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THÈSE

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EN VUE DE L'OBTENTION DU DIPLOME DE DOCTORAT LMD

Thème

Quelques inégalités intégrales relatives aux fonctions quasi-monotones dans les espaces de Lebesgue à exposants variables

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DEDICATE

This work is dedicated
To my father .
To my mother.
To my brothers and sisters .
To all my family.
To my friends and colleagues in Tiaret university and especially to those in department of mathematics.

Abstract

In this thesis, we consider some integral inequalities for classical Lebesgue spaces L_p with $0 < p < 1$ and weighted variable exponent Lebesgue spaces $L_{p(x),w}$ with $0 < p(x) < 1$. First we obtain some new integral inequalities with $0 < p < 1$ under weaker condition than monotonicity via Hardy–Steklov type operators. Second, some integral inequalities were established for the same operators acting from one weighted variable exponent Lebesgue spaces to another weighted exponent Lebesgue spaces with $0 < p(x) < 1$ for nonnegative quasi-monotone functions on $(0, \infty)$. Consequently, some results of A. Senouci et al and R.A.Bandaliev are deduced as particular cases. Finally, we establish some new estimates for the Hardy-Steklov operator for the same spaces and the same functions.
Keywords: Integral inequalities, Hardy-type inequality, Hardy–Steklov operator, Hardy-Steklov type operators, quasi-monotone functions, weighted variable exponent Lebesgue spaces.

mathematics subject classification (2010)
26D10, 26D15, 47G10, 46E30.

ملخــص

في هذه الأطروحة، ندرس بعض المتباينات التكاملية لفضاءات ليبيج الكلاسيكية L_p حيث $0 < p < 1$ و فضاءات ليبيج ذات الأس المتغير $L_{p(x),w}$ مع $0 < p(x) < 1$. أولاً نحصل على بعض المتباينات التكاملية الجديدة من أجل $0 < p < 1$ تحت شروط أضعف من الرتابة لمؤثر Hardy–Steklov . ثانيًا، تم إنشاء بعض المتباينات التكاملية لنفس المؤثرات التي تعمل من أس متغير واحد لفضاءات ليبيج إلى فضاءات ليبيج ذي الأس المتغير $L_{p(x),w}$ مع $0 < p(x) < 1$ للدوال شبه الرتيبة غير السلبية على المجال $(0, \infty)$. وبالتالي، بعض نتائج الاستاذ "A.Senouci " و الاستاذ " R.A.Bandaliev " تصبح على شكل حالات خاصة. أخيرا، قمنا بوضع بعض التقديرات الجديدة لمؤثر " Hardy–Steklov " لنفس الفضاءات ونفس الدوال.

الكلمات المفتاحية: المتباينات التكاملية، متباينة هاردي، مؤثرات هاردي-ستيكلوف، الدوال شبه الرتيبة، فضاءات لوبيغ ذات الاس المتغير.
التصنيفات 26D10, 26D15, 47G10, 26A48 .

Resumé

Dans cette thèse, nous considérons quelques inégalités intégrales pour des espaces de Lebesgue classiques L_p avec $0 < p < 1$ et des espaces de Lebesgue $L_{p(x),w}$ pondérés avec $0 < p(x) < 1$. Nous obtenons d'abord de nouvelles inégalités intégrales avec $0 < p < 1$ sous des conditions plus faibles que la monotonie par l'intermédiaire d'opérateurs de type Hardy-Steklov. Deuxièmement, des inégalités intégrales ont été établies pour les mêmes opérateurs agissant des espaces de Lebesgue à un autre aussi pondéré avec $0 < p(x) < 1$, pour les fonctions quasi-monotones non négatives sur $(0, \infty)$. Par conséquent, certains résultats de A. Senouci et al et de R.A.Bandaliev sont déduits comme cas particuliers. A la fin de ce travail, nous établissons de nouvelles estimations pour l'opérateur de Hardy-Steklov pour les mêmes espaces et les mêmes fonctions.
Mots clés: Inégalités integrales, inégalités de type Hardy, opérateurs de Hardy–Steklov, opérateurs de type Hardy-Steklov, fonctions quasi-monotone, espaces de Lebesgue à exposant variable avec poids.
mathematics subject classification (2010)
26D10, 26D15, 47G10, 26A48.

List of abbreviations and symbols

We use the following notations:

Ω	Subset of $\mathbb{R}^n$.
$\|\Omega\|$	Measure of Ω .
L_p	Lebesgue spaces .
$L_{p(x)}$	Variable Lebesgue spaces.
$L_{p(x),w}$	Weighted variable exponent Lebesgue spaces.
$\|\| \cdot \|\|_E$	Norm of space E.
$C^1(0,\infty)$	The space of functions with continuous derivative.
$C_0^\infty(0,\infty)$	The space of test functions.
$\left(L_{p(x),w}\right)^*$	The dual space of $L_{p(x),w}$.
$\\|.\\|_{L_{p(x)},w}$	The norm in $L_{p(x),w}$.
a.e	Almost every where.

Publications and communications

International publications

1. **Abdelaziz Gherdaoui**, Abdelkader Senouci, Bouharket Benaissa, Some estimates for Hardy-Steklov type operators, Memoirs on Differential Equations and Mathematical Physics, $93\ \ (2024), 99-107$.
2. **Abdelaziz Gherdaoui**, Abdelkader Senouci, Some integral inequalities for Hardy-Steklov operator for quasi-monotone functions with $0 < p(x) < 1$. Submitted.
3. **Abdelaziz Gherdaoui**, Abdelkader Senouci, On Hardy-Steklov type operators for quasi-monotone functions in weighted variable Lebesgue exponent spaces. Submitted.

International communication

1. Some estimates for Hardy-Steklov type operators, during the 6th international colloquium on methids and tools for dicision support [Methode et outils d'aide à la décision -MOAD'2024] held in Mouloud Mammeri university of Tizi ouzou, october 20-22, 2024.

National communication

1. On Hardy-Steklov operators where $0 < p < 1$, The first national colloquium mathematics : [trends and actual aplications (CNMT2A)]. University of Algiers Benyoussef Benkhada, february 22-23, 2025.

Contents

General Introduction

Inequalities are playing a very important role in different areas of mathematics and present a very active and interesting field of research. As example, we have the field of integration which is dominated by inequalities involving the integral operators. Let us cite some famous integral inequalities : the inequalities of Hölder, Minkowski and Hardy. All these inequalities are based on the classical Lebesgue spaces. In turn these spaces have important applications in different branches of mathematics for example Sobolev spaces (see [9]), integral transformations and others. The variable Lebesgue spaces $L_{p(x)}$, are a generalisation of the classical Lebesgue spaces where the constant exponent p is remplaced by a variable exponent function $p(x)$. The spaces $L_{p(x)}$ is a special case of the Musielak-Orlicz spaces (see [28] and [30]). In this work, we are interested in the integral inequalities relating to the integral operators of Hardy, Hardy-Steklov and Hardy-Steklov type acting in the quasi-normed spaces (classical and variable Lebesgue spaces with $0 < p < 1$ and $0 < p(x) < 1$ respectively). The thesis is structured into three chapters, a conclusion and a bibliography.

1. **Chapter one:** In this chapter, we introduce some definitions and related properties to the classical Lebesgue spaces and we recall the Hölder and Minkowski inequalities. In next part of this chapter we consider the classical weighted Hardy inequalities where the summability parameter is $p \geq 1$, and $0 < p < 1$. At the end of the chapter we expose the variable Lebesgue spaces $L_{p(x)}$, the classical inequalities in $L_{p(x)}$ (Hölder's inequalities, Minkowski's inequalities and Hardy's inequalities).

2. **Chapter two:** In the second chapter, we consider the classical weighted Hardy integral inequalities with parameter $0 < p < 1$. We extend the results of [35] to Hardy–Steklov type operators, the chapter includes a work already published under the title ”Some estimates for Hardy-Steklov type operators”(see [18]) and some integral inequalities were established for the same operators for quasi-monotone functions in weighted variable expo-

nent Lebesgue spaces $L_{p(x),w}$ with $0 < p(x) < 1$. This work is submitted under the title " On Hardy-Steklov-type operators for quasi-monotone functions in weighted variable exponent Lebesgue spaces " (see [16]).

3. **Chapter three:** In this chapter we obtain some integral inequalities for Hardy-Steklov operator in weighted variable exponent Lebesgue spaces for nonnegative quasi-monotone and monotone functions with $0 < p(x) < 1$. This work is submitted under the title "Some integral inequalities for Hardy-Steklov operator for quasi-monotone functions with $0 < p(x) < 1$" (see [17]).

   We end this thesis with a conclusion and perspectives and a fairly detailed bibliography.

Chapter 1

Preliminaries

In this chapter, we recall and state some definitions, Lemmas, Corollaries and Theorems that are useful in this thesis.

1.1 Classical Lebesgue spaces.

Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set and f be a Lebesgue measurable function on Ω.

1.1.1 Lemmas and theorems.

Lemma 1.1.1 (Fatou's Lemma [27]). *Let $f_1, f_2, f_3, ...$ be a sequence of non-negative, measurable functions on Ω, and a. e. exists the finite or infinite $\lim_{n\longrightarrow\infty} f_n(x) = f(x)$. Then $f(x) = \underline{\lim}_{n\to\infty} f_n(x)$ is measurable and*

$$\int_{\Omega} f(x)dx \leq \underline{\lim}_{n\to\infty} \int_{\Omega} f_n(x)dx, \tag{1.1}$$

in the sense that the finiteness of the right side implies that f is summable.

Proof. *See [27] and [8].*

Theorem 1.1.1 (Monotone convergence [27]). *Let $\forall n \in \mathbb{N}$, (f_n) be a sequence of nonnegative measurable functions on Ω, moreover $f_k(x) \leq f_{k+1}(x)$ a.e. on Ω. Then*

$$\lim_{n\longrightarrow\infty} \int_{\Omega} f_n(x)dx = \int_{\Omega} \lim_{n\longrightarrow\infty} f_n(x)dx. \tag{1.2}$$

Proof. *See [27] and [8].*

Theorem 1.1.2 (Dominated convergence [27]). *Let* $\forall n \in \mathbb{N}$, (f_n) *be a sequence of measurable functions on* Ω, *and the finite* $\lim_{n \longrightarrow \infty} f_n(x)dx$ *exists a.e on* Ω. *If there exists a summable, nonnegative function* $G(x)$ *on* Ω *such that*

$$\forall n \in \mathbb{N}, \quad |f_n(x)| \leq G(x), \tag{1.3}$$

then $\forall n \in \mathbb{N}$ *the functions* f_n *and* $\lim_{n \longrightarrow \infty} f_n(x) = f(x)$, *are summable on* Ω *and*

$$\lim_{n \longrightarrow \infty} \int_{\Omega} f_n(x)dx = \int_{\Omega} f(x)dx. \tag{1.4}$$

Proof. *See [27] and [8].*

Theorem 1.1.3 (Fubuni's theorem [27]). *Let* E *and* F *be measurable sets* $(E \subset \mathbb{R}^n)$ *and* $(F \subset \mathbb{R}^m)$ *and the function f(x, y) summable on* $E \times F$. *So for almost every* $x \in E$, $f(x, y)$ *is summable on* F, *for almost all* $y \in F$, $f(x, y)$ *is summable on* E *one of two integrals is finite, so the three integrals exist and are equal.*

$$\left(\int_{E \times F} f(x,y)dxdy \right) = \int_E \left(\int_F f(x,y)dy \right) dx = \int_F \left(\int_E f(x,y)dx \right) dy. \tag{1.5}$$

Proof. *See [27] and [8].*

Corollary 1.1.1. *If* $f(x, y)$ *is measurable function on* $E \times F$ *and one of two integrals is finite.*

$$\int_E \left(\int_F |f(x,y)|dy \right) dx, \ \int_F \left(\int_E |f(x,y)|dx \right) dy.$$

So the three integrals in (1.5) exist and are equal.

Remark 1.1.1. *If* f *is not summable over* $E \times F$, *then the iterated integrals may not exist or may exist and be different.*

In the following we recall the definition of Lebesgue spaces.

Definition 1.1.1. *Let* $0 < p < \infty$, Ω *be a measurable subset of* $\mathbb{R}^n$, $f : \Omega \to \mathbb{C}$. *We say that* $f \in L_p(\Omega)$ *if:*

1. f *is measurable on* Ω.

2. $\|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} < \infty$.

Example 1.1.1. *Let* $0 < p < \infty$, $\gamma \in \mathbb{R}$, $r > 0$

$$|x|^{\gamma} \in L_p(B_r) \text{ if and only if } \gamma > -\frac{n}{p}, n \in \mathbb{N}, \tag{1.6}$$

where B_r is the ball with center 0 and radius $r > 0$.
To prove (1.6), we apply the well-known formula:
If $r > 0$, $g(\rho)\rho^{n-1}$ is an integrable function on $(0, r)$, then

$$\int_{B_r} g(|x|)dx = \sigma_n \int_0^r g(\rho)\rho^{n-1}d\rho, \tag{1.7}$$

where $\sigma_n = nV_n$, is the surface of the unitary sphere .
By (1.7), we get

$$\begin{aligned}\||x|^\gamma\|^p_{L_p(B_r)} = &\int_{B_r} |x|^{\gamma p}dx \\ =&\sigma_n \int_0^r \rho^{\gamma p}\rho^{n-1}d\rho \\ =&\frac{\sigma_n}{\gamma p + n} r^{\gamma p+n},\end{aligned}$$

consequently $\||x|^\gamma\|_{L_p(B_r)} < \infty$ if $\gamma p + n > 0$, then $\gamma > -\frac{n}{p}$.

Definition 1.1.2. *Let $e \subset \Omega$ such that $|e| = 0$, then*

$$ess \sup_{x\in\Omega} f(x) = \inf_{e\in\Omega} \sup_{x\in\Omega/e} f(x), \tag{1.8}$$

$$ess \inf_{x\in\Omega} f(x) = \sup_{e\in\Omega} \inf_{x\in\Omega/e} f(x). \tag{1.9}$$

Definition 1.1.3. *We say that $f \in L_\infty(\Omega)$ if f is measurable and*

$$\|f\|_{L_\infty(\Omega)} = ess \sup_{x\in\Omega} |f(x)| < \infty. \tag{1.10}$$

Remark 1.1.2. 1. *By definition we put*

$$\|f\|_{L_\infty(\Omega)} = 0, \textit{ for } |\Omega| = 0.$$

2. *If $f \in L_\infty(\Omega)$, we have*

$$|f(x)| \le \|f\|_{L_\infty(\Omega)}, \textit{ for a.e. } x \in \Omega.$$

Remark 1.1.3. 1. *Let $0 < p < \infty$, since $|\alpha + \beta|^p \le max\left(1, 2^{p-1}\right)\left(|\alpha|^p + |\beta|^p\right)$, $\forall \alpha, \beta \in \mathbb{C}$, then the linear combination $(\alpha f + \beta g) \in L_p(\Omega)$, where $f, g \in L_p(\Omega)$. Consequently $L_p(\Omega)$ is a linear space.*

2. *If $p \ge 1$, $L_p(\Omega)$ is a normed space.*

3. *If $0 < p < 1$, $L_p(\Omega)$ is a quasi-normed space.* [1]

Theorem 1.1.4 (Riesz's theorem [27]). *Let f be a measurable function, then*

$$\lim_{p\to\infty} \|f\|_{L_p(\Omega)} = \|f\|_{L_\infty(\Omega)}. \tag{1.11}$$

Proof. *See [27] and [8].*

[1] $\left(\|x + y\|_{L_p(\Omega)} \le c\left(\|x\|_{L_p(\Omega)} + \|y\|_{L_p(\Omega)}\right)\right.$, where $\left. c \ge 1\right)$.

1.1.2 The classical inequalities.

Theorem 1.1.5 (Young's inequalities [27]). *For all $a, b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

1. *For* $1 \leq p \leq \infty:$
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.12}$$
2. *For* $0 < p < 1:$
$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.13}$$
3. *For* $p < 0:$
$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.14}$$

Proof. *See [27] and [8].*

Since $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$, then $\frac{1}{p/s} + \frac{1}{q/s} = 1$, and by applying inequality (1.12) with $a = x^s$ and $b = y^s$, we get the following Corollary.

Corollary 1.1.2. *Let $p, q, s \geq 1$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$, then*

$$\forall a, b \geq 0, \qquad \left(ab\right)^s \leq \frac{s}{p}a^p + \frac{s}{q}b^q. \tag{1.15}$$

Proof.
$$\left(xy\right)^s = ab \leq \frac{a^{p/s}}{p/s} + \frac{b^{q/s}}{q/s} = \frac{s}{p}x^p + \frac{s}{q}y^q.$$

Theorem 1.1.6 (Hölder's inequality [27]). *Let $0 < p \leq \infty$, $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, then*

1. *If* $1 \leq p \leq \infty$
$$\|fg\|_{L_1(\Omega)} \leq \|f\|_{L_p(\Omega)} \|g\|_{L_{p'}(\Omega)}. \tag{1.16}$$
2. *If* $0 < p < 1$ *or* $p < 0$ *and* $\forall x \in \Omega, g(x) \neq 0$
$$\|fg\|_{L_1(\Omega)} \geq \|f\|_{L_p(\Omega)} \|g\|_{L_{p'}(\Omega)}. \tag{1.17}$$

Proof. *See [27] and [8].*

Corollary 1.1.3. *[11] Let $s > 0$, $p \leq \infty$, $-\infty \leq q \leq +\infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$, then*

1. *If* $s \leq p$
$$\|fg\|_{L_s(\Omega)} \leq \|f\|_{L_p(\Omega)} \|g\|_{L_q(\Omega)}, \tag{1.18}$$
2. *If* $s > p$
$$\|fg\|_{L_s(\Omega)} \geq \|f\|_{L_p(\Omega)} \|g\|_{L_q(\Omega)}. \tag{1.19}$$

Proof. *For the proof of (1.18) and (1.19) we apply (1.16) and (1.17) respectively, with* $\frac{1}{p/s} + \frac{1}{q/s} = 1$.

Proposition 1.1.1. *Let* $p_i \in (1, \infty), i = 1, 2, ..., k$*, where*

$$\sum_{i=1}^{k} \frac{1}{p_i} = 1,$$

$f_i \in L_{p_i}(\Omega)$*, then*

$$\left\| \prod_{i=1}^{k} f_i \right\|_{L_1(\Omega)} \leq \prod_{i=1}^{k} ||f_i||_{L_{p_i}(\Omega)}. \tag{1.20}$$

Proof. *By induction.*

Theorem 1.1.7 (Minkowski's inequality[11]). *Let* $1 \leq p \leq \infty$ *and* $f, g \in L_p(\Omega)$*, then*

$$\|f + g\|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}. \tag{1.21}$$

Proof. *See [11].*

Corollary 1.1.4. *Let* $m \in \mathbb{N}$ *and* $f_k \in L_p(\Omega)$ *for all* $k \in \{1, 2, \dots, m\}$*, then*

$$\left\| \sum_{k=1}^{m} f_k \right\|_{L_p(\Omega)} \leq \sum_{k=1}^{m} \|f_k\|_{L_p(\Omega)}. \tag{1.22}$$

Proof. *By induction.*

Corollary 1.1.5 (Minkowski's inequality for infinite sums[11]). *Let* $f_k \in L_p(\Omega)$ *for all* $k \in \mathbb{N}$*, where*

$$\sum_{k=1}^{\infty} \|f_k\|_{L_p(\Omega)} < \infty,$$

then

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_{L_p(\Omega)} \leq \sum_{k=1}^{\infty} \|f_k\|_{L_p(\Omega)}. \tag{1.23}$$

Proof. *See [11].*

Theorem 1.1.8. *Let* $0 < p < 1$ *and* $f, g \in L_p(\Omega)$*, then*

$$\|f + g\|_{L_p(\Omega)} \leq 2^{\frac{1}{p}-1} \left(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}\right). \tag{1.24}$$

Proof. *We use inequality*

$$(a + b)^p \leq c(a^p + b^p), \quad \forall a, b > 0, \tag{1.25}$$

where $c = max\left(1, 2^{\frac{1}{p}-1}\right)$. *By using this inequality with* $a = |f|$ *and* $b = |g|$, *we have:*

$$\|f+g\|_{L_p(\Omega)} = \left(\int_\Omega |f+g|^p dx\right)^{\frac{1}{p}} \le c\left(\int_\Omega |f|^p dx + \int_\Omega |g|^p dx\right)^{\frac{1}{p}},$$

and $c = max\left(1, 2^{\frac{1}{p}-1}\right) = 2^{\frac{1}{p}-1}$, *then*

$$\begin{aligned}\|f+g\|_{L_p(\Omega)} &\le 2^{\frac{1}{p}-1}\left(\left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} + \left(\int_\Omega |g|^p dx\right)^{\frac{1}{p}}\right)\\ &\le 2^{\frac{1}{p}-1}\left(\|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}\right).\end{aligned}$$

Theorem 1.1.9 (Minkowski's integral inequality[11]). *Let* $E \subset \mathbb{R}^m$ *and* $F \subset \mathbb{R}^n$ *measurable sets,* f *be a measurable function on* $E \times F$, *then*

1. *If* $\quad 1 \le p \le \infty :$

$$\left\|\int_F f(x,y)dy\right\|_{L_p(E)} \le \int_F \|f(x,y)dy\|_{L_p(E)}. \tag{1.26}$$

2. *If* $\quad 0 < p < 1 \quad$ *or* $\quad p < 0 :$

$$\left\|\int_F f(x,y)dy\right\|_{L_p(E)} \ge \int_F \|f(x,y)dy\|_{L_p(E)}. \tag{1.27}$$

Proof. *See [11].*
The Minkowski inequality for $p < 0$ *was established and proved in [34].*

Proposition 1.1.2. *The mapping* $l : L_p(\Omega) \to \mathbb{C}$ *defined by*

$$l(f) = \int_\Omega f(y)g(y)dy, \qquad g \in L_{p'}(\Omega),$$

is a continuous linear functional. The set of linear functionals on $L_p(\Omega)$ is denoted by $(L_p(\Omega))^*$.

Proof. 1. *Linearity, we use the properties of Lebesgue integral.*

2. *Continuity. By applying Hölder's inequality, we get*

$$\begin{aligned}|l(f)| &= \left|\int_\Omega f(y)g(y)dy\right|\\ &\le \int_\Omega |f(y)||g(y)|dy\\ &\le \|f\|_{L_p(\Omega)}\|g\|_{L_{p'}(\Omega)}\\ &\le c\|f\|_{L_p(\Omega)},\end{aligned}$$

where $c = \|g\|_{L_{p'}(\Omega)}$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

1.2 Hardy's inequalities.

1.2.1 Hardy's inequalities in L_p spaces $(p \geq 1)$.

This section presents the classical Hardy inequality, a historical result established by G. H. Hardy in 1925, along with its generalization including power weights, which was also examined by Hardy in 1928.

Theorem 1.2.1 (Discrete Case [20]). *Let $p > 1$, $a_k \geq 0$ and*

$$A_n = \sum_{k=1}^{k=n} a_k,$$

then

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} a_k^p, \tag{1.28}$$

where the constant $\left(\frac{p}{p-1}\right)^p$ is sharp (the best possible).

Proof. *1. See [20].*

2. For sharp constant : see [24].

Theorem 1.2.2 (Continuous Case [20]). *Let $p > 1$, f be a nonnegative measurable function on $(0, \infty)$. The operator H is defined as follows:*

$$(Hf)(x) := \frac{1}{x} \int_0^x f(y)dy,$$

then

$$\int_0^{\infty} \Big((Hf)(x)\Big)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} (f(x))^p dx. \tag{1.29}$$

The constant $\left(\frac{p}{p-1}\right)^p$ is sharp.

Proof. *We choose $y = xs$, we get*

$$\left(\frac{1}{x} \int_0^x f(y)dy\right)^p = \left(\int_0^1 f(xs)ds\right)^p.$$

Using Minkowski's integral inequality (Theorem 1.1.9), gives

$$\begin{aligned}
\left(\int_0^{\infty} \left(\int_0^1 f(xs)ds\right)^p dx\right)^{\frac{1}{p}} &\leq \int_0^1 \left(\int_0^{\infty} f(xs)^p dx\right)^{\frac{1}{p}} ds \\
&= \int_0^1 \left(\int_0^{\infty} f(u)^p \frac{du}{s}\right)^{\frac{1}{p}} ds \\
&= \left(\frac{p}{p-1}\right) \left(\int_0^{\infty} (f(x))^p \, dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Remark 1.2.1. *The constant* $\left(\frac{p}{p-1}\right)^p$ *in Theorem 1.2.2 is sharp.(see [24]).*

The following result originates from G. Hardy's work in 1927.

Theorem 1.2.3 (weighted Hardy's inequality [24]). *Let* $p \geq 1$ *and* f *be a non-negative measurable function on* $(0, \infty)$. *If* $\alpha < p - 1$, *then the inequality*

$$\int_0^\infty \Big((Hf)(x)\Big)^p x^\alpha dx \leq \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty x^\alpha f^p(x)dx, \tag{1.30}$$

is valid and if $-1 < \alpha < p - 1$ *the constant* $\left(\frac{p}{p-1-\alpha}\right)^p$ *is sharp.*

Proof. *Choose* $t = xs$, *we have*

$$\begin{aligned}\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx\right)^{\frac{1}{p}} &= \left(\int_0^\infty \left(\int_0^1 f(xs)ds\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty \left(\int_0^1 f(xs)x^{\frac{\alpha}{p}}ds\right)^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

Now, applying Minkowski's integral inequality, we get

$$\begin{aligned}\left(\int_0^\infty \left(\int_0^1 f(xs)x^{\frac{\alpha}{p}}ds\right)^p dx\right)^{\frac{1}{p}} &\leq \int_0^1 \left(\int_0^\infty f^p(xs)x^\alpha dx\right)^{\frac{1}{p}} ds \\ &= \int_0^1 \left(\int_0^\infty f^p(y)\left(\frac{y}{s}\right)^\alpha \frac{dy}{s}\right)^{\frac{1}{p}} ds \\ &= \left(\int_0^1 s^{-\frac{1}{p}(\alpha+1)}ds\right)\left(\int_0^\infty f^p(y)y^\alpha dy\right)^{\frac{1}{p}} \\ &= \frac{p}{p-\alpha-1}\left(\int_0^\infty f^p(y)y^\alpha dy\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 1.2.2. *If we put* $\alpha = 0, (p > 1)$ *in Theorem 1.2.3, we obtain inequality (1.29) of Theorem 1.2.2.*

1.2.2 Classical Hardy's inequalities for $0 < p < 1$.

The Hardy inequality for $0 < p < 1$ with weight $x^\alpha, \alpha \in \mathbb{R}$ does not take place for any function defined on $(0, \infty)$, on the other hand it is verified with the additional assumption of monotonicity. This result was established by V. I. Burenkov 1989 (see [10]) using a discretization technique based on the following lemma.

Lemma 1.2.1. *[10] Let* $\alpha \in \mathbb{R}^+$ *then there exist constants* c_1, c_2 *such that for any nonnegative monotone function on* $(0, \infty)$, *we have*

$$c_1 \sum_{k=-\infty}^{+\infty} 2^{k(\alpha+1)} f(2^k) \leq \int_0^\infty x^\alpha f(x)dx \leq c_2 \sum_{k=-\infty}^{+\infty} 2^{k(\alpha+1)} f(2^k), \tag{1.31}$$

where

$$c_1 = 2^{-\alpha-1}, \quad c_2 = 2^{\alpha}.$$

Later V. I. Burenkov gave another proof where it specified the sharp constant (see [10]).

Theorem 1.2.4. *Let* $0 < p < 1$.

(a) If $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$*, then for any nonnegative and nonincreasing functions* f *on* $(0,\infty)$*,*

$$\|x^{\alpha}(Hf)(x)\|_{L_p(0,\infty)} \leq \left(1 - \frac{1}{p} - \alpha\right)^{-\frac{1}{p}} \|x^{\alpha} f(x)\|_{L_p(0,\infty)}. \tag{1.32}$$

(b) If $\alpha < \frac{-1}{p}$*, then for any nonnegative and nondecreasing functions* f *on* $(0,\infty)$*,*

$$\|x^{\alpha}(Hf)(x)\|_{L_p(0,\infty)} \leq \left(p\beta\left(p, p\left(\frac{1}{p'} - \alpha\right)\right)\right)^{\frac{1}{p}} \|x^{\alpha} f(x)\|_{L_p(0,\infty)}, \tag{1.33}$$

where $\beta(u,v)$ *is beta function.*

(c) If $\alpha > \frac{1}{p}$*, then for any nonnegative and nonincreasing functions* f *on* $(0,\infty)$*,*

$$\|x^{\alpha}(H^{*}f)(x)\|_{L_p(0,\infty)} \leq \left(p\beta\left(p, p\left(\alpha - \frac{1}{p'}\right)\right)\right)^{\frac{1}{p}} \|x^{\alpha} f(x)\|_{L_p(0,\infty)}, \tag{1.34}$$

where $(H^{*}f)(x) = \frac{1}{x}\int_x^{\infty} f(y)dy$.

We need the following lemma to prove Theorem 1.2.4.

Lemma 1.2.2. *[7]*

(a) Let $-\infty < a < b \leq +\infty$ *and assume that the function* f *is nonnegative and nonincreasing on the interval* (a,b)*. If* $0 < p \leq 1$*, then*

$$\left(\int_a^b f(t)dt\right)^p \leq p\int_a^b f^p(t)(t-a)^{p-1}dt. \tag{1.35}$$

The inequality holds in the reversed direction if $1 \leq p < \infty$.

(b) Let $-\infty < a < b \leq +\infty$ *and* f *be a function that is nonnegative and nondecreasing on the interval* (a,b)*. If* $0 < p \leq 1$*, then*

$$\left(\int_a^b f(t)dt\right)^p \leq p\int_a^b f^p(t)(b-t)^{p-1}dt. \tag{1.36}$$

The inequality holds in the reversed direction if $1 \leq p < \infty$.

(c) The factor p is sharp in these inequalities.

Proof. *See [7].*

Proof (Proof of Theorem 1.2.4). *By applying Lemma 1.2.2 with $a = 0, b = x$ and Fubini's theorem, we obtain*

$$\begin{aligned}\|x^{\alpha}(Hf)(x)\|^{p}_{L_p(0,\infty)} &= \int_0^{\infty}\left(\frac{1}{x}\int_0^{x} f(t)dt\right)^{p} x^{\alpha p}dx\\ &= \int_0^{\infty} x^{(\alpha-1)p}\left(\int_0^{x} f(t)dt\right)^{p} dx\\ &\leq \int_0^{\infty} x^{(\alpha-1)p}p\int_0^{x} t^{p-1}f^{p}(t)dtdx\\ &=p\int_0^{\infty} t^{p-1}f^{p}(t)\left(\int_t^{\infty} x^{(\alpha-1)p}dx\right)dt,\end{aligned}$$

since $\alpha < 1 - \frac{1}{p}$, we get

$$\int_t^{\infty} x^{(\alpha-1)p}dx = \frac{-1}{(\alpha-1)p+1}t^{(\alpha-1)p+1},$$

consequently,

$$\begin{aligned}\int_0^{\infty}\left(\frac{1}{x}\int_0^{x} f(t)dt\right)^{p} x^{\alpha p}dx &\leq \left(\frac{p}{-\alpha p+p-1}\right)\int_0^{\infty} t^{p-1}f^{p}(t)t^{\alpha p-p+1}dt\\ &= \left(\frac{p}{-\alpha p+p-1}\right)\int_0^{\infty} f^{p}(t)t^{\alpha p}dt.\end{aligned}$$

$$\left(\int_0^{\infty}\left(\frac{1}{x}\int_0^{x} f(t)dt\right)^{p} x^{\alpha p}dx\right)^{\frac{1}{p}} \leq \left(\frac{p}{-\alpha p+p-1}\right)^{\frac{1}{p}}\left(\int_0^{\infty} f^{p}(t)t^{\alpha p}dt\right)^{\frac{1}{p}},$$

finally

$$\|x^{\alpha}(Hf)(x)\|_{L_p(0,\infty)} \leq \left(1-\frac{1}{p}-\alpha\right)^{-\frac{1}{p}}\|x^{\alpha}f(x)\|_{L_p(0,\infty)}. \tag{1.37}$$

(1.33) and (1.34) are proved similarly by applying (1.36) and (1.35), respectively.

1.2.3 The Hardy-Steklov Operator.

In this section, we assume that the function f is defined on $(0, \infty)$. Let us start with an example.

Example 1.2.1. *The classical Hardy operator H_1, defined as*

$$(H_1 f)(x) = \frac{1}{x}\int_0^{x} f(t)dt, \qquad 0 < x < \infty, \tag{1.38}$$

it is obviously related to the triangular domain

$$\Delta = \Big\{(x,t); \quad 0 < t < x < \infty\Big\}, \quad (seeFig.01).$$

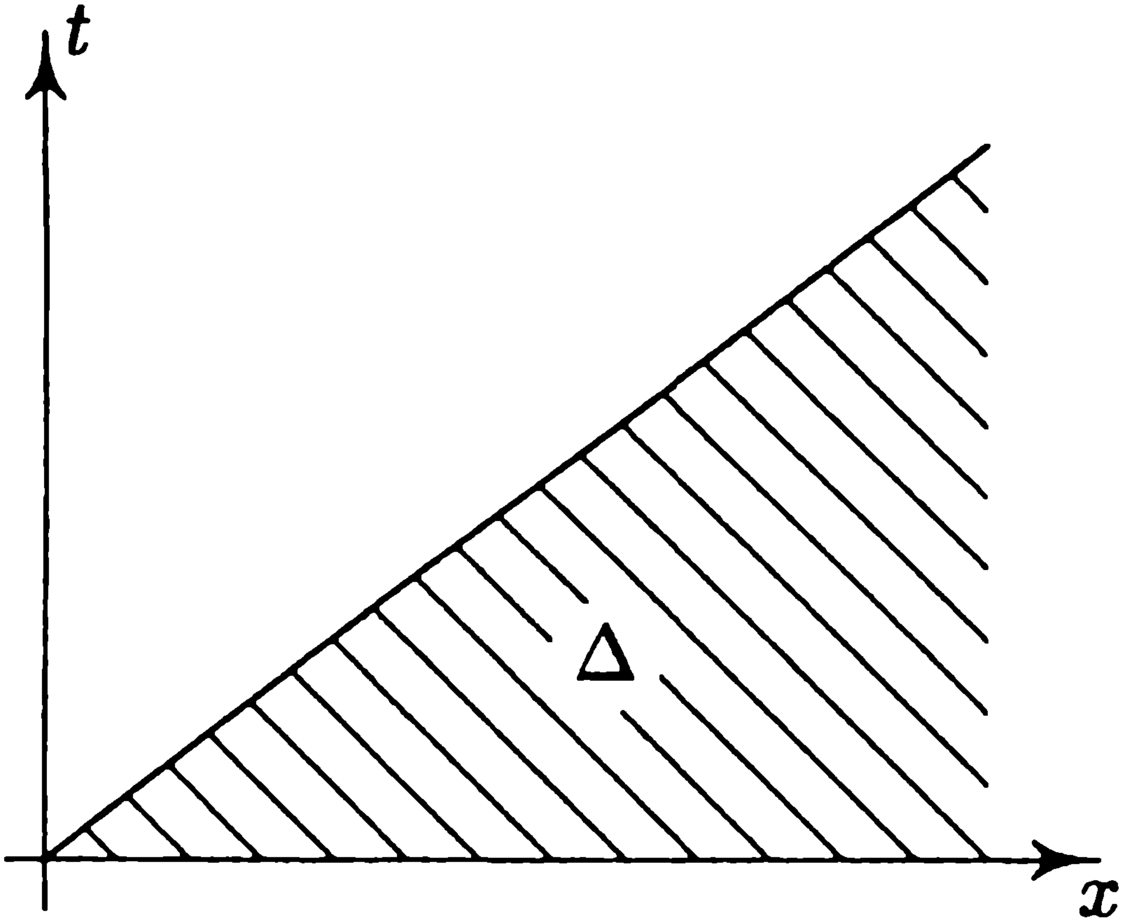

Fig. 01

This can be modified by considering the operator T_1, defined by

$$(T_1 f)(x) = \int_0^{b(x)} f(y)dy, \qquad 0 < x < \infty,$$

with boundary function $b(x)$ satisfying the following conditions:

1. $b(x)$ is differentiable and increasing functions on $(0, \infty)$.
2. $0 < b(x) < \infty$ for $0 < x < \infty$ and $b(0) = 0$, $b(\infty) = \infty$.

The operator T_1 is related to a "perturbed" triangular domain

$$\Delta(b) = \Big\{(x,t): \quad 0 < x < \infty, \quad 0 < t < b(x)\Big\}, \quad (see Fig.02).$$

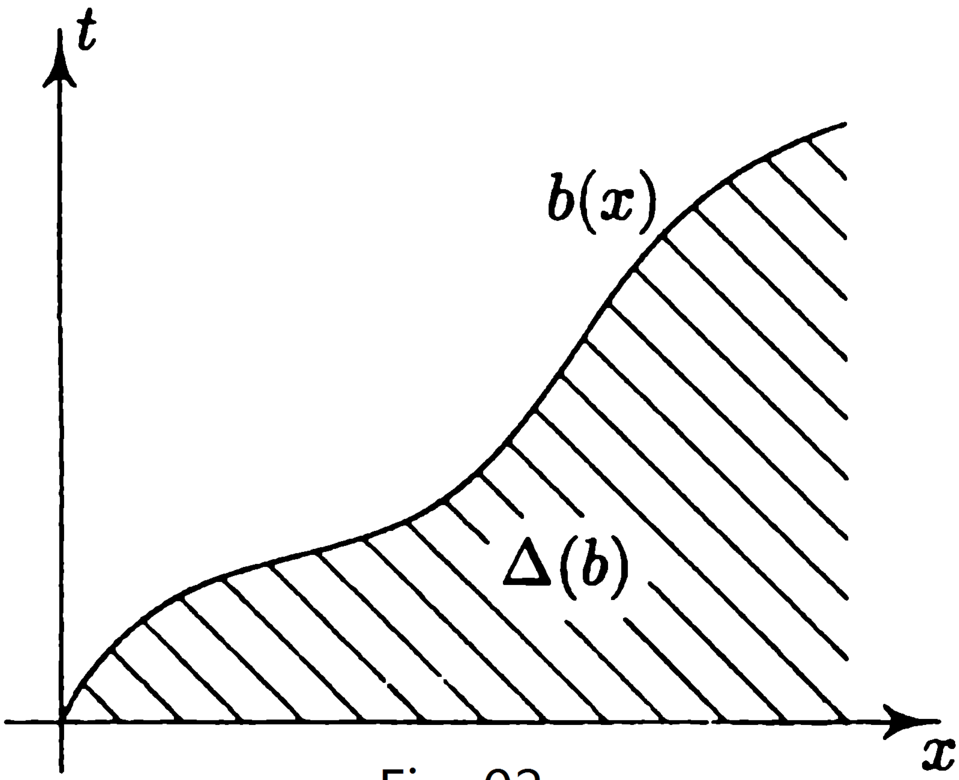

Fig. 02.

Definition 1.2.1 (The Steklov operator[25]). *For functions f defined on $(-\infty, +\infty)$ and for $\delta > 0$, the Steklov operator S_δ is defined as*

$$(S_\delta f)(x) = \int_{x-\delta}^{x+\delta} f(y)dy. \tag{1.39}$$

Definition 1.2.2 (The Hardy-Steklov operator). *(see [25] for more details). The Hardy-Steklov operator is defined as*

$$(Tf)(x) = \frac{1}{x} \int_{a(x)}^{b(x)} f(y)dy, \tag{1.40}$$

with boundary functions $a(x)$, $b(x)$ satisfying the following conditions:

1. $a(x)$, $b(x)$ *are differentiable and increasing functions on* $(0, \infty)$.
2. $0 < a(x) < b(x) < \infty$ *for* $0 < x < \infty$, $a(0) = b(0) = 0$, *and* $a(\infty) = b(\infty) = \infty$,

where f is a nonnegative measurable function on $(0, \infty)$.

1.3 Variable Lebesgue spaces $L_{p(x)}$.

For the first time the variable exponent Lebesgue spaces appeared in the literature already in the thirties of the last century, being introduced by W. Orlicz.

At the beginning these spaces had theoretical interest. Later on the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)$-Laplacian, in Zhikov [42], [44], which in its turn gave an essential impulse for the development of this theory. The extensive investigation of these spaces was also widely stimulated by applications to various problems of Applied Mathematics, e.g., in modelling of electrorheological fluids [32]. Variable Lebesgue spaces appeared as a special case of the Musielak-Orlicz spaces introduced by H. Nakano and developed by J. Musielak and W. Orlicz (see [30]). Further developement of this theory was connected with the theory of modular functions.

1.3.1 Definitions.

Let Ω be a measurable subset of $\mathbb{R}^n$ with $|\Omega| > 0$, and let $\mathcal{P}(\Omega)$ denote the set of summable functions such that $p : \Omega \longrightarrow [1, +\infty]$.
We set:

$$\Omega_a = \Omega_a(p) = \big\{x \in \Omega, p(x) = a, \quad a \in (1, \infty)\big\},$$

in particular:

$$\Omega_1 = \big\{x \in \Omega, p(x) = 1\big\}, \qquad \Omega_\infty = \big\{x \in \Omega, p(x) = \infty\big\},$$

$$\Omega_0 = \Omega/(\Omega_1 \cup \Omega_\infty),$$

$$\overline{p} = ess \sup_{x\in\Omega} p(x), \quad \underline{p} = ess \inf_{x\in\Omega} p(x), \quad if \quad |\Omega_0| > 0,$$

$$c_p = \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty,$$

$$r_p = 1 + \frac{1}{\underline{p}} - \frac{1}{\overline{p}},$$

where χ is the characteristic function of the corresponding sets.

Definition 1.3.1. *[28] We denote by $L_{p(x)}(\Omega)$ the set of measurable functions f such that:*

$$\rho_p(f) = \int_{\Omega/\Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L_\infty(\Omega_\infty)}. \tag{1.41}$$

The functional $\rho_p(f) : L_{p(x)}(\Omega) \longrightarrow [0, \infty)$ is called modular of the space $L_{p(x)}(\Omega)$. In the following, we cite certain properties of $\rho_p(f)$.

Proposition 1.3.1. *[28] Let $p(x) \in \mathcal{P}(\Omega)$*

1. $\rho_p(f) \geq 0, \forall f \in L_{p(x)}(\Omega)$.
2. $\rho_p(f) = 0$ *if and only if* $f(x) = 0$ *a.e.*

3. *If* $\rho_p(-f) = \rho_p(f), \forall f \in L_{p(x)}(\Omega)$.

4. $\rho_p(f)$ *is convexe.*

5. *If* $|f(x)| \geq |g(x)|$ *a.e., and if* $\rho_p(f) < \infty$*, then* $\rho_p(f) \geq \rho_p(g)$.

6. *If* $0 < \rho_p(f) < \infty$ *then the mapping* $\lambda \mapsto \rho_p\left(\frac{f}{\lambda}\right)$ *is continuous and non-ineacresing on* $[1, \infty)$.

Proof. *(1), and (2) are obtained from the properties of the Lebesgue integral.*
(3) Obvious equality.
(4) *See [28]*
(5) *is deduced from a property of the Lebesgue integral.*
(6) *Let* $\lambda_1 \geq \lambda_2 \geq 1$ *then*

$$\frac{1}{\lambda_1} \leq \frac{1}{\lambda_2}$$
$$\frac{|f(x)|}{\lambda_1} \leq \frac{|f(x)|}{\lambda_2}$$
$$\int_\Omega \frac{|f(x)|}{\lambda_1} \leq \int_\Omega \frac{|f(x)|}{\lambda_2},$$

then $\rho_p(\lambda_1) \leq \rho_p(\lambda_2)$.
For the proof of continuity see [28].

Definition 1.3.2. *[28] We define the following norm on* $L_{p(x)}(\Omega)$:

$$\|f\|_{L_{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \quad \rho_p\left(\frac{f}{\lambda}\right) \leq 1\right\}. \tag{1.42}$$

Remark 1.3.1. *If* $p(x) = p$

$$\begin{aligned}\rho_p\left(\frac{f}{\lambda}\right) &= \int \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \\ &= \int \left|\frac{f(x)}{\lambda}\right|^{p} dx \\ &= \frac{1}{\lambda^p}\int |f|^p \leq 1,\end{aligned}$$

so

$$\int |f|^p \leq \lambda^p \text{ implies } \left(\int |f|^p\right)^{\frac{1}{p}} \leq \lambda.$$

Then

$$\begin{aligned}\|f\|_{L_{p(x)}(\Omega)} &= \left\{\inf \lambda > 0, \text{ such that } \rho_p\left(\frac{f}{\lambda}\right) \leq 1\right\} \\ &= \left(\int |f|^p\right)^{\frac{1}{p}} = \|f\|_{L_p}.\end{aligned}$$

Proposition 1.3.2. *Let* $|\Omega_\infty| = 0$*, then for all* s*,* $\frac{1}{\underline{p}} \leq s < \infty$*,*

$$\||f|^s\|_{L_{p(x)}(\Omega)} = \|f\|^s_{L_{sp(x)}(\Omega)}.$$

Proof. *Since* $|\Omega_\infty| = 0$*, and if we set* $\mu = \lambda^{\frac{1}{s}}$*, then*

$$\begin{aligned}\||f|^s\|_{L_{p(x)}(\Omega)} &= \inf\left\{\lambda > 0 : \rho_p\left(\frac{|f|^s}{\lambda}\right) \leq 1\right\}\\ &= \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|^s}{\lambda}\right)^{p(x)} dx \leq 1\right\}\\ &= \inf\left\{\mu^s > 0 : \int_\Omega \left(\frac{|f(x)|}{\mu}\right)^{sp(x)} dx \leq 1\right\}\\ &= \|f\|^s_{L_{sp(x)}(\Omega)}.\end{aligned}$$

Lemma 1.3.1. *Let* $f \in L_{p(x)}(\Omega)$*, then*

$$\rho_p\left(\frac{f}{\|f\|_{L_{p(x)}(\Omega)}}\right) \leq 1, \forall f, \quad where \quad 0 < \|f\|_{L_{p(x)}(\Omega)} < \infty. \tag{1.43}$$

Proof. *1. If* $x \in \Omega_\infty$*, (1.43) is obvious.*

2. If $x \notin \Omega_\infty$*, fix a decreasing sequence* $\{\lambda_n\}$ *such that* $\{\lambda_n\} \longrightarrow \|f\|_{L_{p(x)}(\Omega)}$*, hence the sequence* $\left(\frac{|f|}{\lambda_n}\right)_n$ *is increasing and,*

$$\left(\frac{|f|}{\lambda_n}\right) \longrightarrow \frac{|f|}{\|f\|_{L_{p(x)}(\Omega)}}.$$

Then by Fatou's lemma and the property (6) of proposition 1.3.1 and the definition of the modular, we get

$$\int_\Omega \left(\frac{|f|}{\|f\|_{L_{p(x)}(\Omega)}}\right)^{p(x)} dx \leq \underline{\lim}_{n\to\infty} \int_\Omega \left(\frac{|f|}{\lambda_n}\right)^{p(x)} dx \leq 1.$$

Finally, we have $\rho_p\left(\frac{f}{\|f\|_{L_{p(x)}(\Omega)}}\right) < 1$.

Corollary 1.3.1. *Let* $f \in L_{p(x)}(\Omega)$*, such that* $0 < \|f\|_{L_{p(x)}(\Omega)} < \infty$.

1. *If* $\|f\|_{L_{p(x)}(\Omega)} \leq 1$*, then* $\rho_p(f) \leq \|f\|_{L_{p(x)}(\Omega)}$.
2. *If* $\|f\|_{L_{p(x)}(\Omega)} > 1$*, then* $\rho_p(f) > \|f\|_{L_{p(x)}(\Omega)}$.

Proof. *See [14].*

1.3.2 The classical inequalities on $L_{p(x)}$.

Definition 1.3.3. *Let $p(x) \in [1, \infty)$, then we say that the function $q(x)$ is the conjugate of $p(x)$ if:*

$$q(x) = \begin{cases} \infty & for \quad x \in \Omega_1, \\ 1 & for \quad x \in \Omega_\infty, \\ \frac{p(x)}{p(x)-1} & for \quad x \in \Omega_0. \end{cases}$$

Theorem 1.3.1 (Hölder's Inequality[14]). *Let $p(x), q(x) \in \mathcal{P}(\Omega)$ for all $f \in L_{p(x)}(\Omega)$ and $g \in L_{q(x)}(\Omega)$, $fg \in L_1(\Omega)$, then*

$$\int_\Omega |f(x)g(x)|dx \leq r_p \|f\|_{L_{p(x)}(\Omega)} \|g\|_{L_{q(x)}(\Omega)}, \tag{1.44}$$

holds, where $r_p = 1 + \frac{1}{\underline{p}} - \frac{1}{\overline{p}}$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ and

$$\|f\|_{L_{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_p\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Proof. *See [14].*

Remark 1.3.2. *If $p(x) = p$, $q(x) = q$ then $\overline{p} = \underline{p}$ and $r_p = 1$, so we find the Theorem 1.1.6 (inequality (1.16)).*

Lemma 1.3.2. *[14] Let $0 < r(x) \leq p(x) < \overline{p} < \infty$, $x \in \Omega/\Omega_\infty$, then*

$$\|f\|^{\underline{r}}_{L_{p(x)}(\Omega)} \leq \left\|f^{r(x)}\right\|_{L_{\frac{p(x)}{r(x)}}(\Omega)} \leq \|f\|^{\overline{r}}_{L_{p(x)}(\Omega)}, \qquad for \quad \|f\|_{L_{p(x)}(\Omega)} \geq 1, \tag{1.45}$$

and

$$\|f\|^{\overline{r}}_{L_{p(x)}(\Omega)} \leq \left\|f^{r(x)}\right\|_{L_{\frac{p(x)}{r(x)}}(\Omega)} \leq \|f\|^{\underline{r}}_{L_{p(x)}(\Omega)}, \qquad for \quad \|f\|_{L_{p(x)}(\Omega)} \leq 1. \tag{1.46}$$

Proof. *See [14].*

Proposition 1.3.3. *[14] Let $p(x) \geq 1$, $q(x) \geq 1$ and $r(x) \geq 1$ where $\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{s(x)}$, and let $\sup_{x \in \Omega/\Omega_\infty} s(x) < \infty$, then*

$$\|fg\|_{L_{s(x)}(\Omega)} \leq C\|f\|_{L_{p(x)}(\Omega)} \|g\|_{L_{q(x)}(\Omega)}, \tag{1.47}$$

where $C = \sup \frac{s(x)}{p(x)} + \sup \frac{s(x)}{q(x)}$.

Proof. *See [14].*

Remark 1.3.3. *If $s(x) = 1$, we find the inequality (1.44) with $C = r_p$.*

Remark 1.3.4. *If $p(x) = p$, $q(x) = q$ and $s(x) = s$, then $C = 1$ and we get the Corollary 1.1.3.*

We reference an additional standard definition analogous to a standard established in the context of Orlicz spaces (see [30]).

Definition 1.3.4. *[14] Let $f \in L_{p(x)}(\Omega)$. We define the following norm on $L_{p(x)}(\Omega)$:*

$$\|f\|^{(1)}_{L_{p(x)}(\Omega)} = \sup_{\rho_q(\psi)\leq 1} \left| \int_\Omega f(x)\psi(x)dx \right| < \infty, \tag{1.48}$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Proposition 1.3.4 (Minkowski's inequalities[14]). *If $f, g \in L_{p(x)}(\Omega)$, then*

$$\|f+g\|^{(1)}_{L_{p(x)}(\Omega)} \leq \|f\|^{(1)}_{L_{p(x)}(\Omega)} + \|g\|^{(1)}_{L_{p(x)}(\Omega)}. \tag{1.49}$$

Proof.

$$\begin{aligned}
\|f+g\|^{(1)}_{L_{p(x)}(\Omega)} &= \sup_{\rho_q(\psi)\leq 1} \left| \int_\Omega \Big(f(x)+g(x)\Big)\psi(x)dx \right| \\
&\leq \sup_{\rho_q(\psi)\leq 1} \left| \int_\Omega f(x)\psi(x)dx \right| + \sup_{\rho_q(\psi)\leq 1} \left| \int_\Omega g(x)\psi(x)dx \right| \\
&= \|f\|^{(1)}_{L_{p(x)}(\Omega)} + \|g\|^{(1)}_{L_{p(x)}(\Omega)}.
\end{aligned}$$

Lemma 1.3.3. *Let $n \in \mathbb{N}$, $f_i(x) \in L_{p(x)}$, $i = \{1, 2..., n\}$, then*

$$\left\| \sum_{i=1}^{i=n} fi \right\|^{(1)}_{L_{p(x)}(\Omega)} \leq \sum_{i=1}^{i=n} \|fi\|^{(1)}_{L_{p(x)}(\Omega)}, \tag{1.50}$$

Proof. *By induction.*

Chapter 2

Some estimates for Hardy-Steklov type operators

2.1 Introduction

It is well-known that for L_p spaces with $0 < p < 1$, the Hardy inequality is not satisfied for arbitrary nonnegative measurable functions, but is satisfied for nonnegative monotone functions (see [10]).
In 2013 Rovshan A. Bandaliev et al. established some estimates for Hardy operators for monotone functions in variable exponent Lebesgue spaces with $0 < p(x) < 1$. The aim of this chapter is to establish some new estimates for the Hardy-Steklov type operator. This chapter is structured into two sections:

1. **Section one:** The objective of this section is to establish some new integral inequalities with $0 < p < 1$ for nonnegative function under weaker condition than monotonicity (see [35] for more details), via Hardy–Steklov-type operators (this work is published, see [18]).

2. **Section two:** The investigations of the Hardy inequality in variable exponent Lebesgue spaces $L_{p(x)}$ with $0 < p(x) \leq 1$, are much less known. R.A Bandaliev and A. Senouci et al. have established some weighted inequalities for the classical Hardy operator acting from one weighted variable exponent Lebesgue spaces to another weighted variable exponent Lebesgue spaces with $0 < p(x) \leq 1$ for nonnegative monotone and quasi-monotone functions defined on $(0, \infty)$ (see [2] and [36]). In this section we extend some results of [2] and [36] for the Hardy-Steklov type operators (see [16]).

2.2 Some estimates for Hardy-Steklov type operators for monotone functions with $0 < p < 1$.

Throughout this section, we will assume that the function f is a Lebesgue measurable function on $(0, \infty)$.

2.2.1 Introduction.

The following Lemma and Theorem were proved in [35].

Lemma 2.2.1. *Let $0 < p < 1$, $c_1 > 0$ and f be a nonnegative measurable function on $(0; \infty)$, such that for all $x > 0$,*

$$f(x) \leq \frac{c_1}{x} \left(\int_0^x f^p(y) y^{p-1} dy \right)^{\frac{1}{p}}. \tag{2.1}$$

Then

$$\left(\int_0^x f(y) dy \right)^p \leq c_2 \int_0^x f^p(y) y^{p-1} dy, \tag{2.2}$$

where

$$c_2 = c_1^{p(1-p)}.$$

The classical Hardy operators are defined as follows:

$$(H_1 f)(x) = \frac{1}{x} \int_0^x f(y) dy, \quad (H_2 f)(x) = \frac{1}{x} \int_x^\infty f(y) dy.$$

Theorem 2.2.1 ([35]). *Let $0 < p < 1$, $\alpha < 1 - \frac{1}{p}$ and $c_1 > 0$. If f is nonnegative measurable function on $(0, \infty)$ and satisfies (2.1) for all $x > 0$, then*

$$\|x^\alpha (H_1 f)(x)\|_{L_p(0,\infty)} \leq c_3 \|x^\alpha f(x)\|_{L_p(0,\infty)}, \tag{2.3}$$

where

$$c_3 = c_1^{1-p} \Big(1 - \alpha - \frac{1}{p} \Big)^{-\frac{1}{p}} p^{-\frac{1}{p}}.$$

The constant c_3 is sharp (the best possible).

Remark 2.2.1. *If f is a nonincreasing function on $(0, \infty)$, then (2.1) is satisfied with $c_1 = p^{\frac{1}{p}}$. For such functions inequality (2.3) takes the form*

$$\|x^\alpha (H_1 f)(x)\|_{L_p(0,\infty)} \leq \left(p^p \left(1 - \alpha - \frac{1}{p} \right) \right)^{-\frac{1}{p}} \|x^\alpha f(x)\|_{L_p(0,\infty)}. \tag{2.4}$$

The factor $\left(p^p \left(1 - \alpha - \frac{1}{p} \right) \right)^{-\frac{1}{p}}$ is sharp. Inequality (2.4) was proved earlier (for more details, see [10]).

The well-known Hardy–Steklov operator is defined as follows

$$(Tf)(x) = \frac{1}{x}\int_{a(x)}^{b(x)} f(y)dy.$$

Where f is a nonnegative measurable function on $(0,\infty)$, with the boundary functions $a(x)$, $b(x)$ satisfying the following conditions.

1. $a(x)$, $b(x)$ are differentiable and increasing functions on $(0,\infty)$.
2. $0 < a(x) < b(x) < \infty$ for $0 < x < \infty$, $a(0) = b(0) = 0$ and $a(\infty) = b(\infty) = \infty$.

The objective of this section is to extend the results of [35] to Hardy-Steklov type operators T_1 and T_2 defined as follows:

$$(T_1 f)(x) = \frac{1}{x}\int_{0}^{b(x)} f(y)dy,$$

with boundary function $b(x)$ satisfying the following conditions:

1. $b(x)$ is differentiable and increasing function on $(0,\infty)$.
2. $0 < b(x) < \infty$ for $0 < x < \infty$ and $b(0) = 0$, $b(\infty) = \infty$.

$$(T_2 f)(x) = \frac{1}{x}\int_{a(x)}^{\infty} f(y)dy,$$

with boundary function $a(x)$ satisfying the following conditions:

1. $a(x)$ is differentiable and increasing function on $(0,\infty)$.
2. $0 < a(x) < \infty$ for $0 < x < \infty$ and $a(0) = 0$, $a(\infty) = \infty$.

2.2.2 Main results

Throughout this section, we will assume that the function f is a nonnegative measurable function on $(0,\infty)$.

Theorem 2.2.2. *Let $0 < p < 1$, $\alpha < 1 - \frac{1}{p}$, and $\frac{1}{p} + \frac{1}{p'} = 1$. If f is a nonnegative measurable function on $(0,\infty)$ and satisfies (2.1) for all $x > 0$, then*

$$\|x^{\alpha}(T_1 f)(x)\|_{L_p(0,\infty)} \le c_4 \left\| x^{\frac{1}{p'}} \left(b^{-1}(x)\right)^{\alpha - \frac{1}{p'}} f(x) \right\|_{L_p(0,\infty)},$$

where

$$c_4 = c_1^{1-p}\Big((1-\alpha)p - 1\Big)^{-\frac{1}{p}}.$$

Proof. *Choose* $t = b(x)$, *hence* $x = b^{-1}(t)$, *where* $b^{-1}(t)$ *is the reciprocal function of* $b(t)$. *Applying (2.2) and Fubini's Theorem, we get*

$$\begin{aligned}
\|x^\alpha (T_1 f)(x)\|_{L_p(0,\infty)} &= \left(\int_0^\infty x^{(\alpha-1)p} \left(\int_0^{b(x)} f(y)dy \right)^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left(b^{-1}(t)\right)^{(\alpha-1)p} \left(\int_0^t f(y)dy \right)^p \left(b^{-1}(t)\right)' dt \right)^{\frac{1}{p}} \\
&\le (c_2)^{\frac{1}{p}} \left(\int_0^\infty \left(b^{-1}(t)\right)^{(\alpha-1)p} \left(\int_0^t f^p(y) y^{p-1} dy \right) \left(b^{-1}(t)\right)' dt \right)^{\frac{1}{p}} \\
&= (c_2)^{\frac{1}{p}} \left(\int_0^\infty f^p(y) y^{p-1} \left(\int_y^\infty \left(b^{-1}(t)\right)' \left(b^{-1}(t)\right)^{(\alpha-1)p} dt \right) dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\alpha < 1 - \frac{1}{p}$ *and* $b^{-1}(\infty) = \infty$, *we have*

$$\int_y^\infty \left(b^{-1}(t)\right)' \left(b^{-1}(t)\right)^{(\alpha-1)p} dt = \frac{1}{(1-\alpha)p - 1} \left[b^{-1}(y)\right]^{(\alpha-1)p+1},$$

consequently,

$$\begin{aligned}
\|x^\alpha (T_1 f)(x)\|_{L_p(0,\infty)} &\le \left(\frac{(c_1)^{p(1-p)}}{(1-\alpha)p - 1} \right)^{\frac{1}{p}} \left[\int_0^\infty f^p(y) y^{p-1} \left[b^{-1}(y)\right]^{(\alpha-1)p+1} dy \right]^{\frac{1}{p}} \\
&= (c_1)^{1-p} \Big((1-\alpha)p - 1 \Big)^{-\frac{1}{p}} \left[\int_0^\infty \left(f(y) y^{1-\frac{1}{p}} \left[b^{-1}(y)\right]^{(\alpha-1)+\frac{1}{p}} \right)^p dy \right]^{\frac{1}{p}} \\
&= (c_1)^{1-p} \Big((1-\alpha)p - 1 \Big)^{\frac{-1}{p}} \left[\int_0^\infty \left(f(y) y^{\frac{1}{p'}} \left[b^{-1}(y)\right]^{\alpha-\frac{1}{p'}} \right)^p dy \right]^{\frac{1}{p}} \\
&= c_4 \left\| x^{\frac{1}{p'}} \left(b^{-1}(x)\right)^{\alpha - \frac{1}{p'}} f(x) \right\|_{L_p(0,\infty)}.
\end{aligned}$$

We get the desired inequality.

Remark 2.2.2. *If* f *is a nonincreasing function on* $(0, \infty)$, *we obtain the following inequality:*

$$\|x^\alpha (T_1 f)(x)\|_{L_p(0,\infty)} \le \left(\frac{p^{1-p}}{(1-\alpha)p - 1} \right)^{\frac{1}{p}} \left\| x^{\frac{1}{p'}} \left(b^{-1}(x)\right)^{\alpha - \frac{1}{p'}} f(x) \right\|_{L_p(0,\infty)}.$$

Choosing $b(x) = \beta x$ in Theorem 2.2.2, where $\beta > 0$, we have the following result.

Corollary 2.2.1. *Let f satisfy the assumptions of Theorem 2.2.2 and*

$$(S_1 f)(x) = \frac{1}{x} \int_0^{\beta x} f(y)dy, \text{ for } x > 0,$$

then

$$\|x^\alpha (S_1 f)(x)\|_{L_p(0,\infty)} \le \left(\frac{1}{\beta} \right)^{\alpha - \frac{1}{p'}} c_4 \|x^\alpha f(x)\|_{L_p(0,\infty)}.$$

Remark 2.2.3. *Taking* $\beta = 1$ *in the above corollary, we get Theorem 2.2.1.*

For the next results we need the following

Lemma 2.2.2. *Let* $0 < p < 1$. *Suppose that a nonnegative function* f *satisfies the condition: there is a positive constant* c_5 *such that for all* $x > 0$,

$$f(x) \le \frac{c_5}{x}\Big(\int_x^\infty f^p(y)y^{p-1}dy\Big)^{\frac{1}{p}}, \tag{2.5}$$

then

$$\Big(\int_x^\infty f(y)dy\Big)^p \le c_6 \int_x^\infty f^p(y)y^{p-1}dy, \tag{2.6}$$

where

$$c_6 = c_5^{p(1-p)}.$$

Proof. *Note that*

$$f(x) = (f^p(x)x^p)^{\frac{1}{p}-1} f^p(x)x^{p-1}.$$

Using (2.5), we have

$$x^p f^p(x) \le c_5^p \left(\int_x^\infty f^p(y)y^{p-1}dy\right),$$

therefore,

$$(x^p f^p(x))^{\frac{1}{p}-1} \le c_5^{1-p}\left(\int_x^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}-1}.$$

Multiplying by $f^p(x)x^{p-1}$ *and putting* $0 < t \le x$, *we get*

$$f(x) \le c_5^{1-p}\left(\int_t^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}-1} f^p(x)x^{p-1},$$

consequently

$$\begin{aligned}\int_t^\infty f(x)dx &\le c_5^{1-p}\left(\int_t^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}-1}\int_t^\infty f^p(x)x^{p-1}dx\\ &= c_5^{1-p}\left(\int_t^\infty f^p(x)x^{p-1}dx\right)^{\frac{1}{p}-1}\int_t^\infty f^p(x)x^{p-1}dx\\ &= c_5^{1-p}\left(\int_t^\infty f^p(x)x^{p-1}dx\right)^{\frac{1}{p}}.\end{aligned}$$

Theorem 2.2.3. *Let* $0 < p < 1$, $\alpha > 1 - \frac{1}{p}$. *If* f *is a nonnegative and measurable function on* $(0, \infty)$ *and satisfies (2.5) for all* $x > 0$, *then*

$$\|x^\alpha (T_2 f)(x)\|_{L_p(0,\infty)} \le c_7 \left\| x^{\frac{1}{p'}} \left(a^{-1}(x)\right)^{\alpha - \frac{1}{p'}} f(x)\right\|_{L_p(0,\infty)},$$

where

$$c_7 = c_5^{1-p}\big((\alpha-1)p+1\big)^{-\frac{1}{p}}.$$

Proof. *Put $t = a(x)$, then $a^{-1}(t)$, where $a^{-1}(t)$ is the reciprocal function of $a(t)$. Applying inequality (2.6) and Fubini's Theorem, we get*

$$\begin{aligned}
\|x^{\alpha}(T_2f)(x)\|_{L_p(0,\infty)} &= \left(\int_0^{\infty} x^{(\alpha-1)p}\left(\int_{a(x)}^{\infty} f(y)dy\right)^p dx\right)^{\frac{1}{p}}\\
&= \left(\int_0^{\infty} \left(a^{-1}(t)\right)^{(\alpha-1)p}\left(\int_t^{\infty} f(y)dy\right)^p \left(a^{-1}(t)\right)' dt\right)^{\frac{1}{p}}\\
&\le (c_6)^{\frac{1}{p}}\left(\int_0^{\infty} \left(a^{-1}(t)\right)^{(\alpha-1)p}\left(\int_t^{\infty} f^p(y)y^{p-1}dy\right) \left(a^{-1}(t)\right)' dt\right)^{\frac{1}{p}}\\
&\le (c_6)^{\frac{1}{p}}\left(\int_0^{\infty} f^p(y)y^{p-1}\left(\int_0^{y} \left(a^{-1}(t)\right)'\left(a^{-1}(t)\right)^{(\alpha-1)p} dt\right) dy\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\alpha > 1 - \frac{1}{p}$ and $a^{-1}(0) = 0$, we have

$$\int_0^{y} \left(a^{-1}(t)\right)'\left(a^{-1}(t)\right)^{(\alpha-1)p} dt = \frac{1}{(\alpha-1)p+1}\left[a^{-1}(y)\right]^{(\alpha-1)p+1},$$

consequently,

$$\begin{aligned}
\|x^{\alpha}(T_2f)(x)\|_{L_p(0,\infty)} &\le \left(\frac{c_5^{p(1-p)}}{(\alpha-1)p+1}\right)^{\frac{1}{p}}\left[\int_0^{\infty} f^p(y)y^{p-1}\left[a^{-1}(y)\right]^{(\alpha-1)p+1} dy\right]^{\frac{1}{p}}\\
&= c_5^{1-p}((\alpha-1)p+1)^{-\frac{1}{p}}\left[\int_0^{\infty}\left(f(y)y^{1-\frac{1}{p}}\left[a^{-1}(y)\right]^{(\alpha-1)+\frac{1}{p}}\right)^p dy\right]^{\frac{1}{p}}\\
&= c_7\left\|x^{\frac{1}{p'}}\left(a^{-1}(x)\right)^{\alpha-\frac{1}{p'}} f(x)\right\|_{L_p(0,\infty)}.
\end{aligned}$$

Choosing $a(x) = \lambda x$ in Theorem 2.2.3, where $\lambda > 0$, we obtain the following result.

Corollary 2.2.2. *Let f satisfy the assumptions of Theorem 2.2.3 and*

$$(S_2f)(x) = \frac{1}{x}\int_{\lambda x}^{\infty} f(y)dy \quad for \quad x > 0.$$

Then the inequality.

$$\|x^{\alpha}(S_2f)(x)\|_{L_p(0,\infty)} \le \left(\frac{1}{\lambda}\right)^{\alpha-\frac{1}{p'}} c_7\|x^{\alpha}f(x)\|_{L_p(0,\infty)},$$

holds.

Remark 2.2.4. *Taking $\lambda = 1$, we get*

$$\|x^{\alpha}(H_2f)(x)\|_{L_p(0,\infty)} \le c_7\|x^{\alpha}f(x)\|_{L_p(0,\infty)}.$$

Now, we have obtained the analogue of Theorem 2.2.1 for H_2 which is the dual of Hardy averaging operator H_1.

2.3 On Hardy-Steklov type operators for quasi-monotone functions in weighted variable Lebesgue exponent spaces.

The objective of this section is to extend the results of [36] and [2] to Hardy-Steklov type operators T_1 and T_2 defined as follows :(Submitted work see [16]). Let f be a nonnegative measurable function on $(0, \infty)$.

Let

$$(T_1 f)(x) = \frac{1}{x} \int_0^{b(x)} f(y)dy,$$

with boundary function $b(x)$ satisfying the following conditions:

1. $b(x)$ differentiable and increasing function on $(0, \infty)$.
2. $0 < b(x) < \infty$ for $0 < x < \infty$, and $b(0) = 0$, $b(\infty) = \infty$.

Let

$$(T_2 f)(x) = \frac{1}{x} \int_{a(x)}^{\infty} f(y)dy,$$

with boundary function $a(x)$ satisfying the following conditions:

1. $a(x)$ differentiable and increasing function on $(0, \infty)$.
2. $0 < a(x) < \infty$ for $0 < x < \infty$, and $a(0) = 0$, $a(\infty) = \infty$.

2.3.1 Introduction

We state the following definitions, Lemmas, Corollaries and Theorems that are useful in the proofs of main results.

Definition 2.3.1. *By $L_{p(x),w(x)}(\Omega)$ we denote the set of all measurable function f on Ω such that*

$$\rho_{p(x),w(x)}(f) = \int_\Omega (|f(x)|w(x))^{p(x)}\, dx < \infty. \tag{2.7}$$

Note that the expression

$$\|f\|_{L_{p(x),w(x)}(\Omega)} = \|f\|_{L_{p,w,\Omega}} = \inf\left\{\lambda > 0;\ \int_\Omega \left(\frac{|f(x)|\omega(x)}{\lambda}\right)^{p(x)} dx \le 1\right\}, \tag{2.8}$$

defines a quasi-norm on $L_{p(x),w(x)}(\Omega)$. $L_{p(x),w(x)}(\Omega)$ is a quasi-Banach space equipped with this quasi-norm (see [33]).

The following definition and statement were introduced in [7].

Definition 2.3.2. *We say that a nonnegative function f is quasi-monotone on $(0,\infty)$, if for some real number α, $x^{\alpha}f(x)$ is a decreasing or an increasing function of x. More precisely, given $\beta \in \mathbb{R}$ we say that $f \in Q_{\beta}$ if $x^{-\beta}f(x)$ is nonincreasing and $f \in Q^{\beta}$ if $x^{-\beta}f(x)$ is nondecreasing.*

Proposition 2.3.1. *(a) Let $-\infty < \beta < +\infty$,$f \in Q_{\beta}$, $0 \le a < b \le \infty$ for $\beta > -1$ and $0 < a < b \le \infty$ for $\beta \le -1$. If $0 < p \le 1$ and $\beta \neq -1$, then*

$$\left(\int_a^b f(t)dt\right)^p \le p|\beta+1|^{1-p} \int_a^b \left(\frac{\left|t^{\beta+1}-a^{\beta+1}\right|}{t^{\beta}}\right)^{p-1} f^p(t)dt. \tag{2.9}$$

If $0 < p \le 1$ and $\beta = -1$, then

$$\left(\int_a^b f(t)dt\right)^p \le p \int_a^b \left(t\ln\frac{t}{a}\right)^{p-1} f^p(t)dt. \tag{2.10}$$

The inequalities hold in the reversed direction if $1 \le p < \infty$.

(b) Let $-\infty < \beta < +\infty$,$f \in Q^{\beta}$, $0 \le a < b \le \infty$ for $\beta < -1$ and $0 \le a < b < \infty$ for $\beta \ge -1$. If $0 < p \le 1$ and $\beta \neq -1$, then

$$\left(\int_a^b f(t)dt\right)^p \le p|\beta+1|^{1-p} \int_a^b \left(\frac{\left|t^{\beta+1}-b^{\beta+1}\right|}{t^{\beta}}\right)^{p-1} f^p(t)dt. \tag{2.11}$$

If $0 < p \le 1$ and $\beta = -1$, then

$$\left(\int_a^b f(t)dt\right)^p \le p \int_a^b \left(t\ln\frac{b}{t}\right)^{p-1} f^p(t)dt. \tag{2.12}$$

The inequalities hold in the reversed direction if $1 \le p < \infty$.

(c) The constants in these inequalities are the best possible in all cases.

Remark 2.3.1. *1. If we put $a = 0$ and $b = y$ in (2.9) and (2.11), we have respectively*

(a) If $\beta > -1$, $f \in Q_{\beta}$ and $0 < y \le \infty$.

$$\left(\int_0^y f(t)dt\right)^p \le p(\beta+1)^{1-p} \int_0^y t^{p-1} f^p(t)dt. \tag{2.13}$$

(b) If $\beta > -1$, $f \in Q^{\beta}$ and $0 \le y < \infty$.

$$\left(\int_0^y f(t)dt\right)^p \le p(\beta+1)^{1-p} \int_0^y \left[t^{-\beta}\left(y^{\beta+1}-t^{\beta+1}\right)\right]^{p-1} f^p(t)dt. \tag{2.14}$$

2. *If we put $a = y$ and $b = \infty$ in (2.11), we get*

$$\left(\int_y^\infty f(t)dt\right)^p \leq p|\beta+1|^{1-p} \int_y^\infty t^{p-1} f^p(t)dt, \tag{2.15}$$

where $f \in Q^\beta$ and $\beta < -1$.

The following Corollary, Lemma and Theorems were proved in [2].

Corollary 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and p, q be measurable functions on Ω, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < \infty$ and $r(x) = \frac{p(x)q(x)}{q(x)-p(x)}$. Suppose that w_1 and w_2 are weight functions defined in Ω satisfying the condition*

$$\left\|\frac{w_1}{w_2}\right\|_{L_{r(x)}(\Omega)} < \infty.$$

Then the inequality

$$\|f\|_{L_{p(x),w_1}(\Omega)} \leq \left(A + B + \|\chi_{\Omega_2}\|_{L_\infty(\Omega)}\right)^{\frac{1}{\underline{p}}} \left\|\frac{w_1}{w_2}\right\|_{L_{r(x)}(\Omega)} \|f\|_{L_{q(x),w_2}(\Omega)}, \tag{2.16}$$

holds for every $f \in L_{q(x),w_2(x)}(\Omega)$, where

$$\Omega_1 = \{x \in \Omega : \ p(x) < q(x)\}, \qquad \Omega_2 = \{x \in \Omega : \ p(x) = q(x)\},$$
$$A = \sup_{x\in\Omega_1} \frac{p(x)}{q(x)}, \ B = \sup_{x\in\Omega_1} \frac{q(x)-p(x)}{q(x)}.$$

If $|\Omega_2| = 0$, the constant in (2.16) is sharp. If $|\Omega_2| > 0$, then it is not sharp.

Remark 2.3.2. *An improvement of the constant in (2.16) was obtained in [12].*

$$\|f\|_{L_{p(x),w_1}(\Omega)} \leq (1+M-m)^{\frac{1}{\underline{p}}} \left\|\frac{w_1}{w_2}\right\|_{L_{r(x)}(\Omega)} \|f\|_{L_{q(x),w_2}(\Omega)}, \tag{2.17}$$

where

$$M = ess\sup_{x\in\Omega} \frac{p(x)}{q(x)}, \ m = ess\inf_{x\in\Omega} \frac{p(x)}{q(x)}, \underline{p} = ess\inf_{x\in\Omega} p(x).$$

The constant in (2.17) is sharp for any measurable set Ω.

Lemma 2.3.1. *Let $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ be measurable sets, p be a measurable function on Ω_1 and q be a measurable function on Ω_2, $1 \leq \underline{p} \leq p(x) \leq q(y) \leq \overline{q} < \infty$ for all $x \in \Omega_1 \subset \mathbb{R}^n$ and $y \in \Omega_2 \subset \mathbb{R}^m$. If $p \in C(\Omega_1)$, $q \in C(\Omega_2)$, then the inequality*

$$\left\|\|f\|_{L_{p(x)}(\Omega_1)}\right\|_{L_{q(x)}(\Omega_2)} \leq C_{p,q} \left\|\|f\|_{L_{q(x)}(\Omega_2)}\right\|_{L_{p(x)}(\Omega_1)}, \tag{2.18}$$

is valid, where

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\overline{p}}{\underline{q}} - \frac{\underline{p}}{\overline{\overline{q}}}\right)(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty), \tag{2.19}$$

$\underline{q} = \operatorname{ess\,inf}_{\Omega_2} q(x), \overline{q} = \operatorname{ess\,sup}_{\Omega_2} q(x), \Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2; \ p(x) = q(y)\}, \Delta_2 = (\Omega_1 \times \Omega_2) \backslash \Delta_1$, *$C(\Omega_1)$, $C(\Omega_2)$ are the space of continuous functions in Ω_1, Ω_2 and $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is any measurable function such that $\left\|\|f\|_{L_{q(x)}(\Omega_2)}\right\|_{L_{p(x)}(\Omega_1)} < \infty$.*

Theorem 2.3.1. *Let* p, q *be measurable functions on* $(0,\infty)$, $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$, $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x\in(0,\infty)$ *and* f *be a nonnegative and nonincreasing function defined on* $(0,\infty)$. *Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,\infty)$. *Then for any* $f\in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|H_1 f\|_{L_{q(x),w_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} C_{pq} d_p \left\| \frac{t^{\frac{1}{\underline{p}'}} \left\| \frac{w_2(x)}{x} \right\|_{L_{q(x)}(t,\infty)}}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \quad (2.20)$$

holds, where $(H_1 f)(x)=\frac{1}{x}\int_0^x f(y)dy$ *and*

$$C_{p,q}=\left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)}+\|\chi_{\Delta_2}\|_{L_\infty(0,\infty)}+\underline{p}\left(\frac{1}{\underline{q}}-\frac{1}{\overline{q}}\right)\right)\left(\|\chi_{S_1}\|_{L_\infty(0,\infty)}+\|\chi_{S_2}\|_{L_\infty(0,\infty)}\right),$$

$$S_1=\{x\in(0,\infty):\ p(x)=\underline{p}\},\qquad S_2=(0,\infty)\backslash S_1 \text{ and}$$

$$d_p=\left(1+\frac{\overline{p}-\underline{p}}{\overline{p}}+\|\chi_{S_1}\|_{L_\infty(0,\infty)}\right)^{\frac{1}{\underline{p}}}$$

Theorem 2.3.2. *Let* p, q *be measurable functions on* $(0,\infty)$, $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$, $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x\in(0,1)$ *and* f *be a nonnegative and nondecreasing function defined on* $(0,1)$. *Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,1)$. *Then for any* $f\in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|H_1 f\|_{L_{q(x),w_2(x)}(0,1)} \le$$

$$\underline{p}^{\frac{1}{\underline{p}}} C_{pq} d_p \left\| \left\| \frac{[x-t]^{\frac{1}{\underline{p}'}} w_2(x)}{x} \right\|_{L_{q(x)}(t,1)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,1)}, \quad (2.21)$$

holds, where C_{pq} *and* d_p *are the constants in Theorem 2.3.1.*

Theorem 2.3.3. *Let* p, q *be measurable functions on* $(0,\infty)$, $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$, $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x\in(0,\infty)$ *and* f *be a nonnegative and nonincreasing function defined on* $(0,\infty)$. *Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,\infty)$. *Then for any* $f\in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|H_2 f\|_{L_{q(x),w_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}} C_{pq} d_p$$

$$\times \left\| \left\| \frac{[t-x]^{\frac{1}{\underline{p}'}} w_2(x)}{x} \right\|_{L_{q(x)}(0,t)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \quad (2.22)$$

holds, where $(H_2 f)(x)=\frac{1}{x}\int_x^\infty f(y)dy$ *and* C_{pq} , d_p *are the constants in Theorem 2.3.1.*

The following theorems were proved in [36].

Theorem 2.3.4. *Let* p*,* q *be measurable functions on* $(0,\infty)$*,* $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$*,* $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$*, for* $x\in(0,\infty)$*,* $\beta>-1$ *and* $f\in Q_\beta$*. Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,\infty)$*. Then for any* $f\in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|H_1f\|_{L_{q(x),w_2(x)}(0,\infty)}\le$$

$$\underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{\frac{1}{p'}}C_{pq}d_p\left\|t^{\frac{1}{p'}}\left\|\frac{w_2(x)}{x}\right\|_{L_{q(x)}(t,\infty)}\frac{1}{w_1(x)}\right\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),w_1(x)}(0,\infty)},\qquad(2.23)$$

holds, where C_{pq} *and* d_p *are the constants in Theorem 2.3.1.*

Theorem 2.3.5. *Let* p*,* q *be measurable functions on* $(0,\infty)$*,* $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$*,* $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$*, for* $x\in(0,\infty)$*,* $\beta<-1$ *and* $f\in Q^\beta$*. Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,\infty)$*. Then for any* $f\in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|H_2f\|_{L_{q(x),w_2(x)}(0,\infty)}\le$$

$$\underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{\frac{1}{p'}}C_{pq}d_p\left\|t^{\frac{1}{p'}}\left\|\frac{w_2(x)}{x}\right\|_{L_{q(x)}(0,t)}\frac{1}{w_1(x)}\right\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),w_1(x)}(0,\infty)},\qquad(2.24)$$

holds, where C_{pq} *and* d_p *are the constants in Theorem 2.3.1.*

Theorem 2.3.6. *Let* p*,* q *be measurable functions on* $(0,\infty)$*,* $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$*,* $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$*, for* $x\in(0,1)$*,* $\beta>-1$ *and* $f\in Q^\beta$*. Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,1)$*. Then for any* $f\in L_{p(x),w_1}(0,1)$ *the inequality*

$$\|H_1f\|_{L_{q(x),w_2(x)}(0,1)}\le \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{\frac{1}{p'}}C_{pq}d_p$$

$$\times\left\|\left\|\frac{\left[t^{-\beta}(x^{\beta+1}-t^{\beta+1})\right]^{\frac{1}{p'}}w_2(x)}{x}\right\|_{L_{q(x)}(t,1)}\frac{1}{w_1(x)}\right\|_{L_{r(x)}(0,1)}\|f\|_{L_{p(x),w_1(x)}(0,1)},\qquad(2.25)$$

holds, where C_{pq} *and* d_p *are the constants in Theorem 2.3.1.*

Theorem 2.3.7. *Let* p*,* q *be measurable functions on* $(0,\infty)$*,* $0<\underline{p}\le p(x)\le q(x)\le \overline{q}<1$*,* $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$*, for* $x\in(0,\infty)$ *and* $\beta=-1$*. Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,\infty)$*.*

1. *If* $f\in Q_{-1}$*, then the inequality*

$$\|H_2f\|_{L_{q(x),w_2(x)}(0,\infty)}\le$$

$$\underline{p}^{\frac{1}{\underline{p}}}C_{pq}d_p\left\|\frac{\left\|[t\ln\frac{t}{x}]^{\frac{1}{p'}}\frac{w_2(x)}{x}\right\|_{L_{q(x)}(0,t)}}{w_1(x)}\right\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),w_1(x)}(0,\infty)},\qquad(2.26)$$

holds.

2. *If $f \in Q^{-1}$, then the inequality*

$$\|H_1 f\|_{L_{q(x),w_2(x)}(0,\infty)} \leq$$

$$\underline{p}^{\frac{1}{\underline{p}}} C_{pq} d_p \left\| \frac{\left\| [t \ln \frac{x}{t}]^{\frac{1}{\underline{p}'}} \frac{w_2(x)}{x} \right\|_{L_{q(x)}(t,\infty)}}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{2.27}$$

holds.

C_{pq} and d_p are the constants in Theorem 2.3.1.

2.3.2 Main results

Theorem 2.3.8. *Let p, q be measurable functions on $(0,\infty)$, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta > -1$ and $f \in Q_\beta$. Suppose that w_1 and w_2 are weight functions defined on $(0,\infty)$. Then for any $f \in L_{p(x),w_1}(0,\infty)$ the inequality*

$$\|T_1 f\|_{L_{q(x),w_2(x)}(0,\infty)} \leq M_{pq} \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} (1+M-m)^{\frac{1}{\underline{p}}}$$

$$\times \left\| t^{\frac{1}{\underline{p}'}} \left\| \frac{w_2(b^{-1}(x))}{b^{-1}(x)} \right\|_{L_{q(b^{-1}(x))}(t,\infty)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{2.28}$$

holds, where

$$M_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)\right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right),$$

and

$$q_1 = ess \inf_{x\in(0,\infty)} q(b^{-1}(x)), q_2 = ess \sup_{x\in(0,\infty)} q(b^{-1}(x)), S_1 = \{x \in (0,\infty) : p(x) = \underline{p}\}, S_2 = (0,\infty)/S_1.$$

Proof. *Choose $y = b(x)$, hence $x = b^{-1}(y)$, where $b^{-1}(y)$ is the reciprocal function of $b(y)$ and by using (2.13), where $p = \underline{p}$, we get*

$$\begin{aligned}
\|T_1 f\|_{L_{q(x),w_2(x)}(0,\infty)} &= \left\| \frac{w_2(x)}{x} \int_0^{b(x)} f(t)dt \right\|_{L_{q(x)}(0,\infty)} \\
&= \left\| \frac{w_2(b^{-1}(y))}{b^{-1}(y)} \int_0^y f(t)dt \right\|_{L_{q(b^{-1}(y))}(0,\infty)} \\
&\leq \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{\frac{1-\underline{p}}{\underline{p}}} \left\| \frac{w_2(b^{-1}(y))}{b^{-1}(y)} \left(\int_0^y f^{\underline{p}}(t) t^{\underline{p}-1} dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(b^{-1}(y))}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \frac{w_2(b^{-1}(y))}{b^{-1}(y)} \left(\int_0^y f^{\underline{p}}(t) t^{\underline{p}-1} dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(b^{-1}(y))}(0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \left(\int_0^y f^{\underline{p}}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} t^{\underline{p}-1} dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(b^{-1}(y))}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \int_0^y f^{\underline{p}}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} t^{\underline{p}-1} dt \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \int_0^\infty f^{\underline{p}}(t)\chi_{(0,y)}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} t^{\underline{p}-1} dt \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \left\| f^{\underline{p}}(t)\chi_{(0,y)}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)} .
\end{aligned}
$$

Let $p(x) = 1$, $q(x) = \frac{q(b^{-1}(x))}{\underline{p}}$, *in Lemma 2.3.1 (Inequality (2.18)), thus*

$$
\begin{aligned}
M_{pq} &= \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \frac{1}{\underline{\left(\frac{q(b^{-1}(x))}{\underline{p}}\right)}} - \frac{1}{\overline{\left(\frac{q(b^{-1}(x))}{\underline{p}}\right)}} \right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right) \\
&= \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \left(\frac{\underline{p}}{\underline{q(b^{-1}(x))}} - \frac{\underline{p}}{\overline{q(b^{-1}(x))}} \right) \right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right) \\
&= \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p} \left(\frac{1}{\underline{q(b^{-1}(x))}} - \frac{1}{\overline{q(b^{-1}(x))}} \right) \right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right) \\
&= \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \right) \left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)} \right) .
\end{aligned}
$$

Now applying Lemma 2.3.1, we obtain

$$
\begin{aligned}
&\left\| \left\| f^{\underline{p}}(t)\chi_{(0,y)}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)} \\
&\le M_{pq} \left\| \left\| f^{\underline{p}}(t)\chi_{(0,y)}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_1(0,\infty)} \\
&= M_{pq} \left(\int_0^\infty \left\| f^{\underline{p}}(t)\chi_{(0,y)}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} t^{\underline{p}-1} \right\|_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)} dt \right)^{\frac{1}{\underline{p}}} \\
&= M_{pq} \left(\int_0^\infty f^{\underline{p}}(t) t^{\underline{p}-1} \left\| \chi_{(0,y)}(t) \left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)} \right]^{\underline{p}} \right\|_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)} dt \right)^{\frac{1}{\underline{p}}} .
\end{aligned}
$$

The Fubini Theorem gives

$$\left(\int_0^\infty f^{\underline{p}}(t)t^{\underline{p}-1}\left\|\chi_{(0,y)}(t)\left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)}\right]^{\underline{p}}\right\|_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(0,\infty)}dt\right)^{\frac{1}{\underline{p}}}$$

$$=\left(\int_0^\infty f^{\underline{p}}(t)t^{\underline{p}-1}\left\|\left[\frac{w_2(b^{-1}(y))}{b^{-1}(y)}\right]^{\underline{p}}\right\|_{L_{\frac{q(b^{-1}(y))}{\underline{p}}}(t,\infty)}dt\right)^{\frac{1}{\underline{p}}}$$

$$=\left(\int_0^\infty f^{\underline{p}}(t)t^{\underline{p}-1}\left\|\frac{w_2(b^{-1}(y))}{b^{-1}(y)}\right\|^{\underline{p}}_{L_{q(b^{-1}(y))}(t,\infty)}dt\right)^{\frac{1}{\underline{p}}}$$

$$=\left(\int_0^\infty \left(f(t)t^{\frac{1}{\underline{p}'}}\left\|\frac{w_2(b^{-1}(y))}{b^{-1}(y)}\right\|_{L_{q(b^{-1}(y))}(t,\infty)}\right)^{\underline{p}}dt\right)^{\frac{1}{\underline{p}}}$$

$$=\left\|f(t)t^{\frac{1}{\underline{p}'}}\left\|\frac{w_2(b^{-1}(y))}{b^{-1}(y)}\right\|_{L_{q(b^{-1}(y))}(t,\infty)}\right\|_{L_{\underline{p}}(0,\infty)}.$$

Finally, from Remark 2.3.2, follows that

$$\left\|f(t)t^{\frac{1}{\underline{p}'}}\left\|\frac{w_2(b^{-1}(y))}{b^{-1}(y)}\right\|_{L_{q(b^{-1}(y))}(t,\infty)}\right\|_{L_{\underline{p}}(0,\infty)}$$

$$\leq (1+M-m)^{\frac{1}{\underline{p}}}\left\|t^{\frac{1}{\underline{p}'}}\left\|\frac{w_2(b^{-1}(y))}{b^{-1}(y)}\right\|_{L_{q(b^{-1}(y))}(t,\infty)}\frac{1}{w_1(y)}\right\|_{L_{r(y)}(0,\infty)}\|f\|_{L_{p(y),w_1(x)}(0,\infty)},$$

therefore

$$\|T_1f\|_{L_{q(x),w_2(x)}(0,\infty)}\leq M_{pq}\underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}}(1+M-m)^{\frac{1}{\underline{p}}}$$

$$\times\left\|t^{\frac{1}{\underline{p}'}}\left\|\frac{w_2(b^{-1}(x))}{b^{-1}(x)}\right\|_{L_{q(b^{-1}(x))}(t,\infty)}\frac{1}{w_1(x)}\right\|_{L_{r(x)}(0,\infty)}\|f\|_{L_{p(x),w_1(x)}(0,\infty)}.$$

The following Theorem is proved analogously by applying remark 2.3.1 (inequality (2.14)).

Theorem 2.3.9. *Let* p, q *be measurable functions on* $(0,1)$, $0<\underline{p}\leq p(x)\leq q(x)\leq \overline{q}<1$, $r(x)=\frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x\in(0,1)$, $\beta>-1$ *and* $f\in Q^\beta$. *Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,1)$. *Then for any* $f\in L_{p(x),w_1}(0,1)$ *the inequality*

$$\|T_1f\|_{L_{q(x),w_2(x)}(0,1)}\leq M_{pq}\underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}}(1+M-m)^{\frac{1}{\underline{p}}}$$

$$\times\left\|\left\|\frac{\left[t^{-\beta}(x^{\beta+1}-t^{\beta+1})\right]^{\frac{1}{\underline{p}'}}w_2(b^{-1}(x))}{b^{-1}(x)}\right\|_{L_{q(b^{-1}(x))}(t,b(1))}\frac{1}{w_1(x)}\right\|_{L_{r(x)}(0,b(1))}\|f\|_{L_{p(x),w_1(x)}(0,b(1))},\tag{2.29}$$

holds, where M_{pq}, M *and* m *are the constants in Theorem 2.3.8.*

Remark 2.3.3. *If we put $b(x) = x$ in Theorem 2.3.8 and Theorem 2.3.9, we get Theorem 2.3.4 and Theorem 2.3.6 respectively, where d_p is replaced by improved constants (see remark 2.3.2) .*

If we put $\beta = 0$ in Theorem 2.3.8 and Theorem 2.3.9, we get the following Corollaries respectively.

Corollary 2.3.2. *Let $x \in (0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ and f be a nonnegative and nonincreasing function defined on $(0,\infty)$. Suppose that w_1 and w_2 are weight functions defined on $(0,\infty)$. Then for any $f \in L_{p(x),w_1}(0,\infty)$ the inequality*

$$\|T_1 f\|_{L_{q(x),w_2(x)}(0,\infty)} \le M_{pq}\underline{p}^{\frac{1}{\underline{p}}}(1+M-m)^{\frac{1}{\underline{p}}}$$

$$\times \left\| t^{\frac{1}{\underline{p}'}} \left\| \frac{w_2(b^{-1}(x))}{b^{-1}(x)} \right\|_{L_{q(b^{-1}(x))}(t,\infty)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{2.30}$$

holds, where M_{pq}, M and m are the constants in Theorem 2.3.8.

Corollary 2.3.3. *Let $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ and f be a nonnegative and nondecreasing function defined on $(0,1)$. Suppose that w_1 and w_2 are weight functions defined on $(0,1)$. Then for any $f \in L_{p(x),w_1}(0,1)$ the inequality*

$$\|T_1 f\|_{L_{q(x),w_2(x)}(0,1)} \le M_{pq}\underline{p}^{\frac{1}{\underline{p}}}\big(1+M-m\big)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \left\| (x-t)^{\frac{1}{\underline{p}'}} \left[\frac{w_2(b^{-1}(x))}{b^{-1}(x)} \right] \right\|_{L_{q(b^{-1}(x))}(t,b(1))} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,b(1))} \|f\|_{L_{p(x),w_1(x)}(0,b(1))}, \tag{2.31}$$

holds, where M_{pq}, M and m are the constants in Theorem 2.3.8.

Remark 2.3.4. *If we set $b(x) = x$ in Corollary 2.3.2 and Corollary 2.3.3, we get Theorem 2.3.1 and Theorem 2.3.2 respectively, where d_p is replaced by improved constants (see remark 2.3.2).*

Theorem 2.3.10. *Let p, q be measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, for $x \in (0,\infty)$, $\beta < -1$ and $f \in Q^\beta$. Suppose that w_1 and w_2 are weight functions defined on $(0,\infty)$. Then for any $f \in L_{p(x),w_1}(0,\infty)$ the inequality*

$$\|T_2 f\|_{L_{q(x),w_2(x)}(0,\infty)} \le \underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{-\frac{1}{\underline{p}'}} N_{pq}\big(1+M-m\big)^{\frac{1}{\underline{p}}}$$

$$\times \left\| t^{\frac{1}{\underline{p}'}} \left\| \frac{w_2(a^{-1}(x))}{a^{-1}(x)} \right\|_{L_{q(a^{-1}(x))}(0,t)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{2.32}$$

holds, where M and m are the constants in Theorem 2.3.8 and

$$N_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p}\left(\frac{1}{q_3} - \frac{1}{q_4}\right)\right)\left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)}\right),$$

and

$$q_3 = ess \inf_{x\in(0,\infty)} q(a^{-1}(x)), q_4 = ess \sup_{x\in(0,\infty)} q(a^{-1}(x)), S_1 = \{x \in (0,\infty) : p(x) = \underline{p}\}, S_2 = (0,\infty)/S_1.$$

Proof. Choose $y = a(x)$, hence $x = a^{-1}(y)$, where $a^{-1}(y)$ is the reciprocal function of $a(y)$ and by using (2.15) where $p = \underline{p}$, we get

$$\begin{aligned}
\|T_2 f\|_{L_{q(x),w_2(x)}(0,\infty)} &= \left\|\frac{w_2(x)}{x}\int_{a(x)}^{\infty} f(t)dt\right\|_{L_{q(x)}(0,\infty)} \\
&= \left\|\frac{w_2(a^{-1}(y))}{a^{-1}(y)}\int_{y}^{\infty} f(t)dt\right\|_{L_{q(a^{-1}(y))}(0,\infty)} \\
&\le \underline{p}^{\frac{1}{\underline{p}}}\,|\beta+1|^{\frac{1-\underline{p}}{\underline{p}}}\left\|\frac{w_2(a^{-1}(y))}{a^{-1}(y)}\left(\int_{y}^{\infty} f^{\underline{p}}(t)t^{\underline{p}-1}dt\right)^{\frac{1}{\underline{p}}}\right\|_{L_{q(a^{-1}(y))}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{-\frac{1}{\underline{p}'}}\left\|\left(\int_{y}^{\infty} f^{\underline{p}}(t)\left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)}\right]^{\underline{p}} t^{\underline{p}-1}dt\right)^{\frac{1}{\underline{p}}}\right\|_{L_{q(a^{-1}(y))}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}}\,|\beta+1|^{-\frac{1}{\underline{p}'}}\left\|\int_{y}^{\infty} f^{\underline{p}}(t)\left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)}\right]^{\underline{p}} t^{\underline{p}-1}dt\right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(a^{-1}(y))}{\underline{p}}}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{-\frac{1}{\underline{p}'}}\left\|\int_{0}^{\infty} f^{\underline{p}}(t)\chi_{(y,\infty)}(t)\left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)}\right]^{\underline{p}} t^{\underline{p}-1}dt\right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(a^{-1}(y))}{\underline{p}}}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}}|\beta+1|^{-\frac{1}{\underline{p}'}}\left\|\left\|f^{\underline{p}}(t)\chi_{(y,\infty)}(t)\left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)}\right]^{\underline{p}} t^{\underline{p}-1}\right\|_{L_1(0,\infty)}\right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(a^{-1}(y))}{\underline{p}}}(0,\infty)}.
\end{aligned}$$

The rest is similar to the proof of Theorem 2.3.8. □

Theorem 2.3.11. *Let* $x \in (0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \bar{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ *and* f *be a nonnegative and nonincreasing function defined on* $(0,\infty)$. *Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,\infty)$. *Then for any* $f \in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|T_2 f\|_{L_{q(x),w_2(x)}(0,\infty)} \le N_{pq}\underline{p}^{\frac{1}{\underline{p}}}(1+M-m)^{\frac{1}{\underline{p}}}$$

$$\times \left\|\left\|\frac{(t-x)^{\frac{1}{\underline{p}'}} w_2(a^{-1}(x))}{a^{-1}(x)}\right\|_{L_{q(a^{-1}(x))}(0,t)} \frac{1}{w_1(x)}\right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \qquad (2.33)$$

holds, where N_{pq}, M *and* m *are the constants in Theorem 2.3.10.*

Proof. *Let $a = y$, $b = \infty$ and $\beta = 0$ in (2.9), then*

$$\int_y^\infty f(t)dt \le p^{\frac{1}{p}} \left(\int_y^\infty (t-y)^{p-1} f^p(t)dt \right)^{\frac{1}{p}}. \tag{2.34}$$

We apply inequality (2.34) with $p = \underline{p}$ and the rest is similar to the proof of Theorem 2.3.8.

Remark 2.3.5. *If we put $a(x) = x$ in Theorem 2.3.10 and Theorem 2.3.11, we get Theorem 2.3.5 and Theorem 2.3.3 respectively, where d_p is replaced by improved constants (see remark 2.3.2).*

Now we consider the case $\beta = -1$.

Theorem 2.3.12. *Let p, q be measurable functions on $(0,\infty)$, $0 < \underline{p} \le p(x) \le q(x) \le \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ for $x \in (0,\infty)$ and $\beta = -1$. Suppose that w_1 and w_2 are weight functions defined on $(0,\infty)$.*

1. *If $f \in Q_{-1}$, then the inequality*

$$\|T_2 f\|_{L_{q(x),w_2(x)}(0,\infty)} \le N_{pq}\underline{p}^{\frac{1}{\underline{p}}}(1+M-m)^{\frac{1}{\underline{p}}}$$
$$\times \left\| \left\| \frac{\left(t\ln\frac{t}{x}\right)^{\frac{1}{p'}} w_2(a^{-1}(x))}{a^{-1}(x)} \right\|_{L_{q(a^{-1}(x))}(0,t)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{2.35}$$

   holds , where N_{pq}, M and m are the constants in Theorem 2.3.10.

2. *If $f \in Q^{-1}$, then the inequality*

$$\|T_1 f\|_{L_{q(x),w_2(x)}(0,\infty)} \le M_{pq}\underline{p}^{\frac{1}{\underline{p}}}(1+M-m)^{\frac{1}{\underline{p}}}$$
$$\times \left\| \left\| \frac{\left(t\ln\frac{x}{t}\right)^{\frac{1}{p'}} w_2(b^{-1}(x))}{b^{-1}(x)} \right\|_{L_{q(b^{-1}(x))}(t,\infty)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{2.36}$$

   holds, where M_{pq}, M and m are the constants in Theorem 2.3.8.

Proof. 1. *Let $a = y$ and $b = \infty$ in (2.10), then*

$$\int_y^\infty f(t)dt \le p^{\frac{1}{p}} \left(\int_y^\infty \left(t\ln\frac{t}{y}\right)^{p-1} f^p(t)dt \right)^{\frac{1}{p}} \tag{2.37}$$

Choose $y = a(x)$, hence $x = a^{-1}(y)$, where $a^{-1}(y)$ is the reciprocal function of $a(y)$ and by using (2.37), where $p = \underline{p}$, we get

$$\begin{aligned}
\|T_2 f\|_{L_{q(x),w_2(x)}(0,\infty)} &= \left\| \frac{w_2(x)}{x} \int_{a(x)}^{\infty} f(t)dt \right\|_{L_{q(x)}(0,\infty)} \\
&= \left\| \frac{w_2(a^{-1}(y))}{a^{-1}(y)} \int_{y}^{\infty} f(t)dt \right\|_{L_{q(a^{-1}(y))}(0,\infty)} \\
&\le \underline{p}^{\frac{1}{\underline{p}}} \left\| \frac{w_2(a^{-1}(y))}{a^{-1}(y)} \left(\int_y^{\infty} f^{\underline{p}}(t) \left(t \ln \frac{t}{y} \right)^{\underline{p}-1} dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(a^{-1}(y))}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}} \left\| \left(\int_y^{\infty} f^{\underline{p}}(t) \left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)} \right]^{\underline{p}} \left(t \ln \frac{t}{y} \right)^{\underline{p}-1} dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(a^{-1}(y))}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}} \left\| \int_y^{\infty} f^{\underline{p}}(t) \left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)} \right]^{\underline{p}} \left(t \ln \frac{t}{y} \right)^{\underline{p}-1} dt \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(a^{-1}(y))}{\underline{p}}}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}} \left\| \int_0^{\infty} f^{\underline{p}}(t) \chi_{(y,\infty)}(t) \left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)} \right]^{\underline{p}} \left(t \ln \frac{t}{y} \right)^{\underline{p}-1} dt \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(a^{-1}(y))}{\underline{p}}}(0,\infty)} \\
&= \underline{p}^{\frac{1}{\underline{p}}} \left\| \left\| f^{\underline{p}}(t) \chi_{(y,\infty)}(t) \left[\frac{w_2(a^{-1}(y))}{a^{-1}(y)} \right]^{\underline{p}} \left(t \ln \frac{t}{y} \right)^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(a^{-1}(y))}{\underline{p}}}(0,\infty)} .
\end{aligned}$$

The rest is similar to the proof of Theorem 2.3.8.

2. *Let $a = 0$ and $b = y$ in (2.12), then*

$$\int_0^y f(t)dt \le p^{\frac{1}{p}} \left(\int_0^y \left(t \ln \frac{y}{t} \right)^{p-1} f^p(t)dt \right)^{\frac{1}{p}} . \tag{2.38}$$

Finally we apply (2.38) with $p = \underline{p}$ and the rest is similar to the proof of Theorem 2.3.8.

Remark 2.3.6. *If we put $a(x) = x$ and $b(x) = x$ in (3.35) and (3.36), we get inequalities (2.26) and (2.27) of Theorem 2.3.7, respectively, where d_p is replaced by improved constants (see remark 2.3.2).*

Chapter 3

Some integral inequalities for Hardy-Steklov operator for quasi-monotone functions with $0 < p(x) < 1$

3.1 Introduction

In this chapter, we study the Hardy–Steklov operator in variable exponent Lebesgue $L_{p(x)}$ with $0 < p(x) < 1$. The key idea is to obtain analogues of Lemma 2.1 and Proposition 5.1 of [7] where a and b are constants to boundary functions $a(x)$ and $b(x)$. Consequently by applying these results, we get some new integral inequalities for Hardy-Steklov operator in weighted variable exponent Lebesgue spaces, for nonnegative quasi-monotone and monotone functions with $0 < p(x) < 1$. (Submitted work (see [17])).

The following Corollary and Lemma were established in [2].

Corollary 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and p, q be measurable functions on Ω, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < \infty$ and $r(x) = \frac{p(x)q(x)}{q(x)-p(x)}$. Suppose that w_1 and w_2 are weight functions defined in Ω satisfying the condition*

$$\left\| \frac{w_1}{w_2} \right\|_{L_{r(x)}(\Omega)} < \infty.$$

Then the inequality

$$\|f\|_{L_{p(x),w_1}(\Omega)} \leq \left(A + B + \|\chi_{\Omega_2}\|_{L_\infty(\Omega)}\right)^{\frac{1}{\underline{p}}} \left\| \frac{w_1}{w_2} \right\|_{L_{r(x)}(\Omega)} \|f\|_{L_{q(x),w_2}(\Omega)}, \tag{3.1}$$

holds for every $f \in L_{q(x),w_2(x)}(\Omega)$, *where*

$$\Omega_1 = \{x \in \Omega : \ p(x) < q(x)\}, \qquad \Omega_2 = \{x \in \Omega : \ p(x) = q(x)\},$$

$$A = \sup_{x\in\Omega_1} \frac{p(x)}{q(x)}, \ B = \sup_{x\in\Omega_1} \frac{q(x)-p(x)}{q(x)}.$$

If $|\Omega_2| = 0$, *the constant in (3.1) is sharp. If* $|\Omega_2| > 0$, *then it is not sharp.*

Remark 3.1.1. *The improvement of the constant in (3.1) was obtained in [12].*

$$\|f\|_{L_{p(x),w_1}(\Omega)} \le (1+M-m)^{\frac{1}{\underline{p}}} \left\| \frac{w_1}{w_2} \right\|_{L_{r(x)}(\Omega)} \|f\|_{L_{q(x),w_2}(\Omega)}, \tag{3.2}$$

where

$$M = ess\sup_{x\in\Omega} \frac{p(x)}{q(x)}, \ m = ess\inf_{x\in\Omega} \frac{p(x)}{q(x)}, \underline{p} = ess\inf_{x\in\Omega} p(x).$$

The constant in (3.2) is sharp for any measurable set Ω.

Lemma 3.1.1. *Let* $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ *be measurable sets,* p *be a measurable function on* Ω_1 *and* q *be a measurable function on* Ω_2, $1 \le \underline{p} \le p(x) \le q(y) \le \overline{q} < \infty$ *for all* $x \in \Omega_1 \subset \mathbb{R}^n$ *and* $y \in \Omega_2 \subset \mathbb{R}^m$. *If* $p \in C(\Omega_1)$, $q \in C(\Omega_2)$, *then the inequality*

$$\left\| \|f\|_{L_{p(x)}(\Omega_1)} \right\|_{L_{q(x)}(\Omega_2)} \le C_{p,q} \left\| \|f\|_{L_{q(x)}(\Omega_2)} \right\|_{L_{p(x)}(\Omega_1)}, \tag{3.3}$$

is valid, where

$$C_{p,q} = \left(\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + \frac{\overline{p}}{\underline{q}} - \frac{\underline{p}}{\overline{q}} \right) (\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty), \tag{3.4}$$

$\underline{q} = \text{ess}\inf_{\Omega_2} q(x), \overline{q} = \text{ess}\sup_{\Omega_2} q(x), \Delta_1 = \{(x,y) \in \Omega_1 \times \Omega_2; \ p(x) = q(y)\}, \Delta_2 = (\Omega_1 \times \Omega_2) \backslash \Delta_1$, $C(\Omega_1)$, $C(\Omega_2)$ *are the spaces of all continuous functions in* Ω_1, Ω_2, *respectively and* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *is any measurable function such that* $\left\| \|f\|_{L_{q(x)}(\Omega_2)} \right\|_{L_{p(x)}(\Omega_1)} < \infty$.

Both the definition and the statement that are presented below were given in [7].

Definition 3.1.1. *We say that a nonnegative function* f *is quasi-monotone on* $(0,\infty)$, *if for some real number* α, $x^\alpha f(x)$ *is a decreasing or an increasing function of* x. *More precisely, given* $\beta \in \mathbb{R}$ *we say that* $f \in Q_\beta$ *if* $x^{-\beta} f(x)$ *is nonincreasing and* $f \in Q^\beta$ *if* $x^{-\beta} f(x)$ *is nondecreasing (see [7]).*

Lemma 3.1.2. *(a) Let* $-\infty < a < b \le +\infty$ *and assume that the function* f *is nonnegative and nonincreasing on the interval* (a,b). *If* $0 < p \le 1$, *then*

$$\left(\int_a^b f(t)dt \right)^p \le p \int_a^b f^p(t)(t-a)^{p-1}dt. \tag{3.5}$$

(b) *Let* $-\infty < a < b \leq +\infty$ *and* f *be a function that is nonnegative and nondecreasing on the interval* (a,b)*. If* $0 < p \leq 1$*, then*

$$\left(\int_a^b f(t)dt\right)^p \leq p\int_a^b f^p(t)(b-t)^{p-1}dt. \tag{3.6}$$

(c) *The constant* p *is sharp in these inequalities.*

Proposition 3.1.1. *(a) Let* $-\infty < \beta < +\infty$ *,* $f \in Q_\beta$*,* $0 \leq a < b \leq \infty$ *for* $\beta > -1$ *and* $0 < a < b \leq \infty$ *for* $\beta \leq -1$*. If* $0 < p \leq 1$ *and* $\beta \neq -1$*, then*

$$\left(\int_a^b f(t)dt\right)^p \leq p|\beta+1|^{1-p}\int_a^b \left(\frac{|t^{\beta+1}-a^{\beta+1}|}{t^\beta}\right)^{p-1} f^p(t)dt. \tag{3.7}$$

If $0 < p \leq 1$ *and* $\beta = -1$*, then*

$$\left(\int_a^b f(t)dt\right)^p \leq p\int_a^b \left(t\ln\frac{t}{a}\right)^{p-1} f^p(t)dt. \tag{3.8}$$

The inequalities hold in the reversed direction if $1 \leq p < \infty$.

(b) *Let* $-\infty < \beta < +\infty$ *,* $f \in Q^\beta$*,* $0 \leq a < b \leq \infty$ *for* $\beta < -1$ *and* $0 \leq a < b < \infty$ *for* $\beta \geq -1$*. For* $0 < p \leq 1$ *and* $\beta \neq -1$*, the following is valid:*

$$\left(\int_a^b f(t)dt\right)^p \leq p|\beta+1|^{1-p}\int_a^b \left(\frac{|t^{\beta+1}-b^{\beta+1}|}{t^\beta}\right)^{p-1} f^p(t)dt. \tag{3.9}$$

Assuming that $\beta = -1$ *and* $0 < p \leq 1$*, the following is obtained:*

$$\left(\int_a^b f(t)dt\right)^p \leq p\int_a^b \left(t\ln\frac{b}{t}\right)^{p-1} f^p(t)dt. \tag{3.10}$$

The inequalities hold in the reversed direction if $1 \leq p < \infty$.

(c) *The constants in these inequalities are the best possible in all cases.*

3.2 Main results

The Hardy-Steklov operator (see [25] for more details) is defined as

$$(Tf)(x) = \frac{1}{x}\int_{a(x)}^{b(x)} f(y)dy,$$

where f is a nonnegative measurable function defined on the interval $(0,\infty)$, with boundary functions $a(x)$ and $b(x)$ that satisfy the following condition.

1. $a(x)$, $b(x)$ are differentiable and increasing functions on $(0, \infty)$.
2. $0 < a(x) < b(x) < \infty$ for $0 < x < \infty$ and $a(0) = b(0) = 0$, $a(\infty) = b(\infty) = \infty$.

We assume that the functions $a(x)$ and $b(x)$ in Lemmas 3.2.1 and 3.2.2 satisfy the above conditions.
If we replace in Lemma 3.1.2 the constant a and b by $a(x)$ and $b(x)$ respectively, we get the following Lemma.

Lemma 3.2.1. *(a) Let* $0 < p \leq 1$, $0 \leq a(x) < b(x) \leq +\infty$ *and* f *be a nonnegative and nonincreasing function defined on* $(a(x), b(x))$*, then*

$$\left(\int_{a(x)}^{b(x)} f(t)dt \right)^p \leq p \int_{a(x)}^{b(x)} (t - a(x))^{p-1} f^p(t)dt. \tag{3.11}$$

(b) Let $0 < p \leq 1$,$0 \leq a(x) < b(x) < +\infty$ *and* f *be a nonnegative and nondecreasing function defined on* $(a(x), b(x))$*, then*

$$\left(\int_{a(x)}^{b(x)} f(t)dt \right)^p \leq p \int_{a(x)}^{b(x)} (b(x) - t)^{p-1} f^p(t)dt. \tag{3.12}$$

Lemma 3.2.2. *(a) Let* $-\infty < \beta < +\infty$,$f \in Q_\beta$, $0 \leq a(x) < b(x) \leq \infty$ *for* $\beta > -1$ *and* $0 < a(x) < b(x) \leq \infty$ *for* $\beta \leq -1$*. If* $0 < p \leq 1$ *and* $\beta \neq -1$*, then*

$$\left(\int_{a(x)}^{b(x)} f(y)dy \right)^p \leq p|\beta + 1|^{1-p} \int_{a(x)}^{b(x)} \left(\frac{\left| y^{\beta+1} - (a(x))^{\beta+1} \right|}{y^\beta} \right)^{p-1} f^p(y)dy. \tag{3.13}$$

If $0 < p \leq 1$ *and* $\beta = -1$*, then*

$$\left(\int_{a(x)}^{b(x)} f(y)dy \right)^p \leq p \int_{a(x)}^{b(x)} \left(y \ln \left[\frac{y}{a(x)} \right] \right)^{p-1} f^p(y)dy. \tag{3.14}$$

(b) Let $-\infty < \beta < +\infty$,$f \in Q^\beta$, $0 \leq a(x) < b(x) \leq \infty$ *for* $\beta < -1$ *and* $0 \leq a(x) < b(x) < \infty$ *for* $\beta \geq -1$*. If* $0 < p \leq 1$ *and* $\beta \neq -1$*, then*

$$\left(\int_{a(x)}^{b(x)} f(y)dy \right)^p \leq p|\beta + 1|^{1-p} \int_{a(x)}^{b(x)} \left(\frac{\left| y^{\beta+1} - (b(x))^{\beta+1} \right|}{y^\beta} \right)^{p-1} f^p(y)dy. \tag{3.15}$$

If $0 < p \leq 1$ *and* $\beta = -1$*, then*

$$\left(\int_{a(x)}^{b(x)} f(y)dy \right)^p \leq p \int_{a(x)}^{b(x)} \left(y \ln \left[\frac{b(x)}{y} \right] \right)^{p-1} f^p(y)dy. \tag{3.16}$$

Proof. *(a) Let* $0 < p \leq 1$, $f \in Q_\beta$ *and* $h(y) = y^{-\beta} f(y)$.

(1) *If* $\beta > -1$ *and* $0 \leq a(x) < b(x) \leq \infty$.
Choose $t = y^{\beta+1}$*, hence* $y = t^{\frac{1}{\beta+1}}$ *and* $t \in \left([a(x)]^{\beta+1}, [b(x)]^{\beta+1}\right)$.

$$\begin{aligned}\left(\int_{a(x)}^{b(x)} f(y)dy\right)^p &= \left(\int_{a(x)}^{b(x)} y^\beta h(y)dy\right)^p \\ &= \left(\int_{[a(x)]^{\beta+1}}^{[b(x)]^{\beta+1}} h\left(t^{\frac{1}{\beta+1}}\right)\frac{dt}{(\beta+1)}\right)^p \\ &= (\beta+1)^{-p}\left(\int_{[a(x)]^{\beta+1}}^{[b(x)]^{\beta+1}} h\left(t^{\frac{1}{\beta+1}}\right)dt\right)^p.\end{aligned}$$

Since $f \in Q_\beta$ *and* $\beta > -1$*, then* $h\left(t^{\frac{1}{\beta+1}}\right)$ *is nonincreasing. By applying Lemma 3.2.1 (a), we get*

$$\begin{aligned}\left(\int_{[a(x)]^{\beta+1}}^{[b(x)]^{\beta+1}} h\left(t^{\frac{1}{\beta+1}}\right)dt\right)^p &\leq p\int_{[a(x)]^{\beta+1}}^{[b(x)]^{\beta+1}} \left(t-[a(x)]^{\beta+1}\right)^{p-1} h^p\left(t^{\frac{1}{\beta+1}}\right)dt \\ &= p\int_{a(x)}^{b(x)} \left(y^{\beta+1}-[a(x)]^{\beta+1}\right)^{p-1} h^p(y)(\beta+1)y^\beta dy \\ &= p(\beta+1)\int_{a(x)}^{b(x)} \left(y^{\beta+1}-[a(x)]^{\beta+1}\right)^{p-1} y^{-\beta p} f^p(y) y^\beta dy \\ &= p(\beta+1)\int_{a(x)}^{b(x)} \left(y^{\beta+1}-[a(x)]^{\beta+1}\right)^{p-1}\left(y^{-\beta}\right)^{p-1} f^p(y)dy \\ &= p(\beta+1)\int_{a(x)}^{b(x)} \left(\frac{y^{\beta+1}-[a(x)]^{\beta+1}}{y^\beta}\right)^{p-1} f^p(y)dy.\end{aligned}$$

Finally, we find inequality (3.13).

(2) *If* $\beta < -1$, $0 < a(x) < b(x) \leq \infty$ *and* $f \in Q_\beta$*, then* $h\left(t^{\frac{1}{\beta+1}}\right)$ *is nondecreasing. By applying Lemma 3.2.1 (b), we get inequality (3.13). The proof is similar to that of the case (a)(1).*

(3) *If* $\beta = -1$, $0 < a(x) < b(x) \leq \infty$. *Choose* $t = \ln y$*, hence* $y = e^t$ *and* $t \in \left(\ln[a(x)], \ln[b(x)]\right)$*, then*

$$\begin{aligned}\left(\int_{a(x)}^{b(x)} f(y)dy\right)^p &= \left(\int_{a(x)}^{b(x)} h(y)\frac{dy}{y}\right)^p \\ &= \left(\int_{\ln[a(x)]}^{\ln[b(x)]} h(e^t)dt\right)^p.\end{aligned}$$

Since $f \in Q_{-1}$ *then* $h(e^t)$ *is nonincreasing. By using Lemma 3.2.1 (a),* *we get*

$$\begin{aligned}\left(\int_{a(x)}^{b(x)} f(y)dy\right)^p &= \left(\int_{\ln[a(x)]}^{\ln[b(x)]} h(e^t)dt\right)^p \\ &\le p\int_{\ln[a(x)]}^{\ln[b(x)]} (t-\ln[a(x)])^{p-1} h^p(e^t)dt \\ &= p\int_{a(x)}^{b(x)} (\ln y - \ln[a(x)])^{p-1} h^p(y)\frac{dy}{y} \\ &= p\int_{a(x)}^{b(x)} \left(\ln\left[\frac{y}{a(x)}\right]\right)^{p-1} y^{p-1}f^p(y)dy \\ &= p\int_{a(x)}^{b(x)} \left(y\ln\left[\frac{y}{a(x)}\right]\right)^{p-1} f^p(y)dy.\end{aligned}$$

(b) Let $0 < p \le 1$*,* $f \in Q^\beta$ *and* $h(y) = y^{-\beta}f(y)$*.*

(1) If $\beta < -1$ *and* $0 \le a(x) < b(x) \le \infty$ *.*
Choose $t = y^{\beta+1}$*, hence* $y = t^{\frac{1}{\beta+1}}$ *and* $t \in \left([b(x)]^{\beta+1}, [a(x)]^{\beta+1}\right)$*.*

$$\begin{aligned}\left(\int_{a(x)}^{b(x)} f(y)dy\right)^p &= \left(\int_{a(x)}^{b(x)} y^\beta h(y)dy\right)^p \\ &= \left(\int_{[b(x)]^{\beta+1}}^{[a(x)]^{\beta+1}} h\left(t^{\frac{1}{\beta+1}}\right)\frac{dt}{|\beta+1|}\right)^p \\ &= |\beta+1|^{-p}\left(\int_{[b(x)]^{\beta+1}}^{[a(x)]^{\beta+1}} h\left(t^{\frac{1}{\beta+1}}\right)dt\right)^p.\end{aligned}$$

Since $f \in Q^\beta$ *and* $\beta < -1$*, then* $h\left(t^{\frac{1}{\beta+1}}\right)$ *is nonincreasing. By appling Lemma 3.2.1 (a), we get*

$$\left(\int_{[b(x)]^{\beta+1}}^{[a(x)]^{\beta+1}} h\left(t^{\frac{1}{\beta+1}}\right)dt\right)^p \le p\int_{[b(x)]^{\beta+1}}^{[a(x)]^{\beta+1}} \left(t-[b(x)]^{\beta+1}\right)^{p-1} h^p\left(t^{\frac{1}{\beta+1}}\right)dt.$$

Choose $y = t^{\frac{1}{\beta+1}}$*, hence* $t = y^{\beta+1}$ *and* $y \in \left(a(x), b(x)\right)$*.*

$$\begin{aligned}\left(\int_{[b(x)]^{\beta+1}}^{[a(x)]^{\beta+1}} h(t^{\frac{1}{\beta+1}})dt\right)^p &\le p\int_{[b(x)]^{\beta+1}}^{[a(x)]^{\beta+1}} \left(t-[b(x)]^{\beta+1}\right)^{p-1} h^p\left(t^{\frac{1}{\beta+1}}\right)dt \\ &= p\int_{a(x)}^{b(x)} \left(y^{\beta+1}-[b(x)]^{\beta+1}\right)^{p-1} h^p(y)y^\beta|\beta+1|dy \\ &= p|\beta+1|\int_{a(x)}^{b(x)} \left(y^{\beta+1}-[b(x)]^{\beta+1}\right)^{p-1} y^{-\beta p}f^p(y)y^\beta dy \\ &= p|\beta+1|\int_{a(x)}^{b(x)} \left(y^{\beta+1}-[b(x)]^{\beta+1}\right)^{p-1}\left(y^{-\beta}\right)^{p-1} f^p(y)dy \\ &= p|\beta+1|\int_{a(x)}^{b(x)} \left(\frac{y^{\beta+1}-[b(x)]^{\beta+1}}{y^\beta}\right)^{p-1} f^p(y)dy.\end{aligned}$$

Finally

$$\left(\int_{a(x)}^{b(x)} f(y)dy\right)^p \leq p|\beta+1|^{1-p} \int_{a(x)}^{b(x)} \left(\frac{\left|y^{\beta+1}-[b(x)]^{\beta+1}\right|}{y^\beta}\right)^{p-1} f^p(y)dy.$$

(2) *If* $\beta > -1$ *and* $0 \leq a(x) < b(x) < \infty$. *Since* $f \in Q^\beta$ *and* $\beta > -1$, *then* $h\left(t^{\frac{1}{\beta+1}}\right)$ *is nondecreasing. By applying Lemma 3.2.1 (b). The rest is similar to that of the case (a) (2), consequently we get inequality (3.15).*

(3) *If* $\beta = -1$, $0 \leq a(x) < b(x) < \infty$, *similarly to that of the case (a)(3), by applying Lemma 3.2.1(b), we get inequality (3.16).*

Theorem 3.2.1. *Let* p, q *be measurable functions on* $(0,\infty)$, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$, *for* $x \in (0,\infty)$, $\beta > -1$ *and* $f \in Q_\beta$. *Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,\infty)$. *Then for any* $f \in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,\infty)} \leq \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{\frac{1}{\underline{p}'}} c_{pq}\left(1+M-m\right)^{\frac{1}{\underline{p}}}$$

$$\times \left\|\left\|\frac{\left[t^{-\beta}(t^{\beta+1}-(a(x))^{\beta+1})\right]^{\frac{1}{\underline{p}'}} w_2(x)}{x}\right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \frac{1}{w_1(x)}\right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{3.17}$$

holds, where $\underline{p}'$ *is the conjugate of* $\underline{p}$,

$$c_{p,q} = \left(\|\chi_{\Delta_1}\|_{L_\infty(0,\infty)} + \|\chi_{\Delta_2}\|_{L_\infty(0,\infty)} + \underline{p}\left(\frac{1}{\underline{q}}-\frac{1}{\overline{q}}\right)\right)\left(\|\chi_{S_1}\|_{L_\infty(0,\infty)} + \|\chi_{S_2}\|_{L_\infty(0,\infty)}\right),$$

and

$$S_1 = \{x \in (0,\infty):\ p(x) = \underline{p}\},\quad S_2 = (0,\infty)\backslash S_1,$$

$$M = ess\sup_{x\in\Omega}\frac{p(x)}{q(x)},\quad m = ess\inf_{x\in\Omega}\frac{p(x)}{q(x)},\quad \underline{p} = ess\inf_{x\in\Omega} p(x).$$

Proof. *By applying Lemma 3.2.2 (a) (inequality (3.13) with* $p = \underline{p}$ *), we have*

$$\begin{aligned}\|Tf\|_{L_{q(x),w_2(x)}(0,\infty)} &= \|w_2(x)(Tf)(x)\|_{L_{q(x)}(0,\infty)} \\ &= \left\|\frac{w_2(x)}{x}\int_{a(x)}^{b(x)} f(t)dt\right\|_{L_{q(x)}(0,\infty)} \\ &\leq \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{\frac{1-\underline{p}}{\underline{p}}}\left\|\frac{w_2(x)}{x}\left(\int_{a(x)}^{b(x)}\left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} f^{\underline{p}}(t)dt\right)^{\frac{1}{\underline{p}}}\right\|_{L_{q(x)}(0,\infty)}\end{aligned}$$

$$= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \left(\int_{a(x)}^{b(x)} \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} f^{\underline{p}}(t)dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}$$

$$= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \int_{a(x)}^{b(x)} \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} f^{\underline{p}}(t)dt \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}$$

$$= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \int_{0}^{\infty} f^{\underline{p}}(t)\chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} dt \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}$$

$$= \underline{p}^{\frac{1}{\underline{p}}}(\beta+1)^{-\frac{1}{\underline{p}'}} \left\| \left\| f^{\underline{p}}(t)\chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}.$$

Now by using Lemma 3.1.1, we get

$$\left\| \left\| f^{\underline{p}}(t)\chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}$$

$$\leq c_{pq} \left\| \left\| f^{\underline{p}}(t)\chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} \right\|^{\frac{1}{\underline{p}}}_{L_1(0,\infty)}$$

$$= c_{pq} \left(\int_0^\infty \left\| f^{\underline{p}}(t)\chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dt \right)^{\frac{1}{\underline{p}}}$$

$$= c_{pq} \left(\int_0^\infty f^{\underline{p}}(t) \left\| \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} \chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dt \right)^{\frac{1}{\underline{p}}}.$$

Since $a(x) \leq t \leq b(x)$*, thus* $b^{-1}(t) \leq x \leq a^{-1}(t)$*, where* $a^{-1}(t)$ *and* $b^{-1}(t)$ *are the inverses to the boundary functions* $a(t)$ *and* $b(t)$*, respectively. By applying Fubini's theorem (see [25]), we get*

$$\left(\int_0^\infty f^{\underline{p}}(t) \left\| \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} \chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)} dt \right)^{\frac{1}{\underline{p}}}$$

$$= \left(\int_0^\infty f^{\underline{p}}(t) \left\| \left[t^{-\beta}\left(t^{\beta+1}-(a(x))^{\beta+1}\right)\right]^{\underline{p}-1} \left[\frac{w_2(x)}{x}\right]^{\underline{p}} \right\|_{L_{\frac{q(x)}{\underline{p}}}(b^{-1}(t)\ ,\ a^{-1}(t))} dt \right)^{\frac{1}{\underline{p}}}$$

$$= \left(\int_0^\infty f^{\underline{p}}(t) \left\| \left(\left[t^{-\beta} \left(t^{\beta+1} - (a(x))^{\beta+1} \right) \right]^{\frac{\underline{p}-1}{\underline{p}}} \left[\frac{w_2(x)}{x} \right] \right)^{\underline{p}} \right\|_{L_{\frac{q(x)}{\underline{p}}}(b^{-1}(t)\ ,\ a^{-1}(t))} dt \right)^{\frac{1}{\underline{p}}}$$

$$= \left(\int_0^\infty f^{\underline{p}}(t) \left\| \left[t^{-\beta} \left(t^{\beta+1} - (a(x))^{\beta+1} \right) \right]^{\frac{1}{\underline{p}'}} \left[\frac{w_2(x)}{x} \right] \right\|^{\underline{p}}_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} dt \right)^{\frac{1}{\underline{p}}}$$

$$= \left(\int_0^\infty \left(f(t) \left\| \left[t^{-\beta} \left(t^{\beta+1} - (a(x))^{\beta+1} \right) \right]^{\frac{1}{\underline{p}'}} \left[\frac{w_2(x)}{x} \right] \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \right)^{\underline{p}} dt \right)^{\frac{1}{\underline{p}}}$$

$$= \left\| f(t) \left\| \left[t^{-\beta} \left(t^{\beta+1} - (a(x))^{\beta+1} \right) \right]^{\frac{1}{\underline{p}'}} \left[\frac{w_2(x)}{x} \right] \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \right\|_{L_{\underline{p}}(0,\infty)}.$$

Finally, from Remark 3.1.1, it follows

$$\left\| f(t) \left\| \left[t^{-\beta} \left(t^{\beta+1} - (a(x))^{\beta+1} \right) \right]^{\frac{1}{\underline{p}'}} \left[\frac{w_2(x)}{x} \right] \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \right\|_{L_{\underline{p}}(0,\infty)} \leq \left(1 + M - m\right)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \left\| \left[t^{-\beta} \left(t^{\beta+1} - (a(x))^{\beta+1} \right) \right]^{\frac{1}{\underline{p}'}} \left[\frac{w_2(x)}{x} \right] \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}.$$

Thus

$$\|Tf\|_{L_{q(x),w_2(x)}(0,\infty)} \leq \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{\frac{1}{\underline{p}'}} c_{pq} \left(1 + M - m\right)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \left\| \frac{[t^{-\beta}(t^{\beta+1} - (a(x))^{\beta+1})]^{\frac{1}{\underline{p}'}} w_2(x)}{x} \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}.$$

Theorem 3.2.2. *Let p, q be measurable functions on $(0, \infty)$, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x) - \underline{p}}$, for $x \in (0, \infty)$, $\beta < -1$ and $f \in Q^\beta$. Suppose that w_1 and w_2 are weight functions defined on $(0, \infty)$. Then for any $f \in L_{p(x),w_1}(0, \infty)$ the inequality*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,\infty)} \leq \underline{p}^{\frac{1}{\underline{p}}} |\beta+1|^{-\frac{1}{\underline{p}'}} c_{pq} \left(1 + M - m\right)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \left\| \frac{[t^{-\beta}(t^{\beta+1} - (b(x))^{\beta+1})]^{\frac{1}{\underline{p}'}} w_2(x)}{x} \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \tag{3.18}$$

holds, where c_{pq}, M and m are the constants defined in Theorem 3.2.1.

Proof. *By applying Lemma 3.2.2 (b) (inequality (3.15) with $p = \underline{p}$), we have*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,\infty)} = \left\| \frac{w_2(x)}{x} \int_{a(x)}^{b(x)} f(t)dt \right\|_{L_{q(x)}(0,\infty)}$$

$$\leq \underline{p}^{\frac{1}{\underline{p}}} |\beta+1|^{\frac{1-\underline{p}}{\underline{p}}} \left\| \frac{w_2(x)}{x} \left(\int_{a(x)}^{b(x)} \left[t^{-\beta} \left(t^{\beta+1} - (b(x))^{\beta+1} \right) \right]^{\underline{p}-1} f^{\underline{p}}(t) dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}$$

$$= \underline{p}^{\frac{1}{\underline{p}}} |\beta+1|^{-\frac{1}{\underline{p}'}} \left\| \left(\int_{a(x)}^{b(x)} \left[\frac{w_2(x)}{x} \right]^{\underline{p}} \left[t^{-\beta} \left(t^{\beta+1} - (b(x))^{\beta+1} \right) \right]^{\underline{p}-1} f^{\underline{p}}(t) dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,\infty)}$$

$$= \underline{p}^{\frac{1}{\underline{p}}} |\beta+1|^{-\frac{1}{\underline{p}'}} \left\| \int_{a(x)}^{b(x)} \left[\frac{w_2(x)}{x} \right]^{\underline{p}} \left[t^{-\beta} \left(t^{\beta+1} - (b(x))^{\beta+1} \right) \right]^{\underline{p}-1} f^{\underline{p}}(t) dt \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}^{\frac{1}{\underline{p}}}$$

$$= \underline{p}^{\frac{1}{\underline{p}}} |\beta+1|^{-\frac{1}{\underline{p}'}} \left\| \int_0^\infty f^{\underline{p}}(t) \chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x} \right]^{\underline{p}} \left[t^{-\beta} \left(t^{\beta+1} - (b(x))^{\beta+1} \right) \right]^{\underline{p}-1} dt \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}^{\frac{1}{\underline{p}}}$$

$$= \underline{p}^{\frac{1}{\underline{p}}} |\beta+1|^{-\frac{1}{\underline{p}'}} \left\| \left\| f^{\underline{p}}(t) \chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x} \right]^{\underline{p}} \left[t^{-\beta} \left(t^{\beta+1} - (b(x))^{\beta+1} \right) \right]^{\underline{p}-1} \right\|_{L_1(0,\infty)} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,\infty)}^{\frac{1}{\underline{p}}}.$$

The rest is similar to the proof of Theorem 3.2.1.

The following Theorem is proved analogously by applying Lemma 3.2.2 (b) (inequality (3.15) with $p = \underline{p}$).

Theorem 3.2.3. *Let* p*,* q *be measurable functions on* $(0,1)$*,* $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$*,* $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$*, for* $x \in (0,1)$*,* $\beta > -1$ *and* $f \in Q^\beta$*. Suppose that* w_1 *and* w_2 *are weight functions defined on* $(0,1)$*. Then for any* $f \in L_{p(x),w_1}(0,1)$ *the inequality*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,1)} \leq \underline{p}^{\frac{1}{\underline{p}}} (\beta+1)^{-\frac{1}{\underline{p}'}} c_{pq} \left(1 + M - m\right)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \left\| \frac{\left[t^{-\beta} (t^{\beta+1} - (b(x))^{\beta+1}) \right]^{\frac{1}{\underline{p}'}} w_2(x)}{x} \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,1)} \|f\|_{L_{p(x),w_1(x)}(0,1)}, \tag{3.19}$$

holds, where c_{pq}*,* M *and* m *are the constants defined in Theorem 3.2.1.*

By putting $\beta = 0$ in Theorem 3.2.1 and Theorem 3.2.3, we obtain the following Corollary.

Corollary 3.2.1. *Let* p*,* q *be measurable functions on* $(0,\infty)$*,* $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$*,* $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$*.*

1. *Suppose that* f *is nonnegative and nonincreasing function defined on* $(0,\infty)$ *and* w_1*,* w_2 *are weight functions defined on* $(0,\infty)$*. Then for any* $f \in L_{p(x),w_1}(0,\infty)$ *the inequality*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,\infty)} \leq \underline{p}^{\frac{1}{\underline{p}}} c_{pq} \left(1 + M - m\right)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \left\| \frac{(t-a(x))^{\frac{1}{p'}} w_2(x)}{x} \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \quad (3.20)$$

holds.

2. *Suppose that f is nonnegative and nondecreasing function defined on $(0,1)$ and w_1, w_2 are weight functions defined on $(0,1)$. Then for any $f \in L_{p(x),w_1}(0,1)$ the inequality*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,1)} \leq \underline{p}^{\frac{1}{\underline{p}}} c_{pq} \big(1+M-m\big)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \left\| \frac{(t-b(x))^{\frac{1}{p'}} w_2(x)}{x} \right\|_{L_{q(x)}\left(b^{-1}(t)\ ,\ a^{-1}(t)\right)} \frac{1}{w_1(x)} \right\|_{L_{r(x)}(0,1)} \|f\|_{L_{p(x),w_1(x)}(0,1)}, \quad (3.21)$$

holds.

Where c_{pq}, M and m are the constants defined in Theorem 3.2.1.

Now we consider the case $\beta = -1$.

Theorem 3.2.4. *Let p, q be measurable functions on $(0,\infty)$, $0 < \underline{p} \leq p(x) \leq q(x) \leq \overline{q} < 1$, $r(x) = \frac{\underline{p}p(x)}{p(x)-\underline{p}}$ for $x \in (0,\infty)$ and $\beta = -1$. Suppose that w_1 and w_2 are weight functions defined on $(0,\infty)$.*

1. *If $f \in Q^{-1}$ and $0 \leq a(x) < b(x) < \infty$, then the inequality*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,1)} \leq \underline{p}^{\frac{1}{\underline{p}}} c_{pq} \big(1+M-m\big)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \frac{t^{\frac{1}{p'}} \left\| \frac{w_2(x)}{x} \left[\ln\left(\frac{b(x)}{t}\right)\right]^{\frac{1}{p'}} \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))}}{w_1(x)} \right\|_{L_{r(x)}(0,1)} \|f\|_{L_{p(x),w_1(x)}(0,1)}, \quad (3.22)$$

holds.

2. *If $f \in Q_{-1}$and $0 < a(x) < b(x) \leq \infty$, then the inequality*

$$\|Tf\|_{L_{q(x),w_2(x)}(0,\infty)} \leq \underline{p}^{\frac{1}{\underline{p}}} c_{pq} \big(1+M-m\big)^{\frac{1}{\underline{p}}}$$

$$\times \left\| \frac{t^{\frac{1}{p'}} \left\| \frac{w_2(x)}{x} \left[\ln\left(\frac{t}{a(x)}\right)\right]^{\frac{1}{p'}} \right\|_{L_{q(x)}(b^{-1}(t)\ ,\ a^{-1}(t))}}{w_1(x)} \right\|_{L_{r(x)}(0,\infty)} \|f\|_{L_{p(x),w_1(x)}(0,\infty)}, \quad (3.23)$$

holds.

Where c_{pq}, M and m are the constants defined in Theorem 3.2.1.

Proof. *1. By applying Lemma 3.2.2 (b) (inequality (3.16) with $p = \underline{p}$), we have*

$$\begin{aligned}
\|Tf\|_{L_{q(x),w_2(x)}(0,1)} &= \|w_2(x)(Tf)(x)\|_{L_{q(x)}(0,1)} \\
&= \left\| \frac{w_2(x)}{x} \int_{a(x)}^{b(x)} f(t)dt \right\|_{L_{q(x)}(0,1)} \\
&\leq \underline{p}^{\frac{1}{\underline{p}}} \left\| \frac{w_2(x)}{x} \left(\int_{a(x)}^{b(x)} \left[t \ln \left(\frac{b(x)}{t} \right) \right]^{\underline{p}-1} f^{\underline{p}}(t)dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,1)} \\
&= \underline{p}^{\frac{1}{\underline{p}}} \left\| \left(\int_{a(x)}^{b(x)} \left[\frac{w_2(x)}{x} \right]^{\underline{p}} \left[t \ln \left(\frac{b(x)}{t} \right) \right]^{\underline{p}-1} f^{\underline{p}}(t)dt \right)^{\frac{1}{\underline{p}}} \right\|_{L_{q(x)}(0,1)} \\
&= \underline{p}^{\frac{1}{\underline{p}}} \left\| \int_0^1 f^{\underline{p}}(t) \chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x} \right]^{\underline{p}} \left[t \ln \left(\frac{b(x)}{t} \right) \right]^{\underline{p}-1} dt \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,1)}^{\frac{1}{\underline{p}}} \\
&= \underline{p}^{\frac{1}{\underline{p}}} \left\| \left\| f^{\underline{p}}(t) \chi_{(a(x)\ ,\ b(x))}(t) \left[\frac{w_2(x)}{x} \right]^{\underline{p}} \left[t \ln \left(\frac{b(x)}{t} \right) \right]^{\underline{p}-1} \right\|_{L_1(0,1)} \right\|_{L_{\frac{q(x)}{\underline{p}}}(0,1)}^{\frac{1}{\underline{p}}} .
\end{aligned}$$

The rest is similar to the proof of Theorem 3.2.1.

2. *We apply Lemma 3.2.2 (a) (inequality (3.14) with $p = \underline{p}$) and the rest is similar to the proof of Theorem 3.2.1.*

Conclusion

The objective of this work is to extend some integral inequalities involving usual Hardy operators to the Hardy-Steklov and Hardy-Steklov type operators for quasi-monotone functions in classical and weighted variable Lebesgue spaces. When looking at inequalities that are associated with these operators, it is possible to use other spaces, such as Morrey spaces, Marcinkiewicz spaces and Orlicz spaces, as a method of perspective.

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