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PUBLICATIONS

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PREFACE

In light of the increasing prominence of variable-order fractional calculus within contemporary scientific research, and recognizing the substantial attention it has drawn from a wide array of distinguished scholars across the globe, we have made the strategic decision to embark on a comprehensive exploration of this intellectually stimulating and highly relevant topic. The primary objective of our study is to conduct an in-depth examination of the mathematical intricacies associated with the existence, uniqueness, and if feasible the stability of solutions to certain classes of differential equations governed by variable-order fractional derivatives.

Our approach will be rooted in the sophisticated framework of fixed-point theory, which provides powerful tools for analyzing the behavior of such complex systems. Additionally, we intend to apply rigorous stability criteria to ensure that the solutions we identify not only exist and are unique but also exhibit robust stability under various conditions. This dual focus on both the theoretical and practical aspects of variable-order fractional differential equations is aimed at yielding new insights that could significantly advance our understanding of this emerging field, potentially leading to innovative applications across a range of scientific and engineering disciplines. Through this research, we hope to contribute meaningfully to the ongoing development of variable-order fractional calculus, positioning our work at the forefront of this rapidly expanding area of mathematical inquiry.

ABSTRACT

Abstract: In this doctoral dissertation, we investigate the existence, uniqueness, and stability of solutions for various classes of nonlinear initial and boundary value problems (Pantograph, Langevin, Logistic) involving the variable order fractional operators. All conclusions drawn in the present research have been proven utilizing the variable order fractional calculus and fixed point theorem using the piecewise constant functions properties, which are crucial to convert the considered problems into an equivalent standard constant order counterparts. Furthermore, we investigate the stability in terms of Ulam-Hyers-Rassias stability criterion, and under further assumptions on the nonlinear term, we obtain the generalized Lyapunov inequalities.

Keywords and phrases : Variable order fractional differential equations, Boundary value problem, Piecewise constant functions, Fixed point theorem, Pantograph equations, Greens function, Generalized Lyapunov inequalities, Stability criterion, Brownian motion, Langevin equations, Fluctuations, Logistic equation, Mettag-Leffler function, West function.

GENERAL NOTATIONS

E	Banach Space.
L^p	Vector Space of p-Integrable Functions
C	Vector Space of Continuous Functions
$\|\|.\|\|_\infty$	Supremum Norm.
$H(\cdot,\cdot)$	Nonlinear Function.
$\mathcal{N}_H(\cdot)$	Nemytskii's Operator Associated to Function H.
BVP	Boundary Value Problems.
DDE	Delay Differential Equations.
HFPE	Hadamard Fractional Pantograph Equation.
FLBVP	Fractional Langevin Boundary Value Problem.
VOFLE	Variable Order Fraction Logistic Equation.
$EigVle$	Eigen Values.
$EigFct$	Eigen Functions.
$I_{a^+}^{\alpha(t)}$	Variable Order Riemann-Liouville Fractional Integral.
${}^H I_{a^+}^{\alpha(t)}$	Variable Order Hadamard Fractional Integral.
${}^{RL}\mathcal{D}_{a^+}^{\alpha(t)}$	Variable Order Riemann-Liouville Fractional Derivative.
${}^C\mathcal{D}_{a^+}^{\alpha(t)}$	Variable Order Caputo Fractional Derivative.
${}^H\mathcal{D}_{a^+}^{\alpha(t)}$	Variable Order Hadamard Fractional Derivative.
$G(\cdot,\cdot)$	Green Functions.
$E_{\alpha,\beta}(\cdot)$	Two Parameters Mittag–Lefller Function.

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INTRODUCTION

The primary idea toward fractional calculus seeks to replace natural numbers with rational ones in the order of fractional operators. Despite its basic and simple nature, this concept contains extraordinary effects and consequences that characterize some physics, dynamical systems, modeling, control theory, bioengineering, and biomedical applications phenomena. Thus gaining substantial recognition and importance due to its frequent occurrence in several study disciplines and engineering fields (see [2, 6, 7, 18, 19, 23, 42]).

Variable order fractional operators are a variation of their constant order counterparts, which correspond to a more sophisticated operator category whose order is determined by specific variables. In recent years, several individuals have been interested in the existence and uniqueness of solutions to boundary value problems for fractional differential equations of such order. Although there is a substantial amount of literature on solutions to these problems of constant fractional order, few studies deal with the existence of solutions to boundary value problems of variable order and are rarely discussed in literature, we point out few papers [13, 47, 48, 49, 54].

The Hadamard fractional operators, first described in [27] and then generalized to variable fractional order, have recently been examined in [8, 9]

Boundary value problems (BVPs) are basic constructs in many fields, including mathematics, physics, and engineering. They provide a strong framework for modeling real-world processes with well-defined boundary conditions, and so play an important role in recognizing and predicting dynamical system behaviors. By setting these boundary conditions, BVPs enable researchers and engineers to accurately determine solutions inside a specific domain, allowing them to examine system responses under different situations. Furthermore, fostering the development of powerful computational techniques and numerical methods, essential for addressing complex problems that arise in modern science and engineering, to bridge theoretical concepts with practical applications, facilitating advancements in technology, innovation, and scientific understanding.

Given that pantograph equations are often referred to as differential equations with proportional delays, they have grown into a main example of a delay differential equation in recent decades. It has received a great deal of interest in many different domains of pure and practical mathematics, including number theory, dynamical systems, probability, quantum physics, and electro dynamics, and has been thoroughly researched by numerous researchers, (for more information, see [12, 26, 31, 51]). The term "delay" in delay differential equations (DDEs) refers to the fact that these equations involve time delays in the evolution of the system. In traditional differential equations, the evolution of a system at a given time depends only on the state of the system at that specific time. However, in delay differential equations, the evolution of the system at a particular time depends not only on its current state but also on its past states at certain delayed times. This delay reflects a real world phenomenon where the effects of some processes or interactions are not felt immediately but only after a certain amount of time has passed allowing for a more accurate representation of systems dynamics where time lags play a crucial role. In particular, Ockendon and Taylor, [39] investigated how the electric current is gathered by the pantograph of an electric locomotive, hence where it derives its name.

During the last few decades, the stability theory of functional equations has been investigated in many different spaces. In 1940, at the University of Wisconsin, Dr S. Ulam [50] proposed the problem of stability for the following functional equation $f(x+y) = f(x) + f(y)$. Suppose f satisfies this equation only approximately. Then does there exist a linear function which f approximates? To make the statement of the problem precise, let E and F be Banach spaces and let δ be a positive number. A linear function or a transformation f of E into F is called δ-linear if $||f(x+y)-f(x)-f(y)|| \leq \delta$ for all $x, y \in E$. Then the problem may be stated as follows. Does there exist for each $\epsilon > 0$ a $\delta > 0$ such that, to each δ-linear transformation $f : E \longrightarrow F$ there corresponds a linear transformation $l : E \longrightarrow F$ satisfying the inequality $||f(x) - l(x)|| \leq \epsilon$ for all $x \in E$? A year later, Hyers [30] provided the first solution to the

Ulam issue in the context of Banach spaces. As a result, this sort of stability became known as the Ulam-Hyers stability. Rassias [41] offered a generalization of the Ulam-Hyers stability in 1978. Following that, the Ulam-Hyers-Rassias stability has evolved into one of the most important criteria in the field of mathematical analysis, particularly the stability of differential equations.

Eigenvalues are essential mathematical notions having numerous applications in physics, engineering, and data analysis. They represent scalar values associated with linear transformations or operations, such as matrices. They play an important role in understanding the behavior of dynamical systems and serve as indicators to help predict stability and convergence properties, by computing their lower and upper bounds. In the mid 1800s, Laplace was confronted with an unexpected difficulty, an equation derived from Newtons second law

$$-\frac{d}{dt}\left[p(t)\frac{d}{dt}X(t)\right]+q(t)X(t)=H(t), \quad t\geq 0,$$

together with initial conditions of position and velocity, $X(0)=X_0$, $X'(0)=X_0'$, and could not be solved explicitly, despite the efforts of many mathematicians. These types of equations are used in several fields of physics, including planetary dynamics, heat conduction in one-dimensional objects, and wave propagation, among many others problems. Of course, the answers to certain specific circumstances were widely known, as Euler developed the characteristic equation for constant-coefficient situations in 1730. Bessel expanded on Daniel Bernoulli's radial solutions to create Bessel functions around 1820, and Fourier's work, which combined series expansions with variable separation, enabled the solution of several partial differential equations. However, in order to apply the Fourier method for linear operators with nonconstant coefficients in $\Omega=[0,L]\times(0,\infty)$, we need to know the spectrum and eigenvalues of the following second-order problem

$$-\frac{d}{dt}\left[p(t)\frac{d}{dt}X(t)\right]+q(t)X(t)=\mathcal{W}(t)X(t),$$

with some condition induced by the boundary conditions of the original problem, such as the zero Dirichlet or Neumann boundary conditions. However in 1830, Sturm who was working as an assistant to Fourier and Liouville, studied this problem, and obtained the core results of Sturm-Liouville theory stated as:

Theorem.[40] Assume that p, q, and $\mathcal{W}$ are continuous functions such that $0 \leq \mathcal{W} \leq M$, $p > 0$ on a closed bounded real interval $[0, L]$, and consider the eigenvalue problem

$$\begin{cases} -\dfrac{d}{dt}\left[p(t)\dfrac{d}{dt}X(t)\right] + q(t)X(t) = \mathcal{W}(t)X(t), \quad t \in (0, L), \\ X(0) = X(L) = 0. \end{cases}$$

Then

$i)$ There exists a sequence of real eigenvalues $\{EigVle_i\}_{i\geq 1}$ such that

$$EigVle_1 < EigVle_2 < \cdots < EigVle_i < \cdots \longrightarrow +\infty.$$

$ii)$ To each eigenvalue $EigVle_i$ there corresponds a unique eigenfunction $EigFct_i$, which has exactly $i + 1$ zeros in $[0, L]$.

In particular, Sturm obtained a lower bound for the first eigenvalue of the following simple problem in $[0, L]$ involving the $L^\infty(0, L)$ norm of $\mathcal{W}$

$$\begin{cases} \dfrac{d^2}{dt^2}X(t) = EigVle_1\mathcal{W}(t)X(t), \quad t \in (0, L), \\ X(0) = X(L) = 0, \end{cases}$$

given by

$$\frac{\pi^2}{ML^2} \leq EigVle_1.$$

One could ask whether there exists a lower bound for $EigVle_1$ involving different norms of $\mathcal{W}$. Indeed, for the L^1 norm, such a lower bound is known as Lyapunov inequality, which is named after Aleksander Mikhailovich Lyapunov, a Russian mathematician who made significant contributions to the study of stability of nonlinear dynamical systems in the late 19th and early 20th centuries, which are mathematical expressions that play a fundamental role in stability analysis, particularly in the context of dynamic systems by examining the properties of the second term of the differential equation, which are scalar functions that help quantify the system's behavior. These inequalities are crucial tools for determining whether a system, whether it be a mechanical system, an electrical circuit, or a biological model, will remain within a certain stable region, or conversely, whether it will exhibit unpredictable behavior, oscillations, or instability.

The result proven by Lyapunov [35], states that for a continuous function $\mathcal{W} : [a, b] \longrightarrow \mathbb{R}$, if the boundary value problem

$$\begin{cases} \dfrac{d^2}{dt^2}X(t) + \mathcal{W}(t)X(t) = 0, \quad t \in (a, b), \\ X(a) = X(b) = 0, \end{cases}$$

has a non-trivial solution, then

$$\int_a^b |\mathcal{W}(t)|dt > \frac{4}{b-a}, \quad a < b < +\infty.$$

In particular, the first eigenvalue of the above problem, can be bounded as follows

$$\frac{4}{(b-a)\displaystyle\int_a^b |\mathcal{W}(t)|dt} \leq EigVle_1.$$

Brownian motion is the erratic motion of particles floating in a fluid (liquid or gas) caused by their collision with rapid molecules in the considered fluid. This phenomena was named after Scottish botanist Robert Brown, who first discovered it in 1827 while investigating pollen particles floating in water. However, it was mathematically explained and formalized through the work of Albert Einstein and the French mathematician Louis Bachelier. Particle motion is characterized by random changes in direction and speed, so it plays an important role in modeling various phenomena by simplifying the dynamics of real systems and providing a useful framework for understanding particle behavior in a fluctuating environment. It is widely used in various fields that involve random fluctuations such as physics, chemistry, biology, and finance.

In physics, processes are frequently represented mathematically via the Langevin equation, which is a stochastic differential equation that explains the motion of a particle undergoing Brownian motion in the presence of a random force. It is extensively used in the study of statistical mechanics and is named after the French physicist Paul Langevin, who in 1908 formulated the standard version of this equation in terms of ordinary derivatives as mentioned in [34]

$$m\frac{d^2}{dt^2}X(t) + \lambda\frac{d}{dt}X(t) = \eta(t),$$

where m is the mass of the particle, λ is the friction coefficient, and $\eta(t)$ is a random force. However, in complex media, this model did not appear to accurately describe the dynamics of the system. In 1966, Kubo [33] introduced the extended Langevin equation, which includes a fractional memory kernel to describe fractal and memory properties. Mainardi et al. [38] created the fractional Langevin equation in the 1990s. This provided several fascinating discoveries about the existence, uniqueness, and stability of solutions to fractional order Langevin equations (For further information, see [3, 4, 14] and the references therein).

Dynamical systems theory is a branch of mathematics and physics that studies the mathematical behavior and classification of how systems evolve over time. It provides valuable insights into the complexities that surround the development of real-world processes in response to their initial conditions and the governing equations that describe the dynamics, which can range from simple deterministic systems to extremely complicated and chaotic ones. This theory has numerous applications in physics, engineering, biology, economics, and even the social sciences, allowing researchers to investigate system stability, periodicity, and long-term behavior, making it an essential tool for predicting and modeling the behavior of unpredictable events in the natural and social worlds.

The logistic equation is a fundamental mathematical model used for estimating population increase or the spread of phenomena in a limited environment. Pierre François Verhulst introduced it in the nineteenth century, and it has since become a cornerstone in many fields, including ecology, epidemiology, economics, and even the study of social trends, it is expressed in continuous form as a nonlinear ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}X(t) = \kappa X(t)\left(1 - \dfrac{X(t)}{K}\right), \\ X(0) = X_0. \end{cases}$$

The equation takes two essential aspects into account: the Malthusian parameter $\kappa > 0$ expressing the intrinsic growth rate of species, and K represent the carrying capacity of the environment. If we normalize the above equation, then the above equation is reduced in the nonlinear differential equation written as

$$\begin{cases} \dfrac{d}{dt}X(t) = \kappa X(t)\left(1 - X(t)\right), \\ X(0) = X_0. \end{cases}$$

This simple but powerful model captures the idea that growth is initially exponential but eventually flattens as resources become scarce, making it especially useful for predicting and understanding population dynamics, disease outbreaks, market saturation, and other scenarios where growth is constrained by available resources. Because of its wide range of uses and adaptability, the logistic equation has become a vital tool for academics and decision-makers across many fields.

Following is an outline of our dissertation's arrangement, which consists of the follwoing chapters.

- **First chapter:**

  We provide the necessary notations, definitions, lemmas, and fixed-point theorems used throughout the work.

- **Second chapter:**

  We investigate the existence, uniqueness, and Ulam-Hyers-Rassias stability of a solution to the nonlinear pantograph boundary value problem with a Hadamard fractional derivative of variable order:

$$\begin{cases} {}^{H}\mathcal{D}_{1^+}^{\alpha(t)} X(t) = H(t, X(t), X(\lambda t)), \quad t \in [1, T], \\ X(1) = X(T) = 0, \end{cases}$$

  where $1 < \alpha(t) < 2$, $0 < \lambda < 1$, $H : [1, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, for each $t \in [1, T]$, we denote $\vartheta \in C([\lambda, 1], \mathbb{R})$ the history of system state given by

$$\vartheta(t) = X(\lambda t), \quad 0 < \lambda < 1.$$

- **Third chapter:**

  We explore a generalized Lyapunov inequality for nonlinear boundary value problems under additional assumptions on the nonlinear term.

- **Fourth chapter:**

  We analyze the existence and uniqueness of a solution for a Langevin boundary value problem involving variable-order Caputo fractional derivatives:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha(t)} \left({}^{C}\mathcal{D}_{0^+}^{\beta(t)} - \lambda \right) X(t) = H(t, X(t)), \quad t \in [0, T], \\ X(0) = X(T) = 0, \end{cases}$$

where $0 < \alpha(t), \beta(t) < 1$, $\lambda \in \mathbb{R}^+$, and $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

- **Fifth chapter:**

  We examine a variable-order fractional logistic equation over a finite time interval:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha(t)} X(t) = \kappa X(t)(1 - X(t)), \quad t \in [0,T], \\ X(0) = X_0, \end{cases}$$

  with $0 < \alpha(t) < 1$ and $X_0, \kappa \in \mathbb{R}^+$.

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce the main definitions and lemmas that will be utilized throughout this dissertation.

1.1 Notations and definitions

Definition 1.1.1. Let $[a,b]$ be a subset of $\mathbb{R}$

$i)$ Note by $C([a,b],\mathbb{R})$ the Banach space of continuous functions $X:[a,b]\longrightarrow\mathbb{R}$, with the usual supremum norm

$$||X||_{\infty}=\sup\{|X(t)|,\quad t\in[a,b]\}.$$

$ii)$ Let $L^1([a,b],E)$ be the Banach space of measurable functions $X:[a,b]\longrightarrow E$ which are Bochner integrable on $[a,b]$ with values in a Banach space $(E,||\cdot||)$, equipped with the norm

$$||X||_{L^1}=\int_a^b ||X(s)||\,ds.$$

$iii)$ For $1\leq p<\infty$, let $L^p([a,b],E)$ be the Banach space of measurable functions $X:[a,b]\to E$ for which the p-th power are Bochner integrable, equipped with the norm

$$||X||_{L^p}=\left(\int_a^b ||X(s)||^p ds\right)^{\frac{1}{p}}<\infty.$$

Definition 1.1.2. [53] A generalized interval is a subset Ω of $\mathbb{R}$, which is either a standard interval, a point, or $\emptyset$.

Definition 1.1.3. [53] If Ω a generalized interval, a finite set $\mathcal{P}$ consisting of generalized intervals contained in Ω is called a partition provided that every $x \in \Omega$ belongs to exactly one of the generalized intervals in the finite set $\mathcal{P}$.

Example. The set $\mathcal{P} = \{\{1\}, (1,2), [2,6), [6,7), \{7\}, (7,10]\}$ of generalized intervals is a partition of $[1,10]$.

Definition 1.1.4. [53] The function $\alpha : \Omega \longrightarrow \mathbb{R}$ is piecewise constant with respect to the partition $\mathcal{P}$, if α is constant on any generalized subset of Ω.

Example. The function $\alpha : [0,5] \longrightarrow \mathbb{R}$ given by

$$\alpha(t) = \begin{cases} 0, & 0 \leq t \leq 2, \\ 2, & 2 < t \leq 3, \\ 5, & 3 < t \leq 4, \\ 3, & 4 < t \leq 5, \end{cases}$$

is called a piece-wise constant function with respect to the set of generalized intervals $\mathcal{P} = \{[0,2], (2,3], (3,4], (4,5]\}$ of the subset $[0,5]$.

1.2 Variable order fractional calculus

Definition 1.2.1. [44] Let $-\infty < a < b < +\infty$ and $\alpha : [a,b] \longrightarrow (0,+\infty)$. The left hand Riemann-Liouville fractional integral of variable order $\alpha(t)$ for an integrable function X is given by

$$I_{a^+}^{\alpha(t)} X(t) = \int_a^t \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))} X(s) ds, \quad t > a. \tag{1.1}$$

Definition 1.2.2. [9] For $-\infty < a < b < +\infty$, we let $k \in \mathbb{N}$ and $\alpha : [a,b] \longrightarrow (k-1,k)$. The left hand Caputo derivative of variable order $\alpha(t)$ for an integrable function X is given by

$${}^C\mathcal{D}_{a^+}^{\alpha(t)} X(t) = \int_a^t \frac{(t-s)^{k-1-\alpha(t)}}{\Gamma(k-\alpha(t))} X^{(k)}(s) ds, \quad t > a. \tag{1.2}$$

It is obvious that, when the order $\alpha(t)$ is just a constant α, then the variable order fractional operators (i.e., integral and derivative) coincide with its constant order counterparts. Thus the property of semi-group yields the following properties

$$\begin{aligned} I_{a^+}^{\alpha}\, I_{a^+}^{\beta} &= I_{a^+}^{\beta}\, I_{a^+}^{\alpha} \\ &= I_{a^+}^{\alpha+\beta}. \end{aligned}$$

However, recent studies proved that such properties do not hold for variable order fractional operators. Therefore, it is very difficult to transform a differential equation into an equivalent integral equation. Besides,

$$\begin{aligned} I_{a^+}^{\alpha(t)}\, I_{a^+}^{\beta(t)} &\neq I_{a^+}^{\beta(t)}\, I_{a^+}^{\alpha(t)} \\ &\neq I_{a^+}^{\alpha(t)+\beta(t)}, \end{aligned}$$

where $\alpha(t)$ and $\beta(t)$ are general non negative functions. We shall give some examples to prove these claimed arguments.

Example. Let $\alpha(t) = \cos^2(t)$, $\beta(t) = \sin^2(t)$, $X(t) = t$, $0 \leq t \leq \frac{\pi}{2}$.

$$\begin{aligned} I_{0^+}^{\alpha(t)} I_{0^+}^{\beta(t)} X(t) &= \int_0^t \frac{(t-s)^{\cos^2(t)-1}}{\Gamma\left(\cos^2(t)\right)} \left(\int_0^s \frac{(s-h)^{\sin^2(s)-1}}{\Gamma\left(\sin^2(s)\right)} h\, dh \right) ds \\ &= \int_0^t \frac{(t-s)^{\cos^2(t)-1} s^{\sin^2(s)+1}}{\Gamma\left(\cos^2(t)\right)\Gamma\left(\sin^2(s)+2\right)} ds \\ I_{0^+}^{\alpha(t)} I_{0^+}^{\beta(t)} X(t)\Big|_{t=\frac{\pi}{4}} &= \int_0^{\frac{\pi}{4}} \frac{\left(\frac{\pi}{4}-s\right)^{\cos^2\left(\frac{\pi}{4}\right)-1} s^{\sin^2(s)+1}}{\Gamma\left(\cos^2\left(\frac{\pi}{4}\right)\right)\Gamma\left(\sin^2(s)+2\right)} ds \\ &= \int_0^{\frac{\pi}{4}} \frac{s^{\sin^2(s)+1}}{\sqrt{\frac{\pi}{4}-s}\,\sqrt{\pi}\,\Gamma\left(\sin^2(s)+2\right)} ds \\ &\approx 0.37747. \\ I_{0^+}^{\beta(t)} I_{0^+}^{\alpha(t)} X(t) &= \int_0^t \frac{(t-s)^{\sin^2(t)-1}}{\Gamma\left(\sin^2(t)\right)} \left(\int_0^s \frac{(s-h)^{\cos^2(s)-1}}{\Gamma\left(\cos^2(s)\right)} h\, dh \right) ds \\ &= \int_0^t \frac{(t-s)^{\sin^2(t)-1} s^{\cos^2(s)+1}}{\Gamma\left(\sin^2(t)\right)\Gamma\left(\cos^2(s)+2\right)} ds \\ I_{0^+}^{\beta(t)} I_{0^+}^{\alpha(t)} X(t)\Big|_{t=\frac{\pi}{4}} &= \int_0^{\frac{\pi}{4}} \frac{\left(\frac{\pi}{4}-s\right)^{\sin^2\left(\frac{\pi}{4}\right)-1} s^{\cos^2(s)+1}}{\Gamma\left(\sin^2\left(\frac{\pi}{4}\right)\right)\Gamma\left(\cos^2(s)+2\right)} ds \end{aligned}$$

$$= \int_0^{\frac{\pi}{4}} \frac{s^{\cos^2(s)+1}}{\sqrt{\frac{\pi}{4}-s}\,\sqrt{\pi}\,\Gamma\left(\cos^2(s)+2\right)}ds$$

$$\approx 0.26352.$$

$$I_{0^+}^{\alpha(t)+\beta(t)}X(t)\Big|_{t=\frac{\pi}{4}} = \int_0^{\frac{\pi}{4}} sds = \frac{s^2}{2}\Big|_0^{\frac{\pi}{4}} \approx 0.30842.$$

Therefore

$$I_{0^+}^{\alpha(t)}I_{0^+}^{\beta(t)}X(t)\Big|_{t=\frac{\pi}{4}} \neq I_{0^+}^{\beta(t)}I_{0^+}^{\alpha(t)}X(t)\Big|_{t=\frac{\pi}{4}} \neq I_{0^+}^{\alpha(t)+\beta(t)}X(t)\Big|_{t=\frac{\pi}{4}}.$$

Example. Let $\alpha(t) = \dfrac{t+1}{2}$, $\beta(t) = \dfrac{1-t}{2}$, $X(t) = t$, $0 \le t \le 1$.

$$I_{0^+}^{\alpha(t)}I_{0^+}^{\beta(t)}X(t) = \int_0^t \frac{(t-s)^{\frac{t+1}{2}-1}}{\Gamma\left(\frac{t+1}{2}\right)}\left(\int_0^s \frac{(s-h)^{\frac{1-s}{2}-1}}{\Gamma\left(\frac{1-s}{2}\right)}h\,dh\right)ds$$

$$= \int_0^t \frac{(t-s)^{\frac{t-1}{2}}s^{\frac{3-s}{2}}}{\Gamma\left(\frac{t+1}{2}\right)\Gamma\left(\frac{5-s}{2}\right)}ds$$

$$I_{0^+}^{\alpha(t)}I_{0^+}^{\beta(t)}X(t)\Big|_{t=\frac{1}{2}} = \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{-\frac{1}{4}}s^{\frac{3-s}{2}}}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5-s}{2}\right)}ds$$

$$\approx 0.11152.$$

$$I_{0^+}^{\beta(t)}I_{0^+}^{\alpha(t)}X(t) = \int_0^t \frac{(t-s)^{\frac{1-t}{2}-1}}{\Gamma\left(\frac{1-t}{2}\right)}\left(\int_0^s \frac{(s-h)^{\frac{s+1}{2}-1}}{\Gamma\left(\frac{s+1}{2}\right)}h\,dh\right)ds$$

$$= \int_0^t \frac{(t-s)^{\frac{-1-t}{2}}s^{\frac{3+s}{2}}}{\Gamma\left(\frac{1-t}{2}\right)\Gamma\left(\frac{5+s}{2}\right)}ds$$

$$I_{0^+}^{\beta(t)}I_{0^+}^{\alpha(t)}X(t)\Big|_{t=\frac{1}{2}} = \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{-\frac{3}{4}}s^{\frac{3+s}{2}}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5+s}{2}\right)}ds$$

$$\approx 0.13060.$$

$$I_{0^+}^{\alpha(t)+\beta(t)}X(t)\Big|_{t=\frac{1}{2}} = \int_0^{\frac{1}{2}} s\,ds = \frac{s^2}{2}\Big|_0^{\frac{1}{2}} = 0.125.$$

Therefore

$$I_{0^+}^{\alpha(t)}I_{0^+}^{\beta(t)}X(t)\Big|_{t=\frac{1}{2}} \neq I_{0^+}^{\alpha_2(t)}I_{0^+}^{\alpha_1(t)}X(t)\Big|_{t=\frac{1}{2}} \neq I_{0^+}^{\alpha(t)+\beta(t)}X(t)\Big|_{t=\frac{1}{2}}.$$

In the conversion of fractional differential equations into fractional integral equations, it is critical to understand the behavior of the composition of the respective operators in order to determine if one is the one-sided inverse of the other. The following example demonstrates the issue for variable order fractional operators. The response is often negative. Furthermore, results differ from those for constant order fractional operators.

Example. Let $\alpha(t) = \dfrac{t}{3}$, $X(t) = t$, $0 \le t \le 3$.

$$\begin{aligned}
I_{0^+}^{\alpha(t)}\, {}^C\mathcal{D}_{0^+}^{\alpha(t)} X(t) &= \int_0^t \frac{(t-s)^{\frac{t}{3}} - 1}{\Gamma\left(\frac{t}{3}\right)} \int_0^s \frac{(s-h)^{\frac{-s}{3}}}{\Gamma\left(1-\frac{s}{3}\right)} dh\, ds \\
&= \int_0^t \frac{(t-s)^{\frac{t}{3}-1} s^{1-\frac{s}{3}}}{\Gamma\left(\frac{t}{3}\right)\Gamma\left(2-\frac{s}{3}\right)} ds \\
I_{0^+}^{\alpha(t)}\, {}^C\mathcal{D}_{0^+}^{\alpha(t)} X(t)\Big|_{t=2} &= \int_0^2 \frac{(2-s)^{\frac{-1}{3}} s^{1-\frac{s}{3}}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(2-\frac{s}{3}\right)} ds \\
&\approx 1.9011 \\
&\neq X(2) - X(0) = 2.
\end{aligned}$$

Which is a different result than the constant order fractional calculs, that is

$$I_{0^+}^{\alpha}\, {}^C\mathcal{D}_{0^+}^{\alpha} X(t) = X(t) - X(0), \qquad 0 < t \le T < +\infty,$$

where $0 < \alpha < 1$, $X \in C^1([0,T],\mathbb{R})$.

On the other hand

$$\begin{aligned}
{}^C\mathcal{D}_{0^+}^{\alpha(t)} I_{0^+}^{\alpha(t)} X(t) &= \int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} \left(\frac{d}{ds}\right) \left[\int_0^s \frac{(s-h)^{\alpha(s)-1}}{\Gamma(\alpha(s))} h\, dh\right] ds \\
&= \int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} \left(\frac{d}{ds}\right) \left[\frac{s^{\alpha(s)+1}}{\Gamma(\alpha(s)+2)}\right] ds \\
&= \int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} s^{\alpha(s)+1} \\
&\quad \times \left[\frac{\Gamma(\alpha(s)+2)\left(\frac{\alpha(s)+1}{s} + \alpha'(s)\log(s)\right) - \alpha'(s)\Gamma'(\alpha(s)+2)}{\Gamma^2(\alpha(s)+2)}\right] ds
\end{aligned}$$

$${}^C\mathcal{D}_{0^+}^{\alpha(t)} I_{0^+}^{\alpha(t)} X(t)\Big|_{t=2} = \int_0^2 \frac{(2-s)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} s^{1+\frac{s}{3}} \left[\frac{\Gamma\left(2+\frac{s}{3}\right)\left(\frac{1+\frac{s}{3}}{s} + \frac{\log(s)}{3}\right) - \frac{\Gamma'\left(2+\frac{s}{3}\right)}{3}}{\Gamma^2\left(\frac{s}{3}+2\right)}\right] ds$$

$$\approx 1.9055$$

$$\neq X(2) = 2.$$

Which is a different result than the constant order fractional calculus, that is

$$^{C}\mathcal{D}^{\alpha}_{0^+} I^{\alpha}_{0^+} X(t) = X(t) \qquad 0 < t \leq T < +\infty,$$

where $0 < \alpha < 1$, $X \in C^1([0,T], \mathbb{R})$.

Definition 1.2.3. [8, 9] Let $1 \leq a < b < +\infty$ and $\alpha : [a,b] \longrightarrow (0,+\infty)$. The left hand Hadamard fractional integral of variable order $\alpha(t)$ for an integrable function X is given by

$$^{H}I^{\alpha(t)}_{a^+} X(t) = \frac{1}{\Gamma(\alpha(t))} \int_a^t \left(\ln \frac{t}{s} \right)^{\alpha(t)-1} \frac{X(s)}{s} ds, \quad t > a. \tag{1.3}$$

Definition 1.2.4. [8, 9] Let $k \in \mathbb{N}$ and $\alpha : [a,b] \longrightarrow (k-1, k)$. The left hand Hadamard derivative of variable order $\alpha(t)$ for an integrable function X is given by

$$^{H}\mathcal{D}^{\alpha(t)}_{a^+} X(t) = \frac{1}{\Gamma(k-\alpha(t))} \left(t\frac{d}{dt} \right)^k \left[\int_a^t \left(\ln \frac{t}{s} \right)^{k-1-\alpha(t)} \frac{X(s)}{s} ds \right], \quad t > a. \tag{1.4}$$

As expected, recent studies proved that such properties also do not hold for variable order Hadamard fractional operators neither

$$^{H}I^{\alpha(t)}_{1^+}\, ^{H}I^{\beta(t)}_{1^+} \neq\, ^{H}I^{\beta(t)}_{1^+}\, ^{H}I^{\alpha(t)}_{1^+}$$
$$\neq\, ^{H}I^{\alpha(t)+\beta(t)}_{1^+},$$

where $\alpha(t)$ and $\beta(t)$ are general non negative functions. We shall give some examples to prove these claimed arguments.

Example. In this example, we shall prove that

$$^{H}I^{\alpha(t)}_{1^+}\, ^{H}I^{\beta(t)}_{1^+} X(t) \neq\, ^{H}I^{\beta(t)}_{1^+}\, ^{H}I^{\alpha(t)}_{1^+} X(t)$$
$$\neq\, ^{H}I^{\alpha(t)+\beta(t)}_{1^+} X(t).$$

Let $\alpha(t) = t+2$, $\beta(t) = 2-t$, $X(t) = 1$, $1 \leq t \leq 2$.

$$^{H}I^{\alpha(t)}_{1^+}\, ^{H}I^{\beta(t)}_{1^+} X(t) = \frac{1}{\Gamma(t+2)} \int_1^t \frac{1}{s} \left(\ln \frac{t}{s} \right)^{t+2-1} \left(\frac{1}{\Gamma(2-s)} \int_1^s \frac{1}{h} \left(\ln \frac{s}{h} \right)^{2-s-1} dh \right) ds$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(t+2)} \int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{t+1} \frac{(\ln s)^{2-s}}{\Gamma(3-s)} ds \\
{}^H I_{1^+}^{\alpha(t)}\, {}^H I_{1^+}^{\beta(t)} X(t)\Big|_{t=\frac{3}{2}} &= \frac{1}{\Gamma\left(\frac{7}{2}\right)} \int_1^{\frac{3}{2}} \frac{1}{s}\left(\ln\frac{3}{2s}\right)^{\frac{5}{2}} \frac{(\ln s)^{2-s}}{\Gamma(3-s)} ds \\
&\approx 0.0061594. \\
{}^H I_{1^+}^{\beta(t)}\, {}^H I_{1^+}^{\alpha(t)} X(t) &= \frac{1}{\Gamma(2-t)} \int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{2-t-1} \left(\frac{1}{\Gamma(s+2)} \int_1^s \frac{1}{h}\left(\ln\frac{s}{h}\right)^{s+2-1} dh\right) ds \\
&= \frac{1}{\Gamma(2-t)} \int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{1-t} \frac{(\ln s)^{s+2}}{\Gamma(s+3)} ds \\
{}^H I_{1^+}^{\beta(t)}\, {}^H I_{1^+}^{\alpha(t)} X(t)\Big|_{t=\frac{3}{2}} &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_1^{\frac{3}{2}} \frac{1}{s}\left(\ln\frac{3}{2s}\right)^{-\frac{1}{2}} \frac{(\ln s)^{s+2}}{\Gamma(s+3)} ds \\
&\approx 0.56417. \\
{}^H I_{1^+}^{\alpha(t)+\beta(t)} X(t)\Big|_{t=\frac{3}{2}} &= \frac{1}{\Gamma(4)} \int_1^{\frac{3}{2}} \frac{1}{s}\left(\ln\frac{3}{2s}\right)^{3} ds \\
&= 0.0011244.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
{}^H I_{1^+}^{\alpha(t)}\, {}^H I_{1^+}^{\beta(t)} X(t)\Big|_{t=\frac{3}{2}} &\neq {}^H I_{1^+}^{\beta(t)}\, {}^H I_{1^+}^{\alpha(t)} X(t)\Big|_{t=\frac{3}{2}} \\
&\neq {}^H I_{1^+}^{\alpha(t)+\beta(t)} X(t)\Big|_{t=\frac{3}{2}}.
\end{aligned}
$$

The lack of such properties is due to the unsatisfied Sonnine equation. In other words, we say that two operators form a Sonnine pair $\{I_{a^+}^{\alpha(t)},\ \mathcal{D}_{a^+}^{\alpha(t)}\}$ if their respective kernels satisfy the following equation

$$
\int_a^t \mathrm{DffKr}(t-s)\mathrm{IntKr}(s)ds = \frac{t^{k-1}}{k!}, \quad k = [\alpha]+1. \tag{1.5}
$$

Where IntKr, DffKr are the integration and differentiation kernels respectively. Consider the following functions $\alpha(t) = \dfrac{1}{2} + \dfrac{4}{10}\sin(2\pi t)$, $\mathrm{IntKr}(t) = \frac{t^{\alpha(t)-1}}{\Gamma(\alpha(t))}$, and $\mathrm{DffKr}(t) = \frac{t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}$. The following is a representation of the equation (1.5).

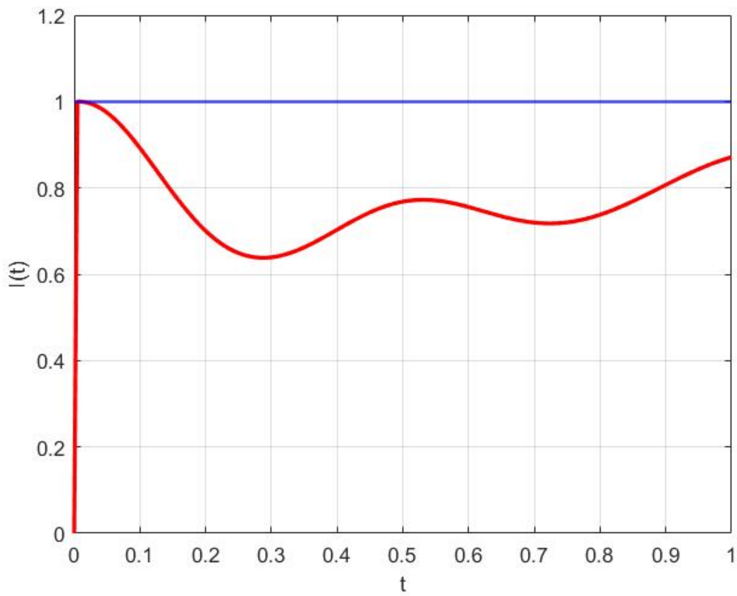

Figure 1.1: The Sonnine condition.

We state the following propositions to prove the well-posedness, and the continuity of the left hand Hadamard variable order integral.

Proposition 1. *[44] If* $\alpha \in C([1,T],(1,2])$*, then for* $X \in C_\gamma([1,T],\mathbb{R})$ *such that* $(\ln(\cdot))^\gamma X \in C([1,T],\mathbb{R})$, $0<\gamma<1$*, the left hand Hadamard variable order integral* ${}^H I_{1^+}^{\alpha(t)} X(t)$ *exists for each* $t \in [1,T]$.

Proof. The function $\Gamma(\alpha(t))$ is a continuous non-zero function on $[1,T]$.

Let $M_\alpha = \max_{t\in[1,T]} \left|\frac{1}{\Gamma(\alpha(t))}\right| > 0$, for $1 \le s \le t \le T$, we have

$$\begin{cases} \left(\ln\frac{t}{s}\right)^{\alpha(t)-1} \le 1, & \text{if } \ 1 \le \frac{t}{s} \le e, \\ \left(\ln\frac{t}{s}\right)^{\alpha(t)-1} \le \left(\ln\frac{t}{s}\right)^{\alpha_{\max}-1}, & \text{if } \ \frac{t}{s} > e. \end{cases}$$

Evidently, for $1 \le \frac{t}{s} < +\infty$, we obtain

$$\left(\ln\frac{t}{s}\right)^{\alpha(t)-1} \le \max\left\{1, \left(\ln\frac{t}{s}\right)^{\alpha_{\max}-1}\right\} = M^*.$$

Then, by applying the function $(\ln(\cdot))^\gamma$ for any $\gamma \in (0,1)$, one can conclude that

$$\left|{}^H I_{1^+}^{\alpha(t)} X(t)\right| = \frac{1}{\Gamma(\alpha(t))}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha(t)-1} \frac{|X(s)|}{s} ds$$

$$\leq M_\alpha \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha(t)-1} (\ln s)^{-\gamma} (\ln s)^{\gamma} \frac{|X(s)|}{s}$$
$$\leq M_\alpha M^* \sup_{t\in[1,T]} |(\ln t)^{\gamma} X(t)| \int_1^t \frac{(\ln s)^{-\gamma}}{s} ds$$
$$\leq M_\alpha M^* \sup_{t\in[1,T]} |(\ln t)^{\gamma} X(t)| \frac{\ln T}{1-\gamma} < +\infty.$$

It yields that the variable-order fractional integral ${}^H I_{1^+}^{\alpha(t)} X(t)$ exists for every $t \in [1,T]$. □

Proposition 2. *[44] Let $\alpha \in C([1,T],(1,2])$. Then, left hand Hadamard variable order integral ${}^H I_{1^+}^{\alpha(t)} X(t) \in C([1,T],\mathbb{R})$ for every $X \in C([1,T],\mathbb{R})$.*

Proof. For $t_1, t_2 \in [1,T],\ t_1 \leq t_2$, and $X \in C([1,T],\mathbb{R})$, we obtain

$$\left|{}^H I_{1^+}^{\alpha(t_1)} X(t_1) - {}^H I_{1^+}^{\alpha(t_2)} X(t_2)\right| = \left|\int_1^{t_1} \frac{1}{\Gamma(\alpha(t_1))} \left(\ln \frac{t_1}{s}\right)^{\alpha(t_1)-1} \frac{X(s)}{s} ds - \int_1^{t_2} \frac{1}{\Gamma(\alpha(t_2))} \left(\ln \frac{t_2}{s}\right)^{\alpha(t_2)-1} \frac{X(s)}{s} ds\right|.$$

Let the following change of variables: $s = \tau(t_i - 1) + 1$, for $i = 1, 2$,

$$= \left|\int_0^1 \frac{1}{\Gamma(\alpha(t_1))} \frac{(t_1-1)}{\tau(t_1-1)+1} \left(\ln \frac{t_1}{\tau(t_1-1)+1}\right)^{\alpha(t_1)-1} X(\tau(t_1-1)+1) d\tau \right.$$
$$\left. - \int_0^1 \frac{1}{\Gamma(\alpha(t_2))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1} X(\tau(t_2-1)+1) d\tau\right|$$
$$= \left|\int_0^1 \left[\frac{1}{\Gamma(\alpha(t_1))} \frac{(t_1-1)}{\tau(t_1-1)+1} \left(\ln \frac{t_1}{\tau(t_1-1)+1}\right)^{\alpha(t_1)-1} X(\tau(t_1-1)+1) \right.\right.$$
$$\left. - \frac{1}{\Gamma(\alpha(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_1}{\tau(t_1-1)+1}\right)^{\alpha(t_1)-1} X(\tau(t_1-1)+1)\right] d\tau$$
$$+ \int_0^1 \left[\frac{1}{\Gamma(\alpha(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_1}{\tau(t_1-1)+1}\right)^{\alpha(t_1)-1} X(\tau(t_1-1)+1) \right.$$
$$\left. - \frac{1}{\Gamma(\alpha(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1} X(\tau(t_1-1)+1)\right] d\tau$$
$$+ \int_0^1 \left[\frac{1}{\Gamma(\alpha(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1} X(\tau(t_1-1)+1) \right.$$
$$\left. - \frac{1}{\Gamma(\alpha(t_2))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1} X(\tau(t_1-1)+1)\right] d\tau$$

$$+\int_0^1 \left[\frac{1}{\Gamma(\alpha(t_2))}\frac{(t_2-1)}{\tau(t_2-1)+1}\left(\ln\frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1} X(\tau(t_1-1)+1)\right.$$
$$\left.-\frac{1}{\Gamma(\alpha(t_2))}\frac{(t_2-1)}{\tau(t_2-1)+1}\left(\ln\frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1} X(\tau(t_2-1)+1)\right] d\tau\Bigg|$$
$$\leq \sup_{t\in[1,T]}|X(t)|\left[\int_0^1 \frac{1}{\Gamma(\alpha(t_1))}\left(\ln\frac{t_1}{\tau(t_1-1)+1}\right)^{\alpha(t_1)-1}\left|\frac{(t_1-1)}{\tau(t_1-1)+1}-\frac{(t_2-1)}{\tau(t_2-1)+1}\right| d\tau\right.$$
$$+\int_0^1 \frac{1}{\Gamma(\alpha(t_1))}\frac{(t_2-1)}{\tau(t_2-1)+1}\left|\left(\ln\frac{t_1}{\tau(t_1-1)+1}\right)^{\alpha(t_1)-1}-\left(\ln\frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1}\right| d\tau$$
$$\left.+\int_0^1 \frac{(t_2-1)}{\tau(t_2-1)+1}\left(\ln\frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1}\left|\frac{1}{\Gamma(\alpha(t_1))}-\frac{1}{\Gamma(\alpha(t_2))}\right| d\tau\right]$$
$$+\int_0^1 \frac{1}{\Gamma(\alpha(t_2))}\frac{(t_2-1)}{\tau(t_2-1)+1}\left(\ln\frac{t_2}{\tau(t_2-1)+1}\right)^{\alpha(t_2)-1}$$
$$\times |X(\tau(t_1-1)+1)-X(\tau(t_2-1)+1)|\, d\tau.$$

Taking into account the continuity of the functions $\frac{(t-1)}{\tau(t-1)+1}$, $\left(\ln\left(\frac{t}{\tau(t-1)+1}\right)\right)^{\alpha(t)-1}$, $\frac{1}{\Gamma(\alpha(t))}$, and $X(t)$, as t_1 tends to t_2, the long expression tends to zero, therefore establishing the continuity of left-hand Hadamard fraction integral of variable order. □

The following Lemmas are crusial in this dissertation for establishing the equivalence between a differential equation and its integral counterpart.

Lemma 1.2.1. *[32] Let $\alpha > 0$, $0 < a < b$, $X \in L^1(a,b)$, and ${}^C\mathcal{D}^{\alpha}_{a^+}X \in L^1(a,b)$. Then the unique solution of the equation*

$${}^C\mathcal{D}^{\alpha}_{a^+}X(t) = 0,$$

is given by

$$X(t) = \varrho_0 + \varrho_1(t-a) + \varrho_2(t-a)^2 + \cdots + \varrho_{k-1}(t-a)^{k-1},$$

where $k = [\alpha]+1$, and $\varrho_i \in \mathbb{R}$ $i \in \{0,1,\ldots,k-1\}$ are arbitrary constants. Moreover,

$$I^{\alpha}_{a^+}\, {}^C\mathcal{D}^{\alpha}_{a^+}X(t) = X(t) + \sum_{i=1}^{k}\varrho_{i-1}(t-a)^{i-1},$$

and

$${}^C\mathcal{D}^{\alpha}_{a^+}\, I^{\alpha}_{a^+}X(t) = X(t).$$

Lemma 1.2.2. *[32] Let $\alpha > 0$, $1 < a < b$, $X \in L^1(a,b)$, and ${}^H\mathcal{D}^{\alpha}_{a^+}X \in L^1(a,b)$. Then the differential equation*

$$ {}^H\mathcal{D}^{\alpha}_{a^+}X(t) = 0 $$

has a solution

$$ X(t) = \varrho_1 \left(\ln \frac{t}{a}\right)^{\alpha-1} + \varrho_2 \left(\ln \frac{t}{a}\right)^{\alpha-2} + \cdots + \varrho_k \left(\ln \frac{t}{a}\right)^{\alpha-k}, $$

where $k = [\alpha] + 1$, and $\varrho_i \in \mathbb{R}, i \in \{1, 2, \ldots, k\}$ are arbitrary constants. Moreover

$$ {}^H I^{\alpha}_{a^+}\, {}^H\mathcal{D}^{\alpha}_{a^+}X(t) = X(t) + \sum_{i=1}^{k} \varrho_i \left(\ln \frac{t}{a}\right)^{\alpha-i}, $$

and

$$ {}^H\mathcal{D}^{\alpha}_{a^+}\, {}^H I^{\alpha}_{a^+}X(t) = X(t). $$

Definition 1.2.5. Let E, F be two Banach spaces. A function $H : [a,b] \times E \longrightarrow F$ is said to be Carathéodory if

$i)$ The map $t \mapsto H(t, X)$ is measurable for any $X \in E$;

$ii)$ The map $X \mapsto H(t, X)$ is continuous for all $t \in [a,b]$.

Let $M([a,b], E)$ be the set of all measurable functions $H : [a,b] \times E \longrightarrow E$. If H is a Carathéodory function, then H defines a mapping $\mathcal{N}_H : M([a,b], E) \longrightarrow M([a,b], E)$ by $\mathcal{N}_H X(t) := H(t, X(t))$, for all $t \in [a,b]$. This mapping is called the Nemytskii's operator associated to H.

Lemma 1.2.3. *[36] Let H be a Carathéodory function, and $p, q \geq 1$. Nemytskii's operator defined on $L^p([a,b], E)$ with values in $L^q([a,b], E)$ is bounded and continuous. Moreover, $\mathcal{N}$ maps all of $L^p([a,b], E)$ into $L^q([a,b], E)$ if and only if the function H satisfies the following condition*

$$ \begin{cases} ||H(t, X(t))|| \leq A(t) + \varrho ||X||^{p/q} \quad \textit{with } A \in L^q,\ \varrho > 0, & q < +\infty, \\ ||H(t, X(t))|| \leq \varrho, & q = +\infty. \end{cases} $$

1.3 Fixed point theorems

Theorem 1.3.1. *[25](Banach Contraction Theorem) Let S be a non-empty closed subset of a Banach space E, then any contraction mapping $\mathcal{Z}$ of S into itself has a unique fixed point.*

Theorem 1.3.2. *[25] (Schauder Fixed Point Theorem) Let E be a Banach space, S be a nonempty bounded convex and closed subset of E, and $\mathcal{Z} : S \to S$ be a compact and continuous map. Then $\mathcal{Z}$ has at least one fixed point in S.*

Theorem 1.3.3. *[46](Kransosel'skii Fixed Point Theorem) Let S be a non-empty bounded closed convex subset of a real Banach space E, and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be operators on S satisfying the following conditions:*

$i)$ $\mathcal{Z}_1(S) + \mathcal{Z}_2(S) \subset S$;

$ii)$ $\mathcal{Z}_1$ *is continuous on* S *and* $\mathcal{Z}_1(S)$ *is a relatively compact subset of* E;

$iii)$ $\mathcal{Z}_2$ *is a strict contraction on* S, *i.e., there exist* $\varrho \in [0, 1)$, *such that*

$$||\mathcal{Z}_2(x) - \mathcal{Z}_2(y)|| \leq \varrho||x - y||, \quad \text{for every } x, y \in S.$$

Then the equation $\mathcal{Z}_1(x) + \mathcal{Z}_2(x) = x$ *has a solution in* S.

CHAPTER 2

EXISTENCE, UNIQUENESS AND STABILITY RESULTS FOR A PANTOGRAPH BVP OF VARIABLE ORDER

2.1 Introduction and motivation

In [28], S. Harikrishnan et al. discussed the nonlocal initial value problems for pantograph equations with $\psi-$Hilfer fractional derivative of the form

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha,\delta;\psi} X(t) = H(t, X(t), X(\lambda t)), \quad 0 < \lambda < 1, \quad t \in J := (a, T] \\ I_{a^+}^{1-\gamma;\psi} X(t)\Big|_{t=a} = \sum_{i=1}^{n} \varrho_i X(T_i), \quad T_i \in (a, T], \end{cases}$$

where $\mathcal{D}_{a^+}^{\alpha,\delta;\psi}$ is $\psi-$Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \delta \leq 1$, $\psi = \alpha + \delta - \alpha\delta$. Let $H : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a given continuous function, T_i are prefixed points satisfying $a < T_1 \leq \cdots \leq T_i < T$ and ϱ_k are real numbers.

In [10], Alzabut et al used Krasnoselskiis and generalized Banach fixed point theorems to investigate the asymptotic stability of solutions of the following discrete fractional pantograph equation

$$\begin{cases} \Delta_*^{\beta}[X](t) = H(t + \beta, X(t + \beta), X(\lambda(t + \beta))), \\ X(0) = p[X], \end{cases}$$

for $t \in \mathbb{N}_{1-\beta}$, where $0 < \beta \leq 1$, $0 < \lambda < 1$, Δ_*^{β} is a Caputo like difference operator, X represents the motion of the pantograph, $H : [0, +\infty) \times C([0, \infty), \mathbb{C}) \times C([0, \infty), \mathbb{C}) \longrightarrow \mathbb{R}$

is continuous with respect to X and t, $p : C([0,+\infty),\mathbb{C}) \longrightarrow \mathbb{R}$ is Lipschitz continuous in X, and $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$.

Using the Hilfer operator, stability properties are investigated in [45] for a certain nonlinear fractional order generalized pantograph equation with discrete time and stability conditions are established using Ulam and Hyers results, and recently, in [16] the authors investigated the existence and uniqueness of positive solutions for nonlinear pantograph Caputo–Hadamard fractional differential equations.

Inspired by work above, we study the existence and uniqueness of solution to the following nonlinear pantograph boundary value problem involving the Hadamard fractional differential equation of variable order:

$$\begin{cases} {}^{H}\mathcal{D}_{1^+}^{\alpha(t)} X(t) = H(t, X(t), X(\lambda t)), \quad t \in [1,T], \\ X(1) = X(T) = 0, \end{cases} \tag{HFPE}$$

where $1 < \alpha(t) < 2$, $0 < \lambda < 1$, $H : [1,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and ${}^{H}\mathcal{D}_{1^+}^{\alpha(t)}$ is the left hand Hadamard fractional derivative of variable order .

2.2 Existence and uniqueness of solution

In this section, we present our main results.

Let $\mathcal{P} = [1,T_1], (T_1,T_2], (T_2,T_3], \cdots, (T_n,T]$ be a partition of the interval $[1,T]$, and let $\alpha(t) : [1,T] \longrightarrow (1,2)$ be the piecewise constant function with respect to P given by

$$\alpha(t) = \sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t) = \begin{cases} \alpha_1 & t \in [0,T_1] \\ \alpha_2 & t \in (T_1,T_2] \\ \vdots & \\ \alpha_n & t \in (T_{n-1},T], \end{cases}$$

where $1 < \alpha_i < 2$, $i \in \{1,2,\ldots,n\}$ are constants, and $\mathbb{I}_i$ is the characteristic function for the interval $(T_{i-1},T_i]$, i.e.,

$$\mathbb{I}_i(t) = \begin{cases} 1 & t \in (T_{i-1},T_i], \\ 0 & \text{otherwise}. \end{cases}$$

Then, for any $t \in (T_{i-1}, T_i]$, $i \in \{1, 2, \ldots, n\}$, the left Hadamard fractional derivative of variable order $\alpha(t)$ for function $X(t)$, could be presented as a sum of left Hadamard fractional derivatives of constant-orders α_i.

Therefore, in the interval $[1, T_1]$, the equation can be written as

$$ {}^{H}\mathcal{D}_{1^+}^{\alpha_1} X(t) = \frac{t^2}{\Gamma(2-\alpha_1)} \frac{d^2}{dt^2} \left[\int_1^t \left(\ln \frac{t}{s} \right)^{1-\alpha_1} \frac{X(s)}{s} ds \right]. \tag{2.1} $$

In the interval $(T_1, T_2]$, it can be written as

$$ \begin{aligned} {}^{H}\mathcal{D}_{1^+}^{\alpha_2} X(t) = & \frac{t^2}{\Gamma(2-\alpha_2)} \frac{d^2}{dt^2} \int_1^{T_1} \left(\ln \frac{t}{s} \right)^{1-\alpha_2} \frac{X(s)}{s} ds \\ & + \frac{t^2}{\Gamma(2-\alpha_2)} \frac{d^2}{dt^2} \int_{T_1}^t \left(\ln \frac{t}{s} \right)^{1-\alpha_2} \frac{X(s)}{s} ds. \end{aligned} \tag{2.2} $$

And in general, in the interval $(T_{i-1}, T_i]$, it can be written as

$$ \begin{aligned} {}^{H}\mathcal{D}_{1^+}^{\alpha_i} X(t) = & \sum_{j=1}^{i-1} \frac{t^2}{\Gamma(2-\alpha_i)} \frac{d^2}{dt^2} \int_{T_{j-1}}^{T_j} \left(\ln \frac{t}{s} \right)^{1-\alpha_i} \frac{X(s)}{s} ds \\ & + \frac{t^2}{\Gamma(2-\alpha_i)} \frac{d^2}{dt^2} \int_{T_{i-1}}^t \left(\ln \frac{t}{s} \right)^{1-\alpha_i} \frac{X(s)}{s} ds. \end{aligned} \tag{2.3} $$

Hence, we have

$$ \begin{aligned} {}^{H}\mathcal{D}_{1^+}^{\alpha(t)} X(t) = & \sum_{j=1}^{i-1} \frac{t^2}{\Gamma(2-\alpha_i)} \frac{d^2}{dt^2} \int_{T_{j-1}}^{T_j} \left(\ln \frac{t}{s} \right)^{1-\alpha_i} \frac{X(s)}{s} ds \\ & + \frac{t^2}{\Gamma(2-\alpha_i)} \frac{d^2}{dt^2} \int_{T_i}^t \left(\ln \frac{t}{s} \right)^{1-\alpha_i} \frac{X(s)}{s} ds \\ = & H(t, X(t), X(\lambda t)). \end{aligned} \tag{2.4} $$

We denote by $E_i = C([\lambda, T_i], \mathbb{R})$ the class of functions that form a Banach space with the norm

$$ ||\widehat{X}||_{E_i} = \sup_{t \in [\lambda, T_i]} |\widehat{X}(t)|, \quad i \in \{1, 2, \ldots, n\}. $$

Let $\widehat{X} \in E_i$ with $\widehat{X}(t) = 0$ for all $t \in [1, T_{i-1}] \cup \{T_i\}$, $i \in \{1, 2, \ldots, n\}$ be the solutions to the above equations. Hence, we consider the auxiliary boundary value problems for Hadamard fractional equations of constant order

$$ \begin{cases} {}^{H}\mathcal{D}_{T_{i-1}^+}^{\alpha_i} \widehat{X}(t) = H(t, \widehat{X}(t), \widehat{X}(\lambda t)), & T_{i-1} \le t \le T_i, \\ \widehat{X}(T_{i-1}) = \widehat{X}(T_i) = 0. \end{cases} \tag{2.5} $$

Definition 2.2.1. We say that the problem (HFPE) has a solution X, if there exist functions X_i, such that: $X_1 \in E_1$ satisfies equation (2.1), and $X_1(1) = X_1(T_1) = 0$; $X_2 \in E_2$ satisfies equation (2.2), and $X_2(T_1) = X_2(T_2) = 0$; $X_i \in E_i$ satisfies equation (2.3), and $X_i(T_{i-1}) = X_i(T_i) = 0$ for $i \in \{3, \ldots, n\}$.

Remark 1. We say problem (HFPE) has a unique solution, if the functions X_i are uniques.

Based on the previous discussion, we have the following results

Lemma 2.2.1. *Let $i \in \{1, 2, \ldots, n\}$. Then the function $\widehat{X}$ is a solution of (2.5) if $\widehat{X}$ is a solution of the integral equation*

$$\begin{aligned} \widehat{X}(t) = &-\left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \left({}^H I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t))\Big|_{t=T_i} \right) \\ &+ {}^H I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t)). \end{aligned} \tag{2.6}$$

Proof. Assume $\widehat{X}$ satisfies (2.5), then we transforme (2.5) into an equivalent integral equation as follows. Let $T_{i-1} < t \leq T_i$, then Lemma 1.2.2 implies

$${}^H I^{\alpha_i}_{T^+_{i-1}} \, {}^H \mathcal{D}^{\alpha_i}_{T^+_{i-1}} \widehat{X}(t) = {}^H I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t))$$

so

$$\widehat{X}(t) = \sum_{j=1}^{2} \varrho_j \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - j} + {}^H I^{\alpha}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t)).$$

Using the boundary conditions $\widehat{X}(T_i) = \widehat{X}(T_{i-1}) = 0$, we obtain

$$\begin{cases} \varrho_1 = -\left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left[\varrho_2 \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 2} + \left({}^H I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t))\Big|_{t=T_i} \right) \right], \\ 0 = \varrho_2 \left[-\left(\ln \frac{T_i}{T_{i-1}}\right)^{\alpha_i - 2} {}^H I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t))\Big|_{t=T_i} \right]. \end{cases}$$

Hence,

$$\begin{cases} \varrho_2 = 0 \\ \varrho_1 = -\left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left({}^H I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t))\Big|_{t=T_i} \right). \end{cases}$$

Therefore, the solution of the auxiliary boundary value problem (2.5) is given by

$$\begin{aligned}\widehat{X}(t) = &-\left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left(\ln\frac{t}{T_{i-1}}\right)^{\alpha_i-1}\left({}^{H}I^{\alpha_i}_{T^+_{i-1}}H(t,\widehat{X}(t),\widehat{X}(\lambda t))\Big|_{t=T_i}\right)\\ &+{}^{H}I^{\alpha_i}_{T^+_{i-1}}H(t,\widehat{X}(t),\widehat{X}(\lambda t)).\end{aligned}$$

A straight forward calculation shows that if $\widehat{X}$ is given by (2.6), then it is a solution of (2.5). □

Before presenting our main results, we first state the following hypotheses that will be needed:

(H1) Let $H:[1,T]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function such that for all $x_1,x_2,y_1,y_2\in\mathbb{R}$, there exists a positive constant $L_1>0$ so that

$$|H(t,x_1,y_1)-H(t,x_2,y_2)|\leq L_1(|x_1-x_2|+|y_1-y_2|);$$

(H2)

$$\frac{4L_1}{\Gamma(\alpha_i+1)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i}<1,\ \textit{for each}\ i\in\{1,2,\dots,n\}.$$

Theorem 2.2.1. *Under conditions **(H1)** and **(H2)**, the boundary value problem* (HFPE) *has a unique solution in* $C([\lambda,T],\mathbb{R})$.

Proof. Consider the operator $\mathcal{Z}:E_i\longrightarrow E_i$ given by

$$\begin{aligned}(\mathcal{Z}\widehat{X})(t) = &-\left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left(\ln\frac{t}{T_{i-1}}\right)^{\alpha_i-1}\left({}^{H}I^{\alpha_i}_{T^+_{i-1}}H(t,\widehat{X}(t),\widehat{X}(\lambda t))\Big|_{t=T_i}\right)\\ &+{}^{H}I^{\alpha_i}_{T^+_{i-1}}H(t,\widehat{X}(t),\widehat{X}(\lambda t)).\end{aligned} \tag{2.7}$$

Clearly, $\mathcal{Z}$ is well defined. For each $i\in\{1,2,\dots,n\}$, let $B_{R_i}=\left\{\widehat{X}\in E_i:||\widehat{X}||_{E_i}\leq R_i\right\}$ be a non-empty, closed, bounded, convex subset of E_i, where

$$R_i\geq\frac{\dfrac{2}{\Gamma(\alpha_i+1)}\sup\limits_{t\in[1,T]}|H(t,0,0)|\left(\ln\dfrac{T_i}{T_{i-1}}\right)^{\alpha_i}}{1-\dfrac{4L_1}{\Gamma(\alpha_i+1)}\left(\ln\dfrac{T_i}{T_{i-1}}\right)^{\alpha_i}}.$$

We split the mapping $\mathcal{Z}$ into two maps $\mathcal{Z}_1$ and $\mathcal{Z}_2$ on B_{R_i} as follows:

$$\begin{cases}(\mathcal{Z}_1\widehat{X})(t) &= -\left(\ln\dfrac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left(\ln\dfrac{t}{T_{i-1}}\right)^{\alpha_i-1}\left({}^{H}I^{\alpha_i}_{T^+_{i-1}}H(t,\widehat{X}(t),\widehat{X}(\lambda t))\Big|_{t=T_i}\right),\\ (\mathcal{Z}_2\widehat{X})(t) &= {}^{H}I^{\alpha_i}_{T^+_{i-1}}H(t,\widehat{X}(t),\widehat{X}(\lambda t)).\end{cases}$$

We will show that the conditions of Theorem 1.3.3 are satisfied.

Step 1: $\mathcal{Z}_1(B_{R_i}) + \mathcal{Z}_2(B_{R_i}) \subset B_{R_i}$. For each $i \in \{1, 2, \ldots, n\}$, we have

$$
\begin{aligned}
&|\mathcal{Z}_1(\widehat{X})(t) + \mathcal{Z}_2(\widehat{X})(t)| \\
&\quad = \Bigg| -\left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \left({}^{H}I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t))\Big|_{t=T_i} \right) \\
&\qquad + {}^{H}I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t)) \Bigg| \\
&\quad \leq \left| \left({}^{H}I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t))\Big|_{t=T_i} \right) + {}^{H}I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t), \widehat{X}(\lambda t)) \right| \\
&\quad \leq \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}(s), \widehat{X}(\lambda s))| ds \\
&\qquad + \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{t}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}(s), \widehat{X}(\lambda s))| ds \\
&\quad \leq \frac{2}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}(s), \widehat{X}(\lambda s)) - H(s, 0, 0)| ds \\
&\qquad + \frac{2}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} |H(s, 0, 0)| ds \\
&\quad \leq \frac{2}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} L_1(|\widehat{X}(s)| + |\widehat{X}(\lambda s)|) ds \\
&\qquad + \frac{2}{\Gamma(\alpha_i)} \sup_{t \in [1,T]} |H(t, 0, 0)|) \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} ds \\
&\quad \leq \frac{4L_1 ||\widehat{X}||_{E_i}}{\Gamma(\alpha_i + 1)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{\alpha_i} \\
&\qquad + \frac{2}{\Gamma(\alpha_{i+1})} \sup_{t \in [1,T]} |H(t, 0, 0)| \left(\ln \frac{T_i}{T_{i-1}}\right)^{\alpha_i} \\
&\quad \leq R_i,
\end{aligned}
$$

which is what we wanted to show.

Step 2: $\mathcal{Z}_1$ is a contraction. Here we see that

$$
\begin{aligned}
&|\mathcal{Z}_1(\widehat{X}_1)(t) - \mathcal{Z}_1(\widehat{X}_2)(t)| \\
&\quad \leq \frac{\left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} \\
&\qquad \times |H(s, \widehat{X}_1(s), \widehat{X}_1(\lambda s)) - H(s, \widehat{X}_2(s), \widehat{X}_2(\lambda s))| ds \\
&\quad \leq \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} L_1 \left[|\widehat{X}_1(s) - \widehat{X}_2(s)| + |\widehat{X}_1(\lambda s) - \widehat{X}_2(\lambda s)| \right] ds
\end{aligned}
$$

$$\leq \frac{2L_1}{\Gamma(\alpha_i+1)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i}||\widehat{X}_1-\widehat{X}_2||_{E_i}.$$

Hence, $\mathcal{Z}_1$ is a contraction by condition **(H2)**.

Step 3: $\mathcal{Z}_2$ is continuous and $\mathcal{Z}_2(B_{R_i})$ is relatively compact. To show that $\mathcal{Z}_2$ is continuous, let $\{\widehat{X}_m\}$ be a sequence such that $\widehat{X}_m \longrightarrow \widehat{X}$ in B_{R_i}. Then for each $t \in [T_{i-1}, T_i]$, $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}
&|\mathcal{Z}_2(\widehat{X}_m)(t)-\mathcal{Z}_2(\widehat{X})(t)|\\
&\qquad\leq \frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t}\frac{1}{s}\left(\ln\frac{t}{s}\right)^{\alpha_i-1}|H(s,\widehat{X}_m(s),\widehat{X}_m(\lambda s))-H(s,\widehat{X}(s),\widehat{X}(\lambda s))|ds\\
&\qquad\leq \frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i}\frac{1}{s}\left(\ln\frac{T_i}{s}\right)^{\alpha_i-1}|H(s,\widehat{X}_m(s),\widehat{X}_m(\lambda s))-H(s,\widehat{X}(s),\widehat{X}(\lambda s))|ds\\
&\qquad\leq \frac{2L_1}{\Gamma(\alpha_i+1)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i}||\widehat{X}_m-\widehat{X}||_{E_i}.
\end{aligned}$$

Thus, we have

$$||\mathcal{Z}_2(\widehat{X}_m)(t)-\mathcal{Z}_2(\widehat{X})(t)||_{E_i}\longrightarrow 0 \quad \text{as} \quad m\to+\infty.$$

Step 4: $\mathcal{Z}_2$ is equicontinuous. Now we have

$$\begin{aligned}
&|\mathcal{Z}_2(\widehat{X})(t_2)-\mathcal{Z}_2(\widehat{X})(t_1)|\\
&\qquad= \left|\frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t_2}\frac{1}{s}\left(\ln\frac{t_2}{s}\right)^{\alpha_i-1}H(s,\widehat{X}(s),\widehat{X}(\lambda s))ds\right.\\
&\qquad\quad\left.-\frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t_1}\frac{1}{s}\left(\ln\frac{t_1}{s}\right)^{\alpha_i-1}H(s,\widehat{X}(s),\widehat{X}(\lambda s))\right|ds\\
&\qquad\leq \frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t_1}\frac{1}{s}\left[\left(\ln\frac{t_2}{s}\right)^{\alpha_i-1}-\left(\ln\frac{t_1}{s}\right)^{\alpha_i-1}\right]|H(s,\widehat{X}(s),\widehat{X}(\lambda s))|ds\\
&\qquad\quad+\frac{1}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}\frac{1}{s}\left(\ln\frac{t_2}{s}\right)^{\alpha_i-1}|H(s,\widehat{X}(s),\widehat{X}(\lambda s))|ds.
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right hand side of the above inequality tends to zero, so the mapping $\mathcal{Z}_2$ is equicontinuous. Since it is uniformly bounded by **Step 1**, by the Ascoli-Arzelá Theorem $\mathcal{Z}_2$ is relatively compact on B_{R_i}.

It then follows from Theorem 1.3.3 that the auxiliary boundary value problem (2.5) has at least one solution in B_{R_i} for each $i \in \{1, 2, \dots, n\}$. Hence, the boundary value problem (HFPE) has a solution $X \in C([\lambda, T], \mathbb{R})$ given by

$$X(t) = \begin{cases} X_1(t), & 1 \leq t \leq T_1, \\ X_2(t), & T_1 \leq t \leq T_2, \\ \vdots \\ X_n(t), & T_{n-1} \leq t \leq T, \end{cases} \tag{2.8}$$

such that

$$X_i(t) = \widehat{X}(t)\Big|_{t \in [T_{i-1}, T_i]}.$$

The uniqueness of the solution obtained above is easy to show by using Banach contraction principle, from which uniqueness of X_i follows for each $i \in \{1, 2, \dots, n\}$. In view of Remark 1, we have uniqueness of solution to (HFPE).

□

2.3 Stability of solutions

One of the most important qualitative results for a given boundary value problem is the stability of solutions. Therefore, in this section, we investigate the Ulam-Hyers-Rassias stability of the solutions to the boundary value problem in hand.

Definition 2.3.1. The boundary value problem (HFPE) is Ulam-Hyers-Rassias stable with respect to the function $\Phi \in C([1, T], \mathbb{R}_+)$ if there exists $\zeta_H \in \mathbb{R}$ such that $\forall \xi > 0$ and $\forall \Psi \in C([\lambda, T], \mathbb{R})$ satisfying

$$|\, {}^H\mathcal{D}_{1^+}^{\alpha(t)} \Psi(t) - H(t, \Psi(t), \Psi(\lambda t))| \leq \xi \Phi(t), \quad t \in [1, T],$$

there exists $X \in C([\lambda, T], \mathbb{R})$ a solution to the boundary value problem (HFPE) with

$$|\Psi(t) - X(t)| \leq \zeta_H \xi \Phi(t), \quad t \in [1, T].$$

We state the folowing hypothesis:

(H3) There exists $\Phi \in C([1,T],\mathbb{R}_+)$ an increasing mapping and there exists $\mu_\Phi > 0$ so that for all $t \in [T_{i-1}, T_i]$ for each $i \in \{1,2,\dots,n\}$

$$ {}^H I^{\alpha_i}_{T^+_{i-1}} \Phi(t) \le \varrho_{\Phi(t)} \Phi(t). $$

Theorem 2.3.1. *Under the conditions **(H1)**, **(H2)** and **(H3)**, the boundary value problem* (HFPE) *is Ulam-Hyers-Rassias stable with respect to Φ.*

Proof. Assuming for all $\xi > 0$, $\Psi \in C([\lambda, T_i],\mathbb{R})$ satisfies the inequality

$$ |{}^H\mathcal{D}^{\alpha_i}_{T^+_{i-1}} \Psi(t) - H(t,\Psi(t),\Psi(\lambda t))| \le \xi\Phi(t), \quad t \in [T_{i-1}, T_i]. \tag{2.9} $$

For any $i \in \{1,2,\dots,n\}$, we propose the following functions

$$ \Psi_i(t) = \begin{cases} 0, & t \in [1, T_{i-1}], \\ \Psi(t), & t \in [T_{i-1}, T_i]. \end{cases} $$

Integrating both sides of the equation (2.9), we obtain, for $t \in [T_{i-1}, T_i]$,

$$ \begin{aligned} &\left| {}^H I^{\alpha_i}_{T^+_{i-1}} \left[{}^H\mathcal{D}^{\alpha_i}_{T^+_{i-1}} \Psi_i(t) - H(t,\Psi_i(t),\Psi_i(\lambda t)) \right] \right| \\ &\quad \le \left| \Psi_i(t) - \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s} \right)^{\alpha_i - 1} H(s,\Psi_i(s),\Psi_i(\lambda s)) ds \right. \\ &\qquad \left. + \left(\ln \frac{T_i}{T_{i-1}} \right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}} \right)^{\alpha_i - 1} \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s} \right)^{\alpha_i - 1} H(s,\Psi_i(s),\Psi_i(\lambda s)) ds \right| \\ &\quad \le \xi \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s} \right)^{\alpha_i - 1} \Phi(s) ds. \end{aligned} $$

In accordance with the argument above, the boundary value problem (HFPE) admits a solution X defined by (2.8). Then, we have, for each $t \in [T_{i-1}, T_i]$, $i \in \{1,2,\dots,n\}$,

$$ \begin{aligned} |\Psi(t) - X(t)| &= |\Psi_i(t) - \widehat{X}(t)| \\ &= \left| \Psi_i(t) + \left(\ln \frac{T_i}{T_{i-1}} \right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}} \right)^{\alpha_i - 1} \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{t}{s} \right)^{\alpha_i - 1} H(s,\widehat{X}(s),\widehat{X}(\lambda s)) ds \right. \\ &\qquad \left. - \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{t}{s} \right)^{\alpha_i - 1} H(s,\widehat{X}(s),\widehat{X}(\lambda s)) ds \right| \\ &\le \left| \Psi_i(t) + \left(\ln \frac{T_i}{T_{i-1}} \right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}} \right)^{\alpha_i - 1} \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{t}{s} \right)^{\alpha_i - 1} H(s,\Psi_i(s),\Psi_i(\lambda s)) ds \right. \end{aligned} $$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t}\frac{1}{s}\left(\ln\frac{t}{s}\right)^{\alpha_i-1}H(s,\Psi_i(s),\Psi_i(\lambda s))ds\Bigg|\\
&+\frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t}\frac{1}{s}\left(\ln\frac{t}{s}\right)^{\alpha_i-1}|H(s,\Psi_i(s),\Psi_i(\lambda s))-H(s,\widehat{X}(s),\widehat{X}(\lambda s))|ds\\
&+\frac{1}{\Gamma(\alpha_i)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left(\ln\frac{t}{T_{i-1}}\right)^{\alpha_i-1}\\
&\times\int_{T_{i-1}}^{t}\frac{1}{s}\left(\ln\frac{t}{s}\right)^{\alpha_i-1}\left|H(s,\Psi_i(s),\Psi_i(\lambda s))-H(s,\widehat{X}(s),\widehat{X}(\lambda s))\right|ds\\
\leq{}& \varrho_{\Phi(t)}\xi\Phi(t)+\frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i}\frac{1}{s}\left(\ln\frac{T_i}{s}\right)^{\alpha_i-1}L_1|\Psi_i(s)-\widehat{X}(s)|+|\Psi_i(\lambda s)-\widehat{X}(\lambda s)|ds\\
&+\frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i}\frac{1}{s}\left(\ln\frac{T_i}{s}\right)^{\alpha_i-1}L_1(|\Psi_i(s)-\widehat{X}(s)|+|\Psi_i(\lambda s)-\widehat{X}(\lambda s)|)ds\\
\leq{}& \varrho_{\Phi(t)}\xi\Phi(t)+\frac{4L_1||\Psi_i-\widehat{X}||_{E_i}}{\Gamma(\alpha_i+1)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i}\\
\leq{}& \varrho_{\Phi(t)}\xi\,\Phi(t)+\rho||\Psi-X||,
\end{aligned}
$$

where

$$\rho=\max_{i\in\{1,2,\ldots,n\}}\frac{4L_1}{\Gamma(\alpha_i+1)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i}.$$

Thus

$$(1-\rho)||\Psi-X||\leq\varrho_{\Phi(t)}\xi\Phi(t).$$

Therefore, we have, for each $t\in[1,T]$,

$$|\Psi(t)-X(t)|\leq\frac{\varrho_{\Phi(t)}\xi\Phi(t)}{1-\rho}=\zeta_H\xi\,\Phi(t).$$

Then, the boundary value problem (HFPE) is U-H-R stable with respect to Φ. □

2.4 Example

In this example, we illustrate the usefulness of the results obtained in this doctoral dissertation. Consider the boundary value problem

$$
\begin{cases}
{}^{H}\mathcal{D}_{1^+}^{\alpha(t)}X(t)=\dfrac{\cos(t)}{t^2+4}X(t)+\dfrac{1}{5\exp(t-1)}X(\frac{1}{4}t), & 1<t\leq e,\\
X(1)=0,\ X(e)=0,
\end{cases}
\tag{2.10}
$$

where $T_1 = 2$, $T_2 = e$, so that our partition of $[1, e]$ becomes $[1, 2]$, $(2, e]$.

Also, we take

$$\alpha(t) = \begin{cases} \dfrac{12}{10}, & t \in [1, 2], \\ \\ \dfrac{17}{10}, & t \in (2, e], \end{cases}$$

and see that $H(t, x, y) = \dfrac{\cos(t)}{t^2 + 4} x + \dfrac{1}{5\exp(t-1)} y$. Since

$$\begin{aligned} &|H(t, x_1, y_1) - H(t, x_2, y_2)| \\ &\quad = \left| \frac{\cos(t)}{t^2+4} x_1 + \frac{1}{5\exp(t-1)} y_1 - \frac{\cos(t)}{t^2+4} x_2 - \frac{1}{5\exp(t-1)} y_2 \right| \\ &\quad \le \frac{|\cos(t)|}{t^2+4} |x_1 - x_2| + \frac{1}{5\exp(t-1)} |y_1 - y_2| \\ &\quad \le \frac{1}{5} \left(|x_1 - x_2| + |y_1 - y_2| \right). \end{aligned}$$

so condition **(H1)** is satisfied with $L_1 = \dfrac{1}{5}$.

Consider the auxiliary boundary value problems

$$\begin{cases} {}^H\mathcal{D}_{1^+}^{\frac{12}{10}} X_1(t) = \dfrac{\cos(t)}{t^2+4} X_1(t) + \dfrac{1}{5\exp(t-1)} X_1(\frac{1}{4}t), \quad 1 < t \le 2, \\ X(1) = 0, \ X(2) = 0, \end{cases}$$

and

$$\begin{cases} {}^H\mathcal{D}_{2^+}^{\frac{17}{10}} X_2(t) = \dfrac{\cos(t)}{t^2+4} X_2(t) + \dfrac{1}{5\exp(t-1)} X_2(\frac{1}{4}t), \quad 2 < t \le e, \\ X(2) = 0, \ X(e) = 0. \end{cases}$$

Now, for $i \in \{1, 2\}$

$$\begin{cases} \dfrac{4L_1}{\Gamma(\alpha_i + 1)} \left(\ln \dfrac{T_i}{T_{i-1}} \right)^{\alpha_i} = \dfrac{4}{5\Gamma\left(\frac{22}{10}\right)} \left(\ln \dfrac{2}{1} \right)^{\frac{12}{10}} \approx 0.46770 < 1, \\ \dfrac{4L_1}{\Gamma(\alpha_i + 1)} \left(\ln \dfrac{T_i}{T_{i-1}} \right)^{\alpha_i} = \dfrac{4}{5\Gamma\left(\frac{27}{10}\right)} \left(\ln \dfrac{e}{2} \right)^{\frac{17}{10}} \approx 0.069519 < 1, \end{cases}$$

so **(H2)** is satisfied. Therefore, by Theorem 2.2.1, the problem (2.10) has a unique solution

$X \in C([\frac{1}{4}, e], \mathbb{R})$ given by

$$X(t) = \begin{cases} X_1(t), & t \in [1,2], \\ X_2(t), & t \in [2,e]. \end{cases}$$

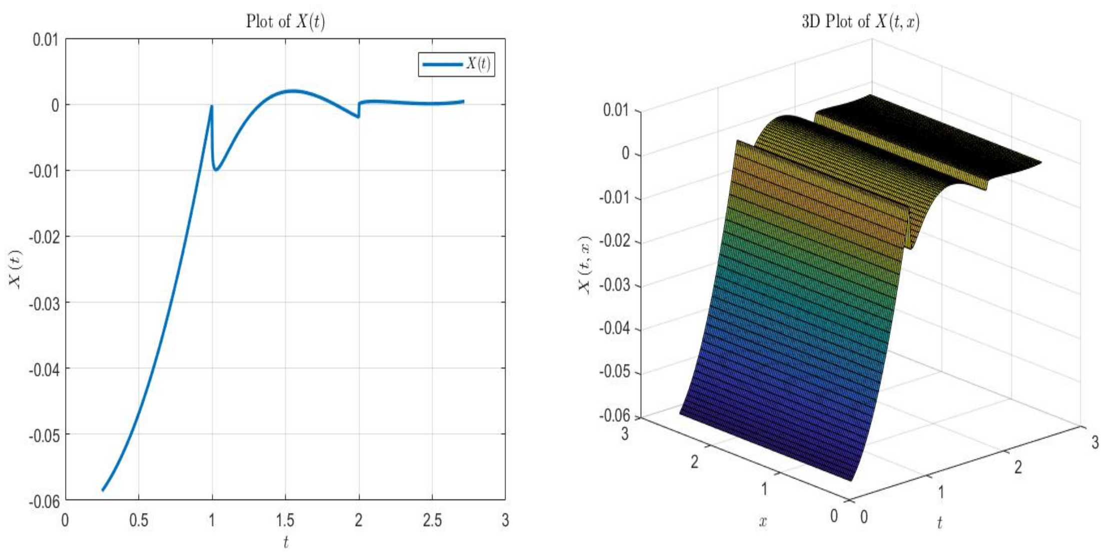

Figure 2.1: 2D and 3D Plots of $X(t)$ with delay function $\vartheta(\tau) = \frac{\tau^2-1}{16}$, $\lambda = \frac{1}{4}$.

In order to determine the U.H.R stability of the solution, we let $\Phi(t) = (\ln t)^{\frac{1}{2}}$. Therefore

$$\begin{aligned} {}^H I_{1^+}^{\alpha_1} \Phi(t) =& \frac{1}{\Gamma\left(\frac{12}{10}\right)} \int_1^t \frac{1}{s}\left(\ln \frac{t}{s}\right)^{\frac{12}{10}-1} (\ln s)^{\frac{1}{2}} ds \\ \leq& \frac{(\ln t)^{\frac{1}{2}}}{\Gamma\left(\frac{12}{10}\right)} \int_1^2 \frac{1}{s}\left(\ln \frac{2}{s}\right)^{\frac{2}{10}} ds \\ \leq& \frac{(\ln 2)^{\frac{12}{10}}}{\Gamma\left(\frac{22}{10}\right)} (\ln t)^{\frac{1}{2}} := \varrho_{\Phi(t)} \Phi(t). \end{aligned}$$

Thus, for $i = 1$, **(H3)** is satisfied for $\Phi(t) = (\ln t)^{\frac{1}{2}}$, and $\varrho_{\Phi(t)} = \dfrac{(\ln 2)^{\frac{13}{10}}}{\Gamma\left(\frac{22}{10}\right)}$,

$$ {}^H I_{2^+}^{\alpha_2} \Phi(t) = \frac{1}{\Gamma\left(\frac{17}{10}\right)} \int_2^t \frac{1}{s}\left(\ln \frac{t}{s}\right)^{\frac{17}{10}-1} \sqrt{\ln s} ds $$

$$\leq \frac{(\ln t)^{\frac{1}{2}}}{\Gamma\left(\frac{17}{10}\right)} \int_2^e \frac{1}{s}\left(\ln \frac{e}{s}\right)^{\frac{7}{10}} ds$$
$$\leq \frac{(\ln \frac{e}{2})^{\frac{17}{10}}}{\Gamma\left(\frac{27}{10}\right)} (\ln t)^{\frac{1}{2}} := \varrho_{\Phi(t)} \Phi(t).$$

Thus, for $i = 2$, **(H3)** is satisfied for $\Phi(t) = (\ln t)^{\frac{1}{2}}$, and $\varrho_{\Phi(t)} = \dfrac{(\ln \frac{e}{2})^{\frac{17}{10}}}{\Gamma\left(\frac{27}{10}\right)}$.

Therefore, the variable order Hadamard pantograph boundary value problem (2.10) is Ulam-Hyers-Rassias stable with respect to Φ.

Notice that, if we take $\lambda = \dfrac{1}{2}$ in the (2.10), we have the following representation

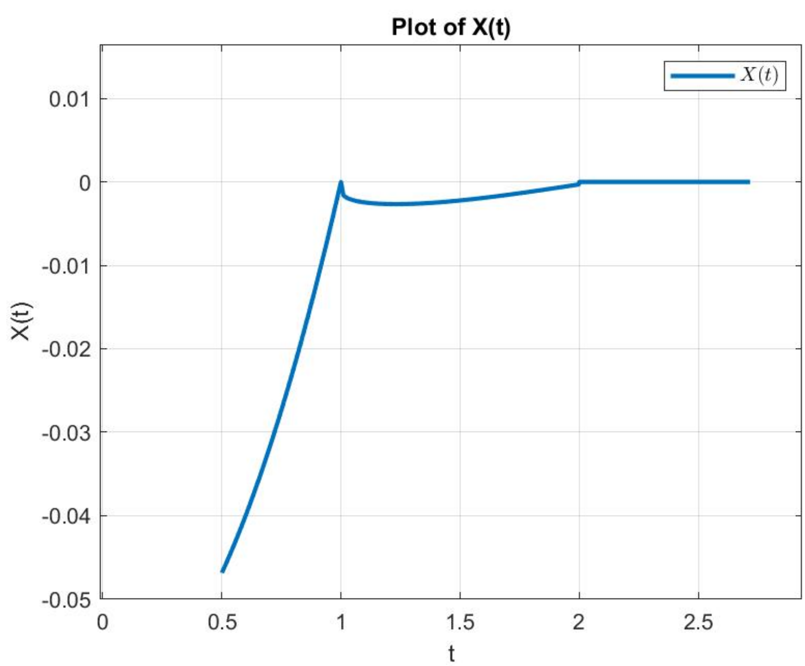

Figure 2.2: 2D Plot of $X(t)$ with delay function $\vartheta(\tau) = \frac{\tau^2-1}{16}$, $\lambda = \frac{1}{2}$.

Remark 2. The nonlinear term $H(t, X(t), X(\lambda t))$ exhibits a dependency structure such that it transitions to different values for different parameters λ. The following table represents a the context-dependent behavior of the function $H(t, X(t), X(\lambda t))$.

Values of λ	Values of t	Nonlinear term in the solution equation
$\lambda = \frac{1}{4}$	$t \in [1, 2]$	$H(t, X(t), X(\lambda t)) := H(t, X_1(t), \vartheta(t))$
	$t \in (2, e]$	$H(t, X(t), X(\lambda t)) := H(t, X_2(t), \vartheta(t))$
$\lambda = \frac{1}{2}$	$t \in [1, 2]$	$H(t, X(t), X(\lambda t)) := H(t, X_1(t), \vartheta(t))$
	$t \in (2, e]$	$H(t, X(t), X(\lambda t)) := H(t, X_2(t), X_1(t))$

Table 2.1: Context-dependent behavior

2.5 Conclusion

The variable order fractional pantograph equation represents a generalized version of its classical one, where the differentiation operator is replaced by a fractional derivative of variable order, thus several aspects can be considered. Firstly, the inclusion of variable order derivatives allows a more flexible modeling of systems with non-local characteristics. Additionally, it provides a richer mathematical framework for analyzing the context-dependent behavior of dynamical systems.

CHAPTER 3

GENERALIZED LYAPUNOV INEQUALITY FOR A PANTOGRAPH BVP OF VARIABLE ORDER

3.1 Introduction and motivation

G. Borg [15] proof of Lyapunov inequality is particulary simple. He starts with inequalities

$$\int_a^b \left|\frac{X''(t)}{X(t)}\right| dt > \left(||X||_\infty\right)^{-1} \int_a^b |X''(t)| dt$$
$$> \left(||X||_\infty\right)^{-1} \left|\int_c^d X''(t) dt\right| = \left(||X||_\infty\right)^{-1} |X'(d) - X'(c)|, \qquad (3.1)$$

for arbitrary $a \leq c < d \leq b$. Now let $||X||_\infty = X(t)$, by Rolle's Theorem we can choose $a < c < t$ and $t < d < b$ such that

$$\begin{cases} X'(c) = \dfrac{||X||_\infty}{t-a}, \\ -X'(d) = \dfrac{||X||_\infty}{b-t}. \end{cases}$$

Combining this with (3.1) we get that

$$\int_a^b \left|\frac{X''(t)}{X(t)}\right| dt > \frac{1}{t-a} + \frac{1}{b-t}.$$

The desired inequality is obtained by minimization of the right hand side of the abvove inequality with respect to t.

Values of t	Sign of $\Lambda'(t)$	Variation of $\Lambda(t)$	Values of $\Lambda(t)$
$t \in (a, \frac{a+b}{2})$	Negative	Decreasing	$\Lambda(t \to a^+) \to +\infty$
$t = \frac{a+b}{2}$	Null	Extremum point	$\Lambda\left(\frac{a+b}{2}\right) = \frac{4}{b-a}$
$t \in (\frac{a+b}{2}, b)$	Positive	Increasing	$\Lambda(t \to b^-) \to +\infty$

Table 3.1: Table of Variations for $\Lambda(t) = \frac{1}{t-a} + \frac{1}{b-t}$.

Lyapunovs inequality has had applications even in fractional calculus, and takes different forms depending on the type of fractional derivative involved [24]. For example, if we have

$$\begin{cases} \mathcal{D}^{\alpha}_{a^+} X(t) + \mathcal{W}(t)X(t) = 0, \quad t \in (a,b), \\ X(a) = X(b) = 0, \end{cases}$$

where $\mathcal{D}^{\alpha}_{a^+}$ is either the Riemann-Liouville or Caputo fractional derivative of order $\alpha \in (1,2]$ and $\mathcal{W} : [a,b] \longrightarrow \mathbb{R}$ is a continuous function, then the Lyapunov inequality takes the fractional form

$$\int_a^b |\mathcal{W}(t)|dt > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}, \quad a < b < +\infty. \tag{3.2}$$

Note that one obtains Lyapunov's classical inequality when $\alpha = 2$.

Recently, in [37], the authors obtained a generalized Lyapunov type inequality for the Hadamard fractional boundary value problem

$$\begin{cases} {}^{H}\mathcal{D}^{\alpha}_{1^+} X(t) + \mathcal{W}(t)X(t) = 0, \quad t \in (1,e), \\ X(1) = X(e) = 0, \end{cases}$$

where ${}^{H}\mathcal{D}^{\alpha}_{a^+}$ is the Hadamard fractional derivative of the order $\alpha \in (1,2]$, and $\mathcal{W} : [1,e] \longrightarrow \mathbb{R}$ is a real continuous function. Therefore, the condition (3.2) becomes

$$\int_1^e |\mathcal{W}(t)|dt > \Gamma(\alpha)\mu^{1-\alpha}(1-\mu)^{1-\alpha}\exp(\mu),$$

where $\mu = \dfrac{2\alpha - 1 - \sqrt{(2\alpha-2)^2+1}}{2}$.

3.2 Generalized Lyapunov Inequality

In this section, we discuss the generalized Lyapunov-type inequalities for the boundary value problem (HFPE).

Proposition 3. *The Green function for the auxiliary boundary value problem* (2.5) *for each* $i \in \{1, 2, \ldots, n\}$ *is given by*

$$G_i(s,t) = \begin{cases} G_{1,i}(s,t), & T_{i-1} \le s \le t \le T_i, \\ G_{2,i}(s,t), & T_{i-1} \le t \le s \le T_i, \end{cases}$$

where

$$G_{1,i}(s,t) = \frac{1}{s\,\Gamma(\alpha_i)} \left[\left(\ln \frac{t}{s}\right)^{\alpha_i - 1} - \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} \right],$$

and

$$G_{2,i}(s,t) = \frac{-1}{s\,\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1}.$$

Proof. From the proof of the Theorem 2.2.1, we have

$$\begin{aligned}
\widehat{X}_i(t) &= \frac{-1}{\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \int_{T_{i-1}}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{t}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s))| ds \\
&= \frac{-1}{\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \left[\int_{T_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s))| ds \right. \\
&\quad \left. + \int_{t}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s))| ds \right] \\
&\quad + \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{t}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s))| ds \\
&= \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} \frac{1}{s} \left[\left(\ln \frac{t}{s}\right)^{\alpha_i - 1} - \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} \right] \\
&\quad \times |H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s))| ds \\
&\quad - \frac{1}{\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \int_{t}^{T_i} \frac{1}{s} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} |H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s))| ds \\
&= \int_{T_{i-1}}^{T_i} G_i(s,t) H(s, \widehat{X}_i(s), \widehat{X}_i(\lambda s)) ds,
\end{aligned}$$

which proves the lemma. □

Lemma 3.2.1. *Let the Green function* G_i *be defined as in Proposition 3. Then, for* $i \in \{1, 2, \ldots, n\}$

$$\max_{t \in [T_{i-1}, T_i]} |G_i(s,t)| \le \frac{1}{\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left[(\mu_i - \ln T_{i-1}) (\ln T_i - \mu_i) \right]^{\alpha_i - 1} \exp(-\mu_i), \quad (3.3)$$

where

$$\mu_i = \frac{2\alpha_i - 2 + \ln T_i T_{i-1} - \sqrt{(2\alpha_i - 2 + \ln T_i T_{i-1})^2 - 4\left[(\alpha_i - 1)\ln T_i T_{i-1} + \ln T_i \ln T_{i-1}\right]}}{2}.$$

Proof. It is easy to see that $G_i(s,t) \leq 0$ for all $T_{i-1} \leq s, t \leq T_i$. Thus $G_{1,i}(T_i, T_i) = G_{2,i}(T_{i-1}, s) = 0$ which are the maximum values of $G_{1,i}$ and $G_{2,i}$ respectively.

For $t \leq s$, we see that

$$\begin{aligned}
G_{2,i}(s,t) &= \frac{-1}{s\,\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{t}{T_{i-1}}\right)^{\alpha_i - 1} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} \\
&\geq \frac{-1}{s\,\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left(\ln \frac{s}{T_{i-1}}\right)^{\alpha_i - 1} \left(\ln \frac{T_i}{s}\right)^{\alpha_i - 1} \\
&\geq \frac{-1}{s\,\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left[\ln T_i \ln s - \ln^2 s - \ln T_{i-1} \ln T_i + \ln T_{i-1} \ln s\right]^{\alpha_i - 1} \\
&\geq \frac{-1}{s\,\Gamma(\alpha_i)} \left(\ln \frac{T_i}{T_{i-1}}\right)^{1-\alpha_i} \left[(\ln T_i T_{i-1}) \ln s - \ln^2 s - \ln T_{i-1} \ln T_i\right]^{\alpha_i - 1}.
\end{aligned}$$

It follows that we need to determine where the maximum value of the function

$$y(s) = \frac{1}{s} \left[(\ln T_i T_{i-1}) \ln s - \ln^2 s - \ln T_{i-1} \ln T_i\right]^{\alpha_i - 1},$$

occurs. Now,

$$\begin{aligned}
y'(s) = &- \frac{1}{s^2} \left[(\ln T_i T_{i-1}) \ln s - \ln^2 s - \ln T_{i-1} \ln T_i\right]^{\alpha_i - 1} \\
&+ \frac{1}{s^2} \left((\ln T_i T_{i-1}) \ln s - 2\ln s\right)(\alpha_i - 1) \left[(\ln T_i T_{i-1}) \ln s - \ln^2 s - \ln T_{i-1} \ln T_i\right]^{\alpha_i - 2} \\
= &\frac{1}{s^2} \left[(\ln T_i T_{i-1}) \ln s - \ln^2 s - \ln T_{i-1} \ln T_i\right]^{\alpha_i - 2} \\
&\times \left[(\alpha_i - 1)(\ln T_i T_{i-1} - 2\ln s) - (\ln T_i T_{i-1}) \ln s - \ln^2 s - \ln T_i \ln T_{i-1}\right] \\
= &\frac{1}{s^2} \left[(\ln T_i T_{i-1}) \ln s - \ln^2 s - \ln T_{i-1} \ln T_i\right]^{\alpha_i - 2} \\
&\times \left[(2 - 2\alpha_i - \ln T_i T_{i-1}) \ln s + \ln^2 s + (\alpha_i - 1) \ln T_i T_{i-1} + \ln T_i \ln T_{i-1}\right],
\end{aligned}$$

and $y'(s) = 0$ if and only if

$$\ln^2 s - (2\alpha_i - 2 + \ln T_i T_{i-1}) \ln s + (\alpha_i - 1) \ln T_i T_{i-1} + \ln T_i \ln T_{i-1} = 0,$$

so

$$\ln s = \frac{2\alpha_i - 2 + \ln T_i T_{i-1} \pm \sqrt{(2\alpha_i - 2 + \ln T_i T_{i-1})^2 - 4\left[(\alpha_i - 1)\ln T_i T_{i-1} + \ln T_i \ln T_{i-1}\right]}}{2}.$$

However, since $\alpha_i \geq 1$, we have

$$\begin{aligned}
\ln s &= \frac{2\alpha_i - 2 + \ln T_i T_{i-1} + \sqrt{(2\alpha_i - 2 + \ln T_i T_{i-1})^2 - 4\left[(\alpha_i - 1)\ln T_i T_{i-1} + \ln T_i \ln T_{i-1}\right]}}{2} \\
&\geq \frac{\ln T_i T_{i-1} + \sqrt{(\ln T_i T_{i-1})^2 - 4\left[\ln T_i \ln T_{i-1}\right]}}{2} \\
&\geq \frac{\ln T_i + \ln T_{i-1} + \ln T_i - \ln T_{i-1}}{2} = \ln T_i,
\end{aligned}$$

which is a contradiction since $s \leq T_i$. Therfore, $\min\limits_{s\in[t,T_i]} G_{2,i}(s,t) = G_{2,i}(s^*, s^*)$, where

$$s^* = \exp\left(\frac{2\alpha_i - 2 + \ln T_i T_{i-1} - \sqrt{(2\alpha_i - 2 + \ln T_i T_{i-1})^2 - 4\left[(\alpha_i - 1)\ln T_i T_{i-1} + \ln T_i \ln T_{i-1}\right]}}{2}\right).$$

For the function $G_{1,i}$, we already know that $G_{1,i} \leq 0$, and that the maximum value of $G_{1,i}$ is 0 for $s = t = T_i$. To obtain the minimum value of $G_{1,i}$, for fixed s, a computation gives

$$\frac{\partial}{\partial t} G_{1,i}(s,t) = \frac{(\alpha_i - 1)}{st\Gamma(\alpha_i)}\left[\left(\ln\frac{t}{s}\right)^{\alpha_i - 2} - \left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left(\ln\frac{t}{T_{i-1}}\right)^{\alpha_i-2}\left(\ln\frac{T_i}{s}\right)^{\alpha_i-1}\right].$$

Observing that $\left(\ln\frac{t}{s}\right) \leq \left(\ln\frac{t}{T_{i-1}}\right)$ and $\alpha_i - 2 \leq 0$, we see that

$$\left(\ln\frac{t}{s}\right)^{\alpha_i-2} \geq \left(\ln\frac{t}{T_{i-1}}\right)^{\alpha_i-2} \geq \left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left(\ln\frac{t}{T_{i-1}}\right)^{\alpha_i-2}\left(\ln\frac{T_i}{s}\right)^{\alpha_i-1}.$$

So $\frac{\partial}{\partial t}G_{1,i}(s,t)$ is increasing with respect to t, which means

$$\begin{aligned}
\min_{t\in[s,T_i]} G_{1,i}(s,t) &= G_{1,i}(s,s) \\
&= \frac{-1}{s\Gamma(\alpha_i)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left(\ln\frac{s}{T_{i-1}}\right)^{\alpha_i-1}\left(\ln\frac{T_i}{s}\right)^{\alpha_i-1} \\
&= \min_{s\in[t,T_i]} G_{2,i}(s,t) \\
&= G_{2,i}(s^*, s^*) \\
&= \frac{-1}{\Gamma(\alpha_i)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left[(\mu_i - \ln T_{i-1})(\ln T_i - \mu_i)\right]^{\alpha_i-1}\exp(-\mu_i).
\end{aligned}$$

This implies that for $i \in \{1, 2, \ldots, n\}$,

$$\max_{t\in[T_{i-1},T_i]} |G_i(s,t)| \leq \frac{1}{\Gamma(\alpha_i)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left[(\mu_i - \ln T_{i-1})(\ln T_i - \mu_i)\right]^{\alpha_i-1}\exp(-\mu_i),$$

and complete the proof of the lemma.

□

We are now ready to present our Lyapunov inequality for our problem.

Theorem 3.2.1. *Assume there exists non-negative continuous function $\mathcal{W}(t) : [1,T] \longrightarrow \mathbb{R}$ such that*

$$|H(t,x,y)| \leq \mathcal{W}(t)(|x|+|y|), \quad 1 \leq t \leq T. \tag{3.4}$$

If the boundary value problem (HFPE) *has a non-trivial solution X, then*

$$\int_1^T \mathcal{W}(s)ds \geq \sum_{i=1}^{n} \frac{\Gamma(\alpha_i)}{2}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i-1}\left[(\mu_i-\ln T_{i-1})(\ln T_i-\mu_i)\right]^{1-\alpha_i}\exp(\mu_i), \tag{3.5}$$

where $\mu_i, i \in \{1,2,\cdots,n\}$ are given in Lemma 3.2.1.

Proof. Let X be a non-trivial solution of (HFPE). Therefore

$$\begin{aligned}
|X_1| &\leq \int_1^{T_1} |G_1(t,s)H(s,X_1(s),X_1(\lambda s))|ds \\
&\leq \int_1^{T_1} \max_{t\in[1,T_1]} |G_1(t,s)|\mathcal{W}(s)(|X_1(s)|+|X_1(\lambda s)|)ds \\
&\leq \frac{2||X_1||_{E_1}}{\Gamma(\alpha_1)}(\ln T_1)^{1-\alpha_1}\left[\mu_1(\ln t_1-\mu_1)\right]^{\alpha_1-1}\exp(-\mu_1)\int_1^{T_1}\mathcal{W}(s)ds,
\end{aligned}$$

which implies that

$$\int_1^{T_1}\mathcal{W}(s)ds \geq \frac{\Gamma(\alpha_1)}{2}(\ln T_1)^{\alpha_1-1}\left[\mu_1(\ln T_1-\mu_1)\right]^{1-\alpha_1}\exp(\mu_1).$$

Similar, for $i \in \{2,3,\ldots,n\}$, we have

$$\begin{aligned}
|X_i| &\leq \int_{T_{i-1}}^{T_i} |G_i(t,s)H(s,X_i(s),X_i(\lambda s))|ds \\
&\leq \int_{T_{i-1}}^{T_i} \max_{t\in[T_{i-1},T_i]} |G_i(t,s)|\mathcal{W}(s)|X_i(s)+X_i(\lambda s)|ds \\
&\leq \frac{2||X_i||_{E_i}}{\Gamma(\alpha_i)}\left(\ln\frac{T_i}{T_{i-1}}\right)^{1-\alpha_i}\left[(\mu_i-\ln T_{i-1})(\ln T_i-\mu_i)\right]^{\alpha_i-1}\exp(-\mu_i)\int_{T_{i-1}}^{T_i}\mathcal{W}(s)ds,
\end{aligned}$$

so

$$\int_{T_{i-1}}^{T_i}\mathcal{W}(s)ds \geq \frac{\Gamma(\alpha_i)}{2}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i-1}\left[(\mu_i-\ln T_{i-1})(\ln T_i-\mu_i)\right]^{1-\alpha_i}\exp(\mu_i).$$

Hence,

$$\int_1^T \mathcal{W}(s)ds \geq \sum_{i=1}^{n}\frac{\Gamma(\alpha_i)}{2}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i-1}\left[(\mu_i-\ln T_{i-1})(\ln T_i-\mu_i)\right]^{1-\alpha_i}\exp(\mu_i),$$

which proves the theorem. □

3.3 Example

Following the obtained results in the above example (2.10). If we take $\mathcal{W}(t) = \dfrac{1}{t}$ on $[1, e]$, then

$$\begin{aligned}|H(t, X(t), X(\lambda t))| &= \left|\frac{\cos(t)}{t^2+4}X(t) + \frac{1}{5\exp(t-1)}X(\frac{1}{4}t)\right| \\ &\leq \frac{|\cos(t)|}{t^2+4}|X(t)| + \frac{1}{5\exp(t-1)}|X(\frac{1}{4}t)| \\ &\leq \frac{1}{t}(|X(t)| + |X(\frac{1}{4}t)|).\end{aligned}$$

Thus, condition (3.4) holds for $\mathcal{W}(t)$. Now, we need to check the conditions (3.3); we have

$$\begin{aligned}\mu_1 &= \frac{\frac{4}{10} + \ln 2 - \sqrt{(\frac{4}{10} + \ln 2)^2 - 4\left[(\frac{2}{10})\ln 2\right]}}{2} \approx 0.14639, \\ \mu_2 &= \frac{\frac{14}{10} + \ln 2e - \sqrt{(\frac{14}{10} + \ln 2e)^2 - 4\left[(\frac{7}{10})\ln 2e + \ln 2\right]}}{2} \approx 0.82991.\end{aligned}$$

Thus

$$\begin{aligned}1 = \int_1^e \frac{1}{s}ds &\geq \sum_{i=1}^{2} \frac{\Gamma(\alpha_i)}{2}\left(\ln\frac{T_i}{T_{i-1}}\right)^{\alpha_i-1}\left[(\mu_i - \ln T_{i-1})(\ln T_i - \mu_i)\right]^{1-\alpha_i}\exp(\mu_i) \\ &\approx 0.30254 + 0.45641 \approx 0.75895.\end{aligned}$$

Therefore, the Lyapunov inquality (3.5) is established.

3.4 Conclusion

Our exploration into Lyapunov inequalities has revealed significant insights that contribute to the understanding of stability in dynamical systems. Through a rigorous analysis and development of inequalities, our study has made several notable contributions to assess system stability by recognizing the challenges posed by traditional methods thus, providing a novel approach and a powerful tool to analyze whether or not the stability is garanteed.

CHAPTER 4

A COMPREHENSIVE STUDY OF THE VARIABLE ORDER FRACTIONAL LANGEVIN BVP

4.1 Introduction and motivation

In [1], Abbas et al. discussed the solvability of the following Langevin equation with two Hadamard fractional derivatives

$$\begin{cases} {}^{H}\mathcal{D}^{\alpha}_{1,t}\left({}^{H}\mathcal{D}^{\beta}_{1,t}-\lambda\right)X(t)=H(t,X(t)), & t\in[1,e],\\ \left({}^{H}\mathcal{D}^{\beta}_{1,t}-\lambda\right)X(e)=0, \quad {}^{H}I^{1-\beta}_{1^+}X(1)=c_0, \end{cases}$$

where $c_0 \in \mathbb{R}$, $\lambda > 0$, ${}^{H}\mathcal{D}^{\alpha}_{1,t}$ and ${}^{H}\mathcal{D}^{\beta}_{1,t}$ denote Hadamard fractional derivatives of orders $0 < \alpha, \beta \leq 1$ respectively, ${}^{H}I^{1-\alpha}_{1^+}$ denotes the left hand Hadamard fractional integral of order $1-\alpha$, and $H : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. The approach they used involved the analysis of a Volterra integral equation and properties of the Mittag-Leffler function.

Recently, Hilal et al. [29] investigated the existence and uniqueness of solution to the following boundary value problem for the Langevin equation with the Hilfer fractional derivative

$$\begin{cases} {}^{H}\mathcal{D}^{\alpha_1,\gamma_1}\left({}^{H}\mathcal{D}^{\alpha_2,\gamma_2}-\lambda\right)X(t)=H(t,X(t)), & a\leq t\leq b,\\ X(a)=0, \quad X(b)=\sum_{i=1}^{2}\mu_i(I^{v_i}(X))(\eta), & a<\eta<b, \end{cases}$$

where ${}^{H}\mathcal{D}^{\alpha_i,\gamma_i}$, $i \in \{1, 2\}$, are Hilfer fractional derivatives of order $0 < \alpha_i < 1$ and parameters

$0 \leq \gamma_i \leq 1$, $\lambda \in \mathbb{R}$, $a \geq 0$, I^{v_i} is the Riemann-Liouville fractional integral of order $v_i > 0$, $\mu_i \in \mathbb{R}$, and $H : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

As far as we know, there are no contributions in the literature on the solutions of fractional Langevin equations of variable order.

Integrating fixed point theory, stochastic analysis, and variable order fractional calculus offers a comprehensive approach to enhancing paper strength. This involves identifying sufficient conditions on the non-linear term H to establish the existence and uniqueness of solutions to problems which address phenomena with memory effects and random fluctuations.

For this purpose, we investigate the Langevin boundary value problem involving variable order Caputo fractional derivatives:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha(t)} \left({}^{C}\mathcal{D}_{0^+}^{\beta(t)} - \lambda \right) X(t) = H(t, X(t)), \quad t \in [0, T], \\ X(0) = X(T) = 0. \end{cases} \tag{FLBVP}$$

Here, $0 < \alpha(t)$, $\beta(t) < 1$, $\lambda \in \mathbb{R}^+$, $H : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and ${}^{C}D_{0^+}^{\alpha(t)}$ and ${}^{C}D_{0^+}^{\beta(t)}$ are Caputo fractional derivatives of variable orders $\alpha(t)$ and $\beta(t)$, respectively, for the function X.

4.2 Existence and uniqueness of solution

Based on the previous discussion, in this section we present our main results.

Let $\mathcal{P} = \{[0, T_1], (T_1, T_2], (T_2, T_3], \ldots, (T_{n-1}, T]\}$ be a partition of the finite interval $[0, T]$, and let $\alpha : [0, T] \to (0, 1]$, and $\beta : [0, T] \to (0, 1]$ be two piecewise constant functions with respect to $\mathcal{P}$ given by

$$\alpha(t) = \sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t) = \begin{cases} \alpha_1, & t \in [0, T_1], \\ \alpha_2, & t \in (T_1, T_2], \\ \vdots & \\ \alpha_n, & t \in (T_{n-1}, T], \end{cases} \qquad \beta(t) = \sum_{i=1}^{n} \beta_i \mathbb{I}_i(t) = \begin{cases} \beta_1, & t \in [0, T_1], \\ \beta_2, & t \in (T_1, T_2], \\ \vdots & \\ \beta_n, & t \in (T_{n-1}, T], \end{cases}$$

where $0 < \alpha_i$, $\beta_i < 1$, $i \in \{1, 2, \ldots, n\}$, are constants, and $\mathbb{I}_i$ is the characteristic function for the interval $(T_{i-1}, T_i]$ for each $i \in \{1, 2, \ldots, n\}$, i.e.,

$$\mathbb{I}_i(t) = \begin{cases} 1, & t \in (T_{i-1}, T_i], \\ 0, & \text{elsewhere}. \end{cases}$$

Hence, we obtain

$$\begin{aligned} &{}^{C}\mathcal{D}_{0^+}^{\alpha(t)} \left({}^{C}\mathcal{D}_{0^+}^{\beta(t)} - \lambda \right) X(t) \\ &\quad = \int_0^t \frac{(t-s)^{-\sum_{i=1}^n \alpha_i \mathbb{I}_i(t)}}{\Gamma\left(1 - \sum_{i=1}^n \alpha_i \mathbb{I}_i(t)\right)} \frac{d}{ds} \left(\int_0^s \frac{(s-w)^{-\sum_{i=1}^n \beta_i \mathbb{I}_i(s)}}{\Gamma\left(1 - \sum_{i=1}^n \beta_i \mathbb{I}_i(s)\right)} X'(w) dw - \lambda X(s) \right) ds. \end{aligned}$$

The equation in the problem (FLBVP) can then be written as

$$\int_0^t \frac{(t-s)^{-\sum_{i=1}^n \alpha_i \mathbb{I}_i(t)}}{\Gamma\left(1 - \sum_{i=1}^n \alpha_i \mathbb{I}_i(t)\right)} \frac{d}{ds} \left(\int_0^s \frac{(s-w)^{-\sum_{i=1}^n \beta_i \mathbb{I}_i(s)}}{\Gamma\left(1 - \sum_{i=1}^n \beta_i \mathbb{I}_i(s)\right)} X'(w) dw - \lambda X(s) \right) ds = H(t, X(t)),$$

for $0 \le t \le T < +\infty$.

We denote by $E_i = C([0, T_i], \mathbb{R})$ the class of functions that form a Banach space with the norm

$$||X||_{E_i} = \sup_{t \in [0, T_i]} |X(t)|, \quad i \in \{1, 2, \ldots, n\}.$$

Let the functions $\widehat{X}_i \in E_i$ be such that $\widehat{X}_i(t) = 0$, and for all $t \in [0, T_{i-1}] \cup \{T_i\}$ for all $i \in \{1, 2, \ldots, n\}$.

Therefore, in the interval $[0, T_1]$, we have

$$\begin{aligned} &{}^{C}\mathcal{D}_{0^+}^{\alpha_1} \left({}^{C}\mathcal{D}_{0^+}^{\beta_1} - \lambda \right) \widehat{X}(t) \\ &\quad = \int_0^t \frac{(t-s)^{-\alpha_1}}{\Gamma(1-\alpha_1)} \frac{d}{ds} \left(\int_0^s \frac{(s-w)^{-\beta_1}}{\Gamma(1-\beta_1)} \widehat{X}'(w) dw - \lambda \widehat{X}(s) \right) ds. \end{aligned} \tag{4.1}$$

Again, in the interval $(T_1, T_2]$,

$$\begin{aligned} &{}^{C}\mathcal{D}_{T_1^+}^{\alpha_2} \left({}^{C}\mathcal{D}_{T_1^+}^{\beta_2} - \lambda \right) \widehat{X}(t) \\ &\quad = \int_{T_1}^t \frac{(t-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} \frac{d}{ds} \left(\int_{T_1}^s \frac{(s-w)^{-\beta_2}}{\Gamma(1-\beta_2)} \widehat{X}'(w) dw - \lambda \widehat{X}(s) \right) ds. \end{aligned} \tag{4.2}$$

Similarly, in $(T_{i-1}, T_i]$,

$$\begin{aligned} &{}^{C}\mathcal{D}_{T_{i-1}^+}^{\alpha_i} \left({}^{C}\mathcal{D}_{T_{i-1}^+}^{\beta_i} - \lambda \right) \widehat{X}(t) \\ &\quad = \int_{T_{i-1}}^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} \frac{d}{ds} \left(\int_{T_{i-1}}^s \frac{(s-w)^{-\beta_i}}{\Gamma(1-\beta_i)} \widehat{X}'(w) dw - \lambda \widehat{X}(s) \right) ds. \end{aligned} \tag{4.3}$$

Thus, for each $i \in \{1, 2, \ldots, n\}$, we consider the auxiliary constant order boundary value problem

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha_i}_{T^+_{i-1}} \left({}^{C}\mathcal{D}^{\beta_i}_{T^+_{i-1}} - \lambda \right) \widehat{X}(t) = H(t, \widehat{X}(t)), \quad T_{i-1} < t \leq T_i, \\ \widehat{X}(T_{i-1}) = \widehat{X}(T_i) = 0. \end{cases} \tag{4.4}$$

Next, we define what we mean by a solution of (FLBVP).

Definition 4.2.1. We say that the problem (FLBVP) has a solution X, if there exist functions X_i, such that: $X_1 \in E_1$ satisfies equation (4.1) with $X_1(0) = X_1(T_1) = 0$; $X_2 \in E_2$ satisfies equation (4.2) with $X_2(T_1) = X_2(T_2) = 0$; $X_i \in E_i$ satisfies equation (4.3) with $X_i(T_{i-1}) = X_i(T_i) = 0$ for $i \in \{3, \ldots, n\}$.

Remark 3. We say problem (FLBVP) has a unique solution if the functions X_i are uniques for each $i \in \{1, 2, \ldots, n\}$.

Based on the previous discussion, we have the following results.

Lemma 4.2.1. *Let $i \in \{1, 2, \ldots, n\}$. Then the function $\widehat{X}$ is a solution of (4.4) if $\widehat{X}$ is a solution of the integral equation*

$$\widehat{X}(t) = -\left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{X}(t) - I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t, \widehat{X}(t)) \right)\Big|_{t=T_i} \left(\frac{t - T_{i-1}}{T_i - T_{i-1}} \right)^{\beta_i} + \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{t} (t-s)^{\beta_i - 1} \widehat{X}(s) ds$$
$$+ \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^{t} (t-s)^{\alpha_i + \beta_i - 1} H(s, \widehat{X}(s)) ds, \tag{4.5}$$

for $t \in (T_{i-1}, T_i]$ for each $i \in \{1, 2, \ldots, n\}$.

Proof. Assume that $\widehat{X}$ satisfies (4.4). We transform (4.4) into an equivalent integral equation as follows. Let $T_{i-1} < t \leq T_i$; then Lemma 1.2.1 implies

$$\left({}^{C}\mathcal{D}^{\beta_i}_{T^+_{i-1}} - \lambda \right) \widehat{X}(t) = I^{\alpha_i}_{T^+_{i-1}} H(t, \widehat{X}(t)) + \varrho_1,$$

so

$$\widehat{X}(t) = \lambda I^{\beta_i}_{T^+_{i-1}} \widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t, \widehat{X}(t)) + \frac{\varrho_1}{\Gamma(\beta_i + 1)} (t - T_{i-1})^{\beta_i} + \varrho_2.$$

Using the boundary conditions $\widehat{X}(T_i) = \widehat{X}(T_{i-1}) = 0$, we obtain

$$\begin{cases} \varrho_2 = 0, \\ \varrho_1 = -\dfrac{\Gamma(\beta_i + 1)}{(T_i - T_{i-1})^{\beta_i}} \left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t, \widehat{X}(t)) \right)\Big|_{t=T_i}. \end{cases}$$

Therefore, the solution of the auxiliary boundary value problem (4.4) is given by

$$\widehat{X}(t) = -\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{X}(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i} + \frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{X}(s)ds$$
$$+\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{X}(s))ds.$$

A straight forward calculation shows that if $\widehat{X}$ is given by (4.5), then it is a solution of (4.4) for each $i \in \{1, 2, \ldots, n\}$. □

Before presenting our main results, we first state the following hypotheses that will be needed:

(A1) Let $H : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that there exist positive constants L_2 and L_3 such that

$$|H(t, x)| \leq L_2|x| + L_3,$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

(A2) The parameter λ satisfies

$$\frac{\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{L_2}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} < \frac{1}{2}.$$

Theorem 4.2.1. *Assume that **(A1)** and **(A2)** hold. Then the boundary value problem* (FLBVP) *has at least one solution in* $C([0, T], \mathbb{R})$.

Proof. Consider the mapping $\mathcal{Z} : E_i \to E_i$ given by

$$\begin{aligned}(\mathcal{Z}\widehat{X})(t) = &-\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{X}(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\\ &+\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{X}(s)ds\\ &+\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{X}(s))ds.\end{aligned}$$

Let the ball $B_{R_i} = \{X \in E_i : ||X||_{E_i} \leq R_i\}$ be a non-empty, closed, bounded, convex subset of E_i, where

$$R_i \geq \frac{\dfrac{2L_3}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i}}{1-\left(\dfrac{2\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \dfrac{2L_2}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i}\right)}.$$

The proof will be given through several steps.

Step 1: For each $i \in \{1, 2, \dots, n\}$, $\mathcal{Z}(B_{R_i}) \subset B_{R_i}$. We have

$$\begin{aligned}
|(\mathcal{Z}\widehat{X})(t)| &= \left| -\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{X}(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\right.\\
&\quad \left. +\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{X}(s)ds + \frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{X}(s))ds\right|\\
&\le \frac{2\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}|\widehat{X}(s)|ds + \frac{2}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}|H(s,\widehat{X}(s))|ds\\
&\le \frac{2\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}|\widehat{X}(s)|ds + \frac{2}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}(L_2|\widehat{X}(s)|+L_3)ds\\
&\le \frac{2\lambda||\widehat{X}||_{E_i}}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}ds + \frac{2L_2||\widehat{X}||_{E_i}}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}ds\\
&\quad + \frac{2L_3}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}ds\\
&\le \frac{2\lambda||\widehat{X}||_{E_i}}{\Gamma(\beta_i+1)}(T_i-T_{i-1})^{\beta_i} + \frac{2L_2||\widehat{X}||_{E_i}}{\Gamma(\alpha_i+\beta_i+1)}(T_i-T_{i-1})^{\alpha_i+\beta_i}\\
&\quad + \frac{2L_3}{\Gamma(\alpha_i+\beta_i+1)}(T_i-T_{i-1})^{\alpha_i+\beta_i}\\
&\le R_i.
\end{aligned}$$

Step 2: $\mathcal{Z}$ is continuous for each $i \in \{1, 2, \dots, n\}$. Let $\{\widehat{X}_m\}$ be a sequence such that $\widehat{X}_m \to \widehat{X}$ in B_{R_i}. Then, for each $t \in [T_{i-1}, T_i]$, $i \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
|(\mathcal{Z}\widehat{X}_m)(t) - (\mathcal{Z}\widehat{X})(t)| &= \left| -\left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{X}_m(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{X}_m(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\right.\\
&\quad + \frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{X}_m(s)ds\\
&\quad +\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{X}_m(s))ds\\
&\quad + \left(\lambda I^{\beta_i}_{T^+_{i-1}}\widehat{X}(t) - I^{\alpha_i+\beta_i}_{T^+_{i-1}}H(t,\widehat{X}(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\\
&\quad - \frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}\widehat{X}(s)ds\\
&\quad \left. -\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,\widehat{X}(s))ds\right|
\end{aligned}$$

$$\leq \frac{2\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{\beta_i - 1} |\widehat{X}_m(s) - \widehat{X}(s)| ds$$
$$+ \frac{2}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^{T_i} (T_i - s)^{\alpha_i + \beta_i - 1} |H(s, \widehat{X}_m(s)) - H(s, \widehat{X}(s))| ds.$$

Taking into account the convergence of the sequence $\{\widehat{X}_m\}$, and the continuity of the function H, the right hand side of above inequality tends to zero as $m \to +\infty$. Therefore,

$$||\mathcal{Z}(\widehat{X}_m)(t) - \mathcal{Z}(\widehat{X})(t)||_{E_i} \to 0 \quad \text{as} \quad m \to +\infty.$$

Step 3: $\mathcal{Z}$ is relatively compact for each $i \in \{1, 2, \ldots, n\}$. In view of **Step 1**, we have that $\mathcal{Z}(B_{R_i}) \subset B_{R_i}$. Thus, $\mathcal{Z}(B_{R_i})$ is uniformly bounded. It remains to show that $\mathcal{Z}$ is equicontinuous for each $i \in \{1, 2, \ldots, n\}$.

Let $t_1, t_2 \in (T_{i-1}, T_i]$. Then,

$$\begin{aligned}
&|(\mathcal{Z}\widehat{X})(t_1) - (\mathcal{Z}\widehat{X})(t_2)| \\
&= \Bigg| - \left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t, \widehat{X}(t))\right)\Big|_{t=T_i} \left(\frac{t_1 - T_{i-1}}{T_i - T_{i-1}}\right)^{\beta_i} \\
&+ \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{t_1} (t_1 - s)^{\beta_i - 1} \widehat{X}(s) ds \\
&+ \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^{t_1} (t_1 - s)^{\alpha_i + \beta_i - 1} H(s, \widehat{X}(s)) ds \\
&+ \left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t, \widehat{X}(t))\right)\Big|_{t=T_i} \left(\frac{t_2 - T_{i-1}}{T_i - T_{i-1}}\right)^{\beta_i} \\
&- \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{t_2} (t_2 - s)^{\beta_i - 1} \widehat{X}(s) ds \\
&- \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^{t_2} (t_2 - s)^{\alpha_i + \beta_i - 1} H(s, \widehat{X}(s)) ds \Bigg| \\
&\leq \left| \left(\lambda I^{\beta_i}_{T^+_{i-1}} \widehat{X}(t) + I^{\alpha_i+\beta_i}_{T^+_{i-1}} H(t, \widehat{X}(t))\right)\Big|_{t=T_i} \left[\left(\frac{t_1 - T_{i-1}}{T_i - T_{i-1}}\right)^{\beta_i} - \left(\frac{t_2 - T_{i-1}}{T_i - T_{i-1}}\right)^{\beta_i} \right] \right| \\
&+ \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{t_1} \left[(t_1 - s)^{\beta_i - 1} - (t_2 - s)^{\beta_i - 1}\right] |\widehat{X}(s)| ds \\
&+ \frac{\lambda}{\Gamma(\beta_i)} \int_{t_1}^{t_2} (t_2 - s)^{\beta_i - 1} |\widehat{X}(s)| ds \\
&+ \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{T_{i-1}}^{t_1} \left[(t_1 - s)^{\alpha_i + \beta_i - 1} - (t_2 - s)^{\alpha_i + \beta_i - 1}\right] |H(s, \widehat{X}(s))| ds \\
&+ \frac{1}{\Gamma(\alpha_i + \beta_i)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha_i + \beta_i - 1} |H(s, \widehat{X}(s))| ds.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Hence, the mapping $\mathcal{Z}$ is equicontinuous. Therefore, in view of the Ascoli-Arzelà Theorem, the mapping $\mathcal{Z}$ is relatively compact on B_{R_i}.

It follows from Theorem 1.3.2 that the auxiliary boundary value problem (4.4) has at least one solution in B_{R_i} for each $i \in \{1, 2, \ldots, n\}$. As a result, the boundary value problem (FLBVP) has a least one solution in $C([0,T], \mathbb{R})$, given by

$$X(t) = \begin{cases} X_1(t), & t \in [0, T_1], \\ X_2(t), & t \in (T_1, T_2], \\ \vdots & \\ X_n(t), & t \in (T_{n-1}, T], \end{cases}$$

such that

$$X_i(t) = \widehat{X}(t)\Big|_{t \in [T_{i-1}, T_i]}.$$

□

In order to prove the uniqueness of solution, we need to introduce an additional hypothesis:

(A3) There exists a positive constant L_4 such that $|H(t,x) - H(t,y)| \leq L_4 |x - y|$ for each $t \in [0,T]$ and $x, y \in \mathbb{R}$.

Theorem 4.2.2. *Assume that condition **(A3)** hold. Then* (4.4) *has a unique solution on* $[T_{i-1}, T_i]$ *for each* $i \in \{1, 2, \ldots, n\}$*, provided that*

$$\left(\frac{2\lambda}{\Gamma(\beta_i + 1)} (T_i - T_{i-1})^{\beta_i} + \frac{2L_4}{\Gamma(\alpha_i + \beta_i + 1)} (T_i - T_{i-1})^{\alpha_i + \beta_i} \right) < 1. \tag{4.6}$$

Proof. As previously in **Step 1** of the proof of Theorem 4.2.1, the mapping $\mathcal{Z} : B_{R_i} \to B_{R_i}$ is uniformly bounded. It remains to show that $\mathcal{Z}$ is a contraction.

Let $i \in \{1, 2, \ldots, n\}$ and let X_i, $X_i^* \in B_{R_i}$. Then,

$$\begin{aligned} &|(\mathcal{Z}X_i)(t) - (\mathcal{Z}X_i^*)(t)| \\ &\quad = \Bigg| - \left(\lambda I_{T_{i-1}^+}^{\beta_i} X_i(t) + I_{T_{i-1}^+}^{\alpha_i + \beta_i} H(t, X_i(t)) \right) \Big|_{t = T_i} \left(\frac{t - T_{i-1}}{T_i - T_{i-1}} \right)^{\beta_i} \\ &\qquad + \frac{\lambda}{\Gamma(\beta_i)} \int_{T_{i-1}}^{t} (t - s)^{\beta_i - 1} X_i(s) ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,X_i(s))ds\\
&+\left(\lambda I_{T_{i-1}^+}^{\beta_i}X_i^*(t)+I_{T_{i-1}^+}^{\alpha_i+\beta_i}H(t,X_i^*(t))\right)\Big|_{t=T_i}\left(\frac{t-T_{i-1}}{T_i-T_{i-1}}\right)^{\beta_i}\\
&-\frac{\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\beta_i-1}X_i^*(s)ds\\
&-\frac{1}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i+\beta_i-1}H(s,X_i^*(s))ds\Big|\\
\leq\;&\frac{2\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}|X_i(s)-X_i^*(s)|ds\\
&+\frac{2}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}|H(s,X_i(s))-H(s,X_i^*(s))|ds\\
\leq\;&\frac{2\lambda}{\Gamma(\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\beta_i-1}|X_i(s)-X_i^*(s)|ds\\
&+\frac{2L_4}{\Gamma(\alpha_i+\beta_i)}\int_{T_{i-1}}^{T_i}(T_i-s)^{\alpha_i+\beta_i-1}|X_i(s)-X_i^*(s)|ds\\
\leq\;&\left(\frac{2\lambda}{\Gamma(\beta_i+1)}(T_i-T_{i-1})^{\beta_i}+\frac{2L_4}{\Gamma(\alpha_i+\beta_i+1)}(T_i-T_{i-1})^{\alpha_i+\beta_i}\right)||X_i-X_i^*||_{E_i}.
\end{aligned}
$$

In view of (4.6), $\mathcal{Z}$ is a contraction for each $i\in\{1,2,\dots,n\}$. As a consequence of Banach's fixed point theorem, the operator $\mathcal{Z}$ has a unique fixed point, which corresponds to a unique solution of (4.4) on $(T_{i-1},T_i]$ for each $i\in\{1,2,\dots,n\}$. In view of Remark 3, we have uniqueness of solutions to (FLBVP). □

4.3 Example

In this section, we illustrate the applicability of the results obtained in this paper. Consider the fractional Langevin boundary value problem

$$
\begin{cases}
{}^C\mathcal{D}_{0^+}^{\alpha(t)}\left({}^C\mathcal{D}_{0^+}^{\beta(t)}-\frac{1}{8}\right)X(t)=\sin\left(\frac{1}{8}X(t)\right)+\frac{1}{(t+2)^2}, \quad 0\leq t\leq 4,\\
X(0)=0,\; X(4)=0.
\end{cases}
\tag{4.7}
$$

Here, $\lambda = \frac{1}{8}$, $H(t,x) = \sin\left(\frac{x}{8}\right) + \frac{1}{(t+2)^2}$, $T_1 = 2$, and $T_2 = 4$ so that our partition of $[0,4]$ becomes $\{[0,2],(2,4]\}$. We take

$$\alpha(t) = \begin{cases} \frac{2}{10}, & t \in [0,2], \\ \\ \frac{7}{10}, & t \in (2,4], \end{cases} \qquad \beta(t) = \begin{cases} \frac{3}{10}, & t \in [0,2], \\ \\ \frac{5}{10}, & t \in (2,4]. \end{cases}$$

Since $|H(t,x)| \leq \frac{1}{8}|x| + \frac{1}{4}$, in view of **(A1)**, we see that $L_2 = \frac{1}{8}$ and $L_3 = \frac{1}{4}$. Consider the auxiliary boundary value problems

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\frac{2}{10}}\left({}^{C}\mathcal{D}_{0^+}^{\frac{3}{10}} - \frac{1}{8}\right)X(t) = \sin\left(\frac{1}{8}X(t)\right) + \frac{1}{(t+2)^2}, & 0 \leq t \leq 2, \\ X(0) = 0,\ X(2) = 0, \end{cases}$$

and

$$\begin{cases} {}^{C}\mathcal{D}_{2^+}^{\frac{7}{10}}\left({}^{C}\mathcal{D}_{2^+}^{\frac{5}{10}} - \frac{1}{8}\right)X(t) = \sin\left(\frac{1}{8}X(t)\right) + \frac{1}{(t+2)^2}, & 2 < t \leq 4, \\ X(2) = 0,\ X(4) = 0. \end{cases}$$

Now, for $i \in \{1,2\}$, we have

$$\begin{cases} \frac{\lambda}{\Gamma(\beta_i+1)}\left(T_i - T_{i-1}\right)^{\beta_i} + \frac{L_2}{\Gamma(\alpha_i+\beta_i+1)}\left(T_i - T_{i-1}\right)^{\alpha_i+\beta_i} \approx 0,37093 < \frac{1}{2}, \\ \\ \frac{\lambda}{\Gamma(\beta_i+1)}\left(T_i - T_{i-1}\right)^{\beta_i} + \frac{L_2}{\Gamma(\alpha_i+\beta_i+1)}\left(T_i - T_{i-1}\right)^{\alpha_i+\beta_i} \approx 0,47880 < \frac{1}{2}. \end{cases}$$

So **(A2)** is satisfied. Therefore, by Theorem 4.2.1, the problem (4.7) has at least one solution given by

$$X(t) = \begin{cases} X_1(t), & t \in [0,2], \\ X_2(t), & t \in [2,4]. \end{cases}$$

To illustrate Theorem 4.2.2, in the above problem we take $H(t,x) = \frac{|x|}{(10+t^2)(1+|x|)}$ for all $t \in [0,4]$. Clearly, $|H(t,x) - H(t,y)| \leq \frac{1}{10}|x-y|$. Thus, **(A3)** is satisfied with $L_4 = \frac{1}{10}$.

Direct computations give

$$\begin{cases} \frac{2\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{2L_4}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \approx 0,50250 < 1, \\ \\ \frac{2\lambda}{\Gamma(\beta_i+1)}(T_i - T_{i-1})^{\beta_i} + \frac{2L_4}{\Gamma(\alpha_i+\beta_i+1)}(T_i - T_{i-1})^{\alpha_i+\beta_i} \approx 0,60743 < 1. \end{cases}$$

For $i \in \{1, 2\}$, which proves the theorem.

4.4 Conclusion

Studying the variable order fractional Langevin boundary value problem involves exploring a complex mathematical model that combines fractional calculus and stochastic processes to help understand systems with memory-dependent and stochastic characteristics. This problem addresses phenomena with memory effects and random fluctuations, and its solutions lead to advancements across multiple disciplines like physics, engineering, finance, and biology.

CHAPTER 5

A MATHEMATICAL INSIGHT OF FRACTIONAL LOGISTIC EQUATION OF VARIABLE ORDER ON FINITE INTERVALS

5.1 Introduction and motivation

In [52], B. J. West studied a more generalized version of the logistic equation by incorporating memory through the use of fractional derivatives in continuous time

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0^+}X(t) = \kappa^{\alpha}X(t)(1-X(t)), \\ X(0) = X_0, \end{cases} \tag{5.1}$$

where $0 < \alpha < 1$, $X_0 \in \mathbb{R}$, and ${}^{C}D^{\alpha}_{0^+}$ illustrate the Caputo fractional derivative operator of order α.

The author provided an exact solution to this extension of equation, which has been mentioned as West function, given by

$$X(t) = \sum_{n=0}^{\infty} \left(\frac{X_0 - 1}{X_0}\right)^{n} E_{\alpha}(-n\kappa^{\alpha}t^{\alpha}), \tag{5.2}$$

where E_α denotes the so–called one parameter Mittag–Leffer function, denoted by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0,\ z \in \mathbb{C}, \tag{5.3}$$

which was first proposed by G. M. Mittag-Leffler and may be thought of as a generalization of the exponential function.

Shortly after, I. Area et al [11] in a short note, showed that the real function (5.2) proposed by B. J. West [52] is not an exact solution for the fractional logistic equation.

In summary, finding exact solutions to the fractional logistic equation explicitly can be challenging. For this purpose, researchers typically resort to numerical methods to explore the system's behavior and dynamics, such as finite difference methods or spectral methods. These methods can provide valuable insights into the behavior of the system described by the fractional logistic equation (for more details see [5, 21, 22] and the refernces therein).

Recently, in [20] K. Devendra et al, analyzed the logistic equation with the novel fractional derivative given by Caputo Fabrizio

$$\begin{cases} {}^{CF}\mathcal{D}^{\alpha}_{0;t}X(t) = \kappa X(t)(1 - X(t)), \\ X(0) = X_0, \end{cases} \tag{5.4}$$

Motivated by the works aforementioned, we study the dynamical properties of the following fractional logistic equation involving the variable order Caputo fractional derivative

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha(t)}_{0^+}X(t) = \kappa X(t)(1 - X(t)), \quad t \in [0, T], \\ X(0) = X_0. \end{cases} \tag{\textit{VOFLE}}$$

Where $0 < \alpha(t) < 1$, $X_0 \in \mathbb{R}^+$, $\kappa > 0$, and ${}^{C}D^{\alpha(t)}_{0^+}$ illustrate the Caputo fractional derivative operator of variable order $\alpha(t)$ for the function $X(t)$.

5.2 Existence of solutions

Based on the previous discussion, in this section we present our main results.

Let $\mathcal{P} = \{[0, T_1],\ (T_1, T_2],\ (T_2, T_3], \ldots,\ (T_{n-1}, T]\}$ be a partition of the finite interval $[0, T]$, and let $\alpha(t) : [0, T] \longrightarrow (0, 1]$ be a piecewise constant function with respect to P given by

$$\alpha(t) = \sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t) = \begin{cases} \alpha_1 & t \in [0, T_1] \\ \alpha_2 & t \in (T_1, T_2] \\ \vdots & \\ \alpha_n & t \in (T_{n-1}, T], \end{cases}$$

where $0 < \alpha_i < 1$, $i \in \{1, 2, \ldots, n\}$ are constants, and $\mathbb{I}_i$ is the characteristic function for the interval $[T_{i-1}, T_i]$, i,e,.

$$\mathbb{I}_i(t) = \begin{cases} 1, & t \in [T_{i-1}, T_i], \\ 0, & \text{elsewhere.} \end{cases}$$

To reach our primary conclusions, we first do basic analysis on the equation of (*VOFLE*)

Since

$$\alpha(t) = \sum_{i=1}^{n} \alpha_i \mathbb{I}_i(t).$$

Hence, we get

$$^{C}\mathcal{D}_{0^+}^{\alpha(t)} X(t) = \int_0^t \frac{(t-s)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} X'(s)\, ds = \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} X'(s)\, ds. \tag{5.5}$$

So, The equation of the problem (*VOFLE*) can be written as the following

$$^{C}\mathcal{D}_{0^+}^{\alpha(t)} X(t) = \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} X'(s)\, ds = \kappa X(t)(1 - X(t)), \quad 0 \le t \le T < +\infty.$$

Therefore, in the interval $[0, T_1]$ it can be written as

$$^{C}\mathcal{D}_{0^+}^{\alpha_1} X(t) = \int_0^t \frac{(t-s)^{-\alpha_1}}{\Gamma(1-\alpha_1)} X'(s)\, ds = \kappa X(t)(1 - X(t)), \quad 0 < t \le T_1. \tag{5.6}$$

Again, in the interval $(T_1, T_2]$ it can be written as

$$^{C}\mathcal{D}_{0^+}^{\alpha_2} X(t) = \int_0^t \frac{(t-s)^{-\psi_2}}{\Gamma(1-\alpha_2)} X'(s)\, ds = \kappa X(t)(1 - X(t)), \quad T_1 < t \le T_2. \tag{5.7}$$

In the same way, in the interval $(T_{i-1}, T_i]$ it can be written as

$$^{C}\mathcal{D}_{0^+}^{\alpha_i} X(t) = \int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} X'(s)\, ds = \kappa X(t)(1 - X(t)), \quad T_{i-1} < t \le T_i. \tag{5.8}$$

We denote by $E_i = (C([0, T_i], \mathbb{R}), ||\cdot||_{E_i})$ the class of functions that form a Banach space with the equivalent norm

$$||X||_{E_i} = \sup_{t \in [0, T_i]} e^{-Nt} |X(t)|, \quad N > 0, \quad i \in \{1, 2, \ldots, n\}.$$

Thus, we may consider the following auxiliary initial value problems of constant order defined on any interval $[T_{i-1}, T_i],\ i \in \{1, 2, \ldots, n\}$ as following

$$\begin{cases} ^{C}\mathcal{D}_{0^+}^{\alpha_i} X_i(t) = \kappa X_i(t)(1 - X_i(t)), & T_{i-1} < t \le T_i, \\ X_i(0) = X_0. \end{cases} \tag{5.9}$$

Definition 5.2.1. We will define X_i for all $i \in \{1, 2, \ldots, n\}$ to be a solution of the initial value problem (5.9) if

1) $(t, X_i) \in D$, such that $D = [0, T_i] \times B_{R_i}$, where $B_{R_i} = \{X_i \in E_i : |X_i| \leq R_i\}$,

2) $X_i(t)$ satises the integral equation for each $i \in \{1, 2, \ldots, n\}$.

Definition 5.2.2. We say that the problem (*VOFLE*) has a solution X, if there exist functions X_i, such that $X_1 \in C([0, T_1], \mathbb{R})$ satisfying equation (5.6), and $X_1(0) = X_0$; $X_2 \in C([0, T_2], \mathbb{R})$ satisfying equation (5.7), and $X_2(0) = X_0$; $X_i \in C([0, T_i], \mathbb{R})$ satisfying equation (5.8), and $X_i(0) = X_0$, for all $i \in \{3, \ldots, n\}$.

Remark 4. We say problem (*VOFLE*) has one unique solution, if the functions X_i are uniques.

Theorem 5.2.1. *The auxiliary initial value problem* (5.9) *has a unique solution in* $C([0, T_i], \mathbb{R})$, *for all* $i \in \{1, 2, \ldots, n\}$.

Proof. For all $i \in \{1, 2, \ldots, n\}$, and from the properties of fractional calculus, the fractional order differential equation in (5.9) can be written as

$$I^{1-\alpha_i} \frac{d}{dt} X_i(t) = \kappa X_i(t)(1 - X(t)),$$

using Lemma 1.2.1, we integrate the above equation α_i-times. Therefore we obtain

$$X_i(t) = X_0 + I_{0^+}^{\alpha_i} \left(\kappa X_i (1 - X_i)\right)(t). \tag{5.10}$$

Define the operator $\mathcal{Z} : E_i \longrightarrow E_i$ by

$$(\mathcal{Z} X_i)(t) = X_0 + I_{0^+}^{\alpha_i} \left(\kappa X_i (1 - X_i)\right)(t).$$

Then

$$\begin{aligned}
e^{-Nt} |\mathcal{Z}(X_i) - \mathcal{Z}(X_i^*)| &= \kappa \left| e^{-Nt} I_{0^+}^{\alpha_i} \left[(X_i(t) - X_i^*(t)) - (X_i^2(t) - \tilde{X}_l^2(t)) \right] \right| \\
&\leq \kappa \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-Nt} |X_i(t) - X_i^*(t)| (1 + |X_i(t)| + |X_i^*(t)|) \, ds \\
&\leq \kappa \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-N(t-s)} e^{-Ns} |X_i(t) - X_i^*(t)| (1 + |X_i(t)| + |X_i^*(t)|) \, ds \\
&\leq \kappa (1 + 2R_i) ||X_i - X_i^*||_{E_i} \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-N(t-s)} \, ds
\end{aligned}$$

$$\leq \frac{\kappa(1+2R_i)}{N^{\alpha_i}}||X_i - X_i^*||_{E_i} \int_0^{Nt} \frac{\tau^{\alpha_i-1}e^{-\tau}}{\Gamma(\alpha_i)}\, d\tau.$$

This implies that

$$||\mathcal{Z}(X_i) - \mathcal{Z}(X_i^*)||_{E_i} \leq \frac{\kappa(1+2R_i)}{N^{\alpha_i}}||X_i - X_i^*||_{E_i},$$

and it can be proved by vertue of the Banach contraction principle, that if we choose N such that $N^{\alpha_i} > \kappa(1+2R_i)$, we obtain that the operator $\mathcal{Z}$ has a unique xed point for all $i \in \{1, 2, \ldots, n\}$.

In view of Remark 4, we have uniqueness of solution to (*VOFLE*).

□

5.3 Uniform stability

Theorem 5.3.1. *For all* $i \in \{1, 2, \ldots, n\}$*, The solutions* X_i *of the initial value problem* (5.9) *are uniformly stables in the sense that*

$$|X_0 - X_0^*| \leq \delta \implies ||X_i - X_i^*||_{E_i} \leq \epsilon,$$

where X_i^* *is the solution of the initial value problem* (5.9) *with the initial condition*

$$X_i^*(0) = X_0^*.$$

Remark 5. We say problem (*VOFLE*) is uniformaly stable, if the functions X_i are uniformaly stables.

Proof. Direct computation gives

$$||X_i - X_i^*||_{E_i} \leq |X_0 - \tilde{X}_0| + \frac{\kappa(1+2R_i)}{N^{\alpha_i}}||X_i - X_i^*||_{E_i},$$

which umplies that

$$||X_i - X_i^*||_{E_i} \leq \left(1 - \frac{\kappa(1+2R_i)}{N^{\alpha_i}}\right)^{-1} |X_0 -^* X_0| \leq \epsilon,$$

where $\epsilon = \left(1 - \frac{\kappa(1+2R_i)}{N^{\alpha_i}}\right)^{-1} \delta.$

In view of Remark 5, we have uniform stability of solution to (*VOFLE*). □

5.4 Numerical methods and results

We have following fractional logistic equation involving the variable order Caputo fractional derivative

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha(t)}X(t) = \kappa X(t)(1-X(t)), t \in [0,T], \\ X(0) = X_0. \end{cases}$$

Where $0 < \alpha(t) < 1$, $X_0 \in \mathbb{R}^+$, $\kappa > 0$. In this problem, we will choose the finite difference method with different space steps h.

The Finite Difference Method (FDM) can be extended to solve fractional differential equations (FDEs). Unlike standard FDM, which approximates local integer-order derivatives using a few neighboring points, fractional derivatives are inherently nonlocal. This results in discrete approximations that involve weighted sums over a large number of previous time steps.

Step 1: Discretization of the Time-Fractional Derivative: The L1 Scheme Let $h > 0$ be the time step size. Define the grid points:

$$t_i = ih, \quad i \in \{0, 1, \dots, N\}, \quad N = \frac{T}{h}.$$

Denote $X_i \approx X(t_i)$, and $\alpha_i = \alpha(t_i)$. Now, we use the L1 scheme for the variable-order Caputo derivative

$${}^{C}\mathcal{D}_{0^+}^{\alpha_i}X(t_i) \approx \frac{h^{-\alpha_i}}{\Gamma(1-\alpha_i)} \sum_{k=0}^{i-1} b_{i-k-1}^{(\alpha_i)}(X_{k+1} - X_k), \tag{5.11}$$

where the coefficients are given by

$$b_j^{(\alpha_i)} = (j+1)^{1-\alpha_i} - j^{1-\alpha_i}, \quad j \in \{0, 1, \dots, i-1\}.$$

Substituting the approximation into the differential equation at $t = t_n$

$$\frac{h^{-\alpha_i}}{\Gamma(1-\alpha_i)} \sum_{k=0}^{i-1} b_{i-k-1}^{(\alpha_i)}(X_{k+1} - X_k) = \kappa X_i(1 - X_i). \tag{5.12}$$

Therefore

$$\sum_{k=0}^{i-1} b_{i-k-1}^{(\alpha_i)}(X_{k+1} - X_k) = \underbrace{\sum_{k=0}^{i-2} b_{i-k-1}^{(\alpha_i)}(X_{k+1} - X_k)}_{S_i} + (X_i - X_{i-1}),$$

where S_i involves only known values $X_0, X_1, \ldots, X_{i-1}$. Thus

$$\frac{h^{-\alpha_i}}{\Gamma(1-\alpha_i)}[S_i + X_i - X_{i-1}] = \kappa X_i(1 - X_i).$$

Multiply both sides by $\dfrac{\Gamma(1-\alpha_i)}{h^{-\alpha_i}}$, we get

$$S_i + X_i - X_{i-1} = \kappa\Gamma(1-\alpha_i)h^{\alpha_i}X_i(1 - X_i).$$

Rearrange all terms to one side, we obtain a quadratic equation in X_i:

$$C_i X_i^2 + (1 - C_i)X_i + (S_i - X_{i-1}) = 0, \tag{5.13}$$

where $C_i = \kappa\Gamma(1-\alpha_i)h^{\alpha_i}$.

Step 2: Since the exact solution to this problem where $\alpha(t) = 1$ is

$$X_1 = \frac{X_0}{X_0 + (1 - X_0)\exp(-\kappa t)}.$$

We applied the finite difference method in this system with size $h = 0.001$ and $T = 10$ for different $\alpha(t)$ and X_0 which they have presented in the following images:

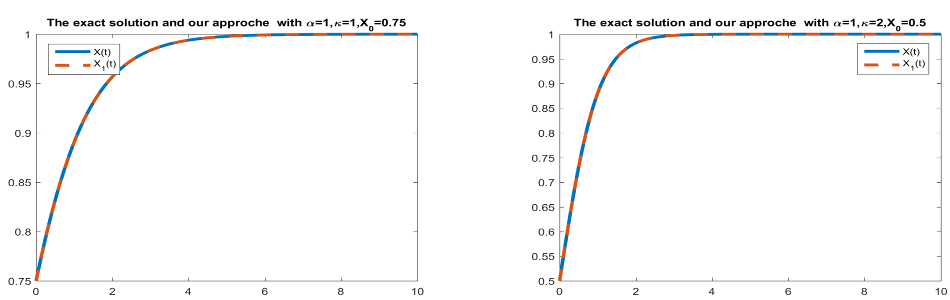

Figure 5.1: The exacte solution compared to our approach.

We observe that the solution with this method is the same as the exacte solution.

Step 3: Now, we calculate the solution of system (*VOFLE*) with the variable order $\alpha(t)$ in two case (increasing and decreasing cases) in the $]0, 1]$ for different X_0 and size $h = 0.001$

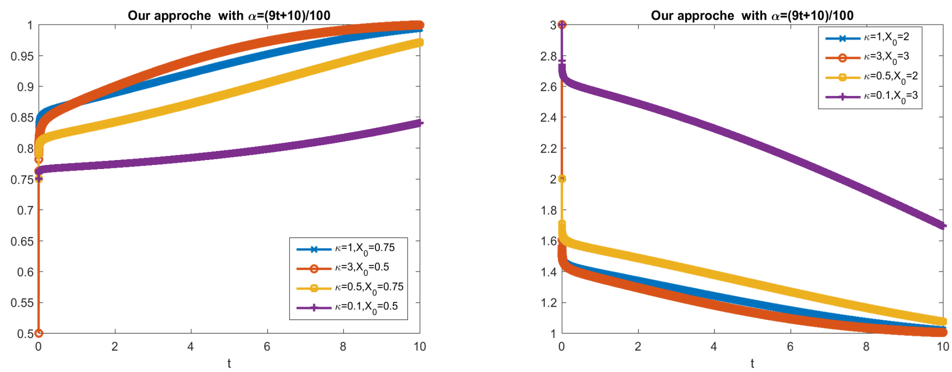

Figure 5.2: The solution X for different κ, X_0 and $\alpha(t) = \frac{9t+10}{100}$.

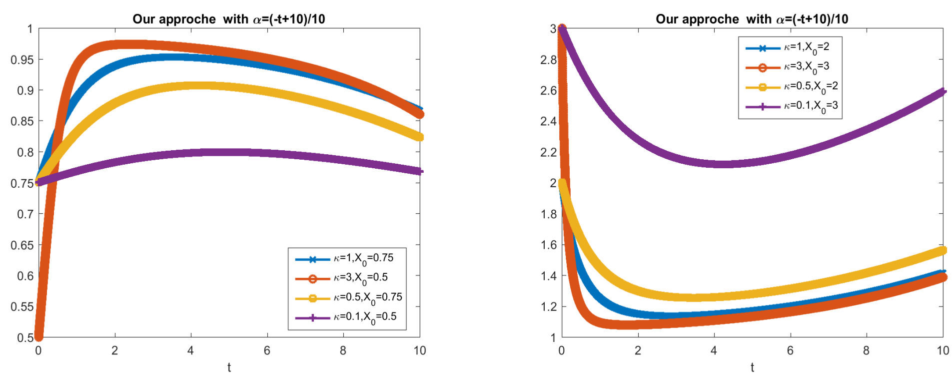

Figure 5.3: The solution X for different κ, X_0 and $\alpha(t) = \frac{-t+10}{10}$.

5.5 Conclusion

Studying the logistic equation using variable-order fractional calculus provides a flexible and generalized framework to model dynamic systems with varying growth rates and memory effects. This approach enhances the classical logistic model by incorporating non-local, history-dependent behaviors, and adapting the fractional order over time or space to better capture complex real-world phenomena. The results reveal that variable-order fractional calculus improves model accuracy and predictive capabilities, demonstrating its potential for advancing nonlinear system analysis.

GENERAL CONCLUSION AND PERSPECTIVES

In this doctoral dissertation, we have established the existence and uniqueness of solutions for some classes of nonlinear boundary value problem involving the fractional differential equations of variable order. By using the standard fixed point theorems such as Banach contraction principle, Schauders fixed point, Krasnoselskiis fixed point theorem, we established the existence and uniqueness of solutions for divers boundary value problem, and the stability in the sense of Ulam-Hyers-Rassias, after that we obtained a generalized version of Lyapunov inequality by using the variable order fractional calculus and Green functions, then we turned our focus to the existence and uniqueness of solutions for an initial value problem of the logistic equation in the variable order form. In summary, the utilization of the variable order fractional calculus yields a more comprehensive and accurate modeling framework compared to the previous approaches. It provides enhanced capabilities for capturing complex dynamics, accounting for memory effects, and advancing our understanding of systems with intricate behaviors. Therefore, all results in this work show a great potential to be applied in various applications of multidisciplinary sciences, and we encourage fellow researchers to build upon our work, exploring new applications, refining methodologies, and pushing the boundaries of knowledge through collective efforts, thus unlocking further insights into the intricate dynamics of complex systems.

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الملخص : في هذه أطروحة الدكتوراه، نقوم بالتحقيق في وجود، وحدانية، واستقرار الحلول لفئات مختلفة من مسائل القيمة الأولية و القيم الحدية غير الخطية (بانتوغراف، لانغفين، لوجستي) التي تنطوي على عوامل الاشتقاق الكسرية لكابوتو و هادامارد ذات الترتيب المتغير. تم إثبات جميع الاستنتاجات المستخلصة في هذا البحث باستخدام حساب التفاضل والتكامل الكسري المتغير ونظرية النقطة الثابتة باستخدام خصائص الدوال الثابتة الجزئية، والتي تعتبر ضرورية لتحويل المسألة الكسرية ذات الترتيب المتغير المدروسة الى معادلة تكاملية مكافئة. علاوة على ذلك، فإننا نحقق في الاستقرار من حيث معيار استقرار أولام هايرز راسياس، وتحت افتراضات أخرى حول المصطلح غير الخطي، نحصل على متراجحات ليابونوف المعممة.

Abstract : In this doctoral dissertation, we investigate the existence, uniqueness, and stability of solutions for various classes of nonlinear initial and boundary value problems (Pantograph, Langevin, Logistic) involving the variable order fractional operators. All conclusions drawn in the present research have been proven utilizing the variable order fractional calculus and fixed point theorem using the piecewise constant functions properties, which are crucial to convert the considered problems into an equivalent standard constant order counterparts. Furthermore, we investigate the stability in terms of Ulam-Hyers-Rassias stability criterion, and under further assumptions on the nonlinear term, we obtain the generalized Lyapunov inequalities.

Résumé : Dans cette thèse de doctorat, nous étudions l'existence, l'unicité et la stabilité des solutions pour diverses classes de problèmes à valeur initiale et aux limites non linéaires (Pantographe, Langevin, Logistique) qui utilisent les opérateurs fractionnaires d'ordre variable. Toutes les conclusions validées dans la présente recherche ont été prouvées en utilisant le calcul fractionnaire d'ordre variable et la théorie du point fixe en se basant sur les propriétés des fonctions constantes par morceaux, qui jouent un rôle essentiel dans la conversion des problèmes fractionnaires considérés en équivalents standard d'ordre constant. De plus, nous étudions la stabilité en termes du critère d'Ulam-Hyers-Rassias, et sous d'autres hypothèses sur le terme non linéaire, nous obtenons les inégalités de Lyapunov généralisées.