(\tau)f(\tau) - \varphi_2(\rho)f(\rho) \\
&+ \quad \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho)f(\rho). \qquad (3.35)
\end{aligned}
$$

Mutiplying (3.35) *by* $(\ln t/\tau))^{\alpha-1}/\tau\Gamma(\alpha),\ \tau \in (1,t),\ t > 1$ *and integrating the resulting with respect to* τ *from 1 to* t, *we get*

$$
\begin{aligned}
& \quad (\varphi_2(\rho) - f(\rho))({}_H J^\alpha f(t) -_H J^\alpha \varphi_1(t)) \\
&+ \quad ({}_H J^\alpha \varphi_2(t) -_H J^\alpha f(t))(f(\rho) - \varphi_1(\rho)) \\
&- \quad {}_H J^\alpha((\varphi_2 - f)(f - \varphi_1)(t)) - (\varphi_2(\rho) - f(\rho)) \\
&\times \quad (f(\rho) - \varphi_1(\rho))\frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \\
&= \quad {}_H J^\alpha f^2(t) + f^2(\rho)\frac{(\ln t)^\alpha}{\Gamma(\alpha+1)} \\
&- \quad 2f(\rho)_H J^\alpha f(t) + \varphi_2(\rho)_H J^\alpha f(t) \\
&+ \quad f(\rho)_H J^\alpha \varphi_1(t) - \varphi_2(\rho)_H J^\alpha \varphi_1(t) \\
&+ \quad f(\rho)_H J^\alpha \varphi_2(t) + \varphi_1(\rho)_H J^\alpha f(t) \\
&- \quad \varphi_1(\rho)_H J^\alpha \varphi_2(t) -_H J^\alpha \varphi_2 f(t) +_H J^\alpha \varphi_1\varphi_1(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \ {}_{H}J^{\alpha}\varphi_1 f(t) - \varphi_1(\rho) f(\rho)\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \ \varphi_1(\rho)\varphi_2(\rho)\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} - \varphi_1(\rho) f(\rho)\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}. \qquad (3.36)
\end{aligned}
$$

Mutiplying (3.36) *by* $(\ln t/\rho))^{\alpha-1}/\rho\Gamma(\alpha)$, $\rho \in (1,t)$, $t > 1$ *and integrating the resulting with respect to* ρ *from 1 to* t, *we obtain the inequality* (3.34). □

Remark 3.5. *Applying Lemma 3.1 for* $\varphi_1 = m$, $\varphi_2 = M$, *we obtain equality* (2.28) *in Lemma 2.1.*

Conclusion

In this memoir, we have considered some fractional integral inequalities, via Hadamard's fractional integral.

New integral inequalities are obtained including a Gruss-type Hadamard fractional integral inequality, by using Young and weighted AM-GM inequality. Many other cases are also discussed.

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