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THÈSE

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Thème

Sur des Inégalités Intégrales Dans certaines Classes De Fonctions

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Dedication

I dedicate this work to my dear and beloved mother, and to my cherished and loving father, may his soul rest in peace, my role model in knowledge and perseverance. To my life companion, my beloved wife. To the apples of my heart: Aisha, Rawan, Abdel Qader, Abrar, and Younes. To my dear brother Younes, may his soul rest in peace, who always encouraged me to continue my studies, and to all my siblings. To all my family and friends , Dr. Soufyane Mokhtari. and to everyone who encouraged, inspired, and supported me in achieving this milestone.

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I may have inadvertently omitted some others, and I apologize if that is the case.

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Benguessoum Adel.

Abstract

In this study, we focus on proving and developing fractional integral inequalities for h-convex functions and functions whose absolute value of derivatives exhibits h-strong convexity. These concepts extend classical integral inequalities to fractional orders. By leveraging the properties of h-convexity within the fractional integral framework, we establish new inequalities related to the Hermite-Hadamard type. Additionally, we derive estimates and bounds for integral transforms and provide bounds for the left and right sides of Riemann-Liouville integrals. These findings contribute to broadening the theoretical applications of both classical and fractional integrals across various types.

ملخص

في هذه الدراسة نركز على إثبات بعض المتباينات في التكاملات الكسرية للدوال المحدبة (h-convex) والدوال ذات المشتقات التي تتميز بخاصية قوية - التقعر، تمتد هذه المفاهيم لتشمل تعميمات المتباينات الكلاسكية في التكامل وصولا إلى رتب كسرية، مع استخدام مؤثرات تكاملية جديدة مثل التكامل الكسري (ريمان ليوفيل)، من خلال خصائص التحدب وإطار التكامل الكسري، تم إثبات متباينات جديدة تتعلق بنوع هيرميت - هادا مارد – Hermite- hadamard بالإضافة إلى التوصل إلى تقديرات وحدود لتحولات لابلاس وبعض الحدود للطرفين الأيسر والأيمن لتكاملات ريمان ليوفيل. تسهم هذه النتائج في توسيع نطاق التطبيقات النظرية للتكاملات الكلاسكية والكسرية بمختلف أنواعها.

Résumé

Dans cette étude, nous nous concentrons sur la démonstration et le développement de certaines inégalités intégrales fractionnaires pour les fonctions h-convexes et les fonctions dont les dérivées en valeur absolue présentent une proprieté de h-convexité forte. Ces concepts étendent les inégalités intégrales classiques aux ordres fractionnaires. En exploitant les proprietés de la h-convexité dans le cadre des intégrales fractionnaires, nous établissons de nouvelles ingalités intégrales liées au type Hermite-Hadamard inequality.

Contents

Introduction

Fractional calculus,is an extension of classical calculus that deals with non-integer integrals and derivatives, has developed as a valuable tool for mathematical modeling in a variety of scientific and engineering fields. When it comes to explaining phenomena with memory effects, anomalous diffusion, and complex dynamics, traditional integer-order calculus frequently falls short. Fractional integration, which has applications in physics, engineering, signal processing, and artificial intelligence, offers the flexibility required to capture these complex processes by generalizing the idea of integration. The genesis of fractional calculus can be traced back to the 17th century with Leibniz's inquiry into the meaning of fractional order derivatives. This initial curiosity spurred the interest of mathematicians like Fourier and Laplace, who explored its applications in areas such as heat transfer and wave phenomena. The 19th century witnessed the formal development of fractional integration by Riemann and Liouville, laying the groundwork for understanding systems with inherent memory. Later, Hadamard introduced alternative definitions using logarithmic transformations, proving valuable for analyzing irregular or delayed behaviors. Recent advancements have further expanded the applicability of fractional calculus, leading to the development of new forms of fractional integrals like the Caputo derivative (suited for differential equations with well-defined initial conditions), integrals with generalized kernels (enhancing model accuracy), and integrals in functional spaces (broadening the theoretical scope). These developments have solidified fractional calculus as a crucial component of modern science and engineering. A significant area within fractional calculus is the study of fractional integral inequalities. These inequalities play a vital role in establishing the existence and uniqueness of solutions to differential and integral equations, as well as in analyzing their stability and optimization properties. Research in this domain often focuses on inequalities involving specific classes of functions, such as h-convex functions, which possess unique mathematical characteristics. A fundamental inequality in this context is the Hermite-Hadamard (H-H) inequality, providing valuable approximations for convex functions and finding widespread use in optimization and numerical analysis. This study aims to contribute to the existing body of knowledge by exploring new inequalities and expanding classical results within this framework.

The work is organized into three distinct chapters, each addressing specific aspects of these investigations.

Chapter one lays the foundation by presenting definitions and fundamental concepts related to convexity, classical integral inequalities (including Hôlder's inequality and Minkowski's inequality), and the basic principles of fractional calculus. This chapter serves as a necessary introduction for the subsequent analyses.

In chapter two,We present fractional integrals functions which generalizes than of Riemann-Liouville fractional integrals, characterized by two parameters, and two non-negative locally integrable functions. This study leads to establish some fractional integral inequalities via the class of h-convex.As consequence, some estimates and bounds for some functions are obtained, also bounds for left hand side and right of Riemann-Liouville integrals, which lead to the well-known Hermite-Hadamard inequality. This was the subject of a publication that appeared in the journal

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http://nfaa.kyungnam.ac.kr/journal-nfaa.

Chapter three presents the concept of strongly h-convex functions and investigates some of their properties.We apply the same integral functions to this class of functions.This was the subject of a publication that appeared in the journal :Arabian journal of Mathematics

Arab. J. Math. https://doi.org/10.1007/s40065-025-00556-6.

In summary, this thesis underscores the theoretical and practical significance of fractional integration as a powerful analytical approach with far-reaching applications across diverse scientific domains. By concentrating its efforts on exploring fractional integral inequalities in conjunction with convex functions, this research aims to generate meaningful contributions to ongoing advancements in the field, offering both theoretical insights and potential applications in allied disciplines.

In conclusion,we find a fairly recent and detailed bibliography.

Chapter 1

PRELIMINARY CONCEPTS

1.1 Definitions

Convex functions have been the object of attention in recent decades and the original notion has been extended and generalized in various directions, such functions are important in many parts of analysis and geometry and their properties have been studied in detail. Readers interested can consult , where a panorama, practically complete, of these branches is presented.

1.1.1 Some type of Convexities

- **Convexity**

Definition 1.1.1 *A function* $\varphi : [\alpha, \beta] \to (-\infty, +\infty)$ *is said to be a convex function if it satisfies the following inequality*

$$\varphi(\tau\xi + (1-\tau)\zeta) \leq \tau\varphi(\xi) + (1-\tau)\varphi(\zeta), \forall \zeta, \xi \in [\alpha, \beta].$$

where $0 \leq \tau \leq 1$.

- P**-convex**

Definition 1.1.2 *Let* I *an interval in* $\mathbb{R}$, *we say that a function* $\psi : I \to \mathbb{R}$ *is of* P *type, or that* ψ *belongs to the class* $P(I)$, *if* ψ *is nonnegative and for all* $a, b \in I$ *and* $\rho \in [0,1]$ *we have:*

$$\psi(\rho a + (1-\rho)b) \leq \psi(a) + \psi(b). \tag{1.1}$$

- J**- convex**

Definition 1.1.3 *Let* I *an interval in* $\mathbb{R}$, *we say that a function* $\psi : I \to \mathbb{R}$ *is Jensen-convex or(mid-convex) function or shortly (*J*-convex), that is function satisfying the condition:*

$$\forall a, b \in I, \psi\left(\frac{a+b}{2}\right) \leq \frac{\psi(a) + \psi(b)}{2}. \tag{1.2}$$

- s**-convex**

Two definitions of s-convexity $(0 < s < 1)$ of real-valued functions are known in the literature.It is proved among others that s-convexity in the second sense is essentially stronger than the s-convexity in the first, original, sense whenever $0 < s < 1$.

Definition 1.1.4 *A function* $f : \mathbb{R}^+ \to \mathbb{R}$, *where* $R^+ = [0, +\infty)$, *is said to be* s*-convex in the first sense if the function satisfying the condition:*

$$\psi(\alpha a + \rho b) \leq \alpha^s \psi(a) + \rho^s \psi(b). \tag{1.3}$$

for all $a, b \in [0, \infty)$ and all $\alpha, \rho > 0$ with $\alpha^s + \rho^s = 1$. This class of functions is denoted by K_s^1-functions, was introduced by Orlicz (1961) .

Definition 1.1.5 *A function $\psi : \mathbb{R}^+ \to \mathbb{R}$, where $R^+ = [0, +\infty)$, is said to be s-convex in the second sense if the function satisfying the condition:*

$$\psi(\alpha a + \rho b) \leq \alpha^s \psi(a) + \rho^s \psi(b). \tag{1.4}$$

for all $a, b \in [0, \infty)$ and all $\alpha, \rho > 0$ with $\alpha + \rho = 1$. This class of functions is denoted by K_s^2-functions.

Remark 1.1.6

- *Of course, both s-convexities mean just the convexity when $s = 1$.*
- *For more information on the classes K_s^1 and K_s^2 see [4].*

- **Strongly convex function**

Definition 1.1.7 *[26]. Let I be an interval in $\mathbb{R}$. We say that a function $\psi : I \to \mathbb{R}$ is strongly convex with modulus $w > 0$ if ψ is nonnegative and for all $x, y \in I$ and $\rho \in (0, 1)$, we have:*

$$\psi(\rho x + (1-\rho)y) \leq \rho\psi(x) + (1-\rho)\psi(y) - w\rho(1-\rho)(x-y)^2. \tag{1.5}$$

Strongly convex functions have been introduced by Polyak [26]. They have useful properties in optimization theory. For instance, if ψ is strongly convex, then it is bounded from below, its level sets $\{x \in I : \psi(x) \leq \lambda\}$ are bounded for each λ and ψ has a unique minimum on every closed subinterval of I (cf. [28], p. 268). Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions.

A function $\psi : I \to \mathbb{R}$ is said to be strongly convex with modulus w if and only if the function $g : I \to \mathbb{R}$ defined by $g(x) = \psi(x) - wx^2$ is convex.

For a twice differentiable function ψ is strongly convex with modulus w, we have

$$\psi^{"}(x) \geq 2w.$$

- **h-Convex function**

Definition 1.1.8 *[35]. Let I be an interval in $\mathbb{R}$ and $h : [0,1] \subset J \to (0,\infty)$ be a given function. A function $\psi : I \to \mathbb{R}$ is called h- convex if for all $x, y \in I$ and $\rho \in [0,1]$*

$$\psi(\rho x + (1-\rho)y) \leq h(\rho)\psi(x) + h((1-\rho))\psi(y). \tag{1.6}$$

holds. If (1.6) is reversed,then ψ is said h-concave.

This notion was introduced by S.Varosanec and generalizes the classes of non-negative convex functions, $s-$ convex functions (in the second sence), Godunova-Levin functions and $P-$functions, which are obtained by taking in (1.6) $h(t) = t, h(t) = t^s(s \in (0.1)), h(t) = 1/t$ and $h(t) = 1$, respectively.

- **Strongly h-convex**

Definition 1.1.9 *Let I be an interval in $\mathbb{R}$ and $h : [0,1] \to (0,\infty)$ be a given function.A function $\psi : I \to \mathbb{R}$ is said strongly h-convex function with modulus*

$w > 0$, *if*

$$\psi(\rho c + (1-\rho)d) \leq h(\rho)\psi(c) + h(1-\rho)\psi(d) - w\rho(1-\rho)(d-c)^2. \qquad (1.7)$$

1.2 Some Fundamental Integral Inequalities

Definition 1.2.1 *[29]. For* $1 \leq p \leq \infty, -\infty \leq \delta < \Delta \leq \infty$. *We denote by* $L_p := L_p([\delta, \Delta])$, *the set of all Lebesgue measurable functions* ϑ, *real valued for which*

$$\int_{\delta}^{\Delta} |\vartheta(\xi)|^p \, d\xi < \infty.$$

If $p = \infty$, $L_\infty([\delta, \Delta])$ *is defined as the set of all essentially bounded functions for which*

$$ess\sup|\vartheta(\xi)| := \inf\{M > 0 : meas(\{\xi : \vartheta(\xi) \geq M\}) = 0\} < \infty,$$

where esssup $|\vartheta(x)|$ *is an essential supremum of the function* $|\vartheta(x)|$.

Theorem 1.2.2 *[29]. For* $1 \leq p \leq \infty$, *the spaces* $L_p(.)$ *are Banach spaces (complete normed spaces) under the norms:*

$$\|\vartheta\|_\theta = \left(\int_{\delta}^{\Delta} |\vartheta(\xi)|^\theta \, d\xi\right)^{\frac{1}{\theta}} < \infty,$$

$$\|\vartheta\|_\infty = ess\sup|\vartheta(\xi)| < \infty.$$

1.2.1 Hôlder's Inequality

The Hôlder's[1]inequality and its corollaries in the theory of Lebesgue spaces L_p are fundamental inequalities.

Definition 1.2.3 *[29](Hôlder conjugates). Let* $1 \leq p, q \leq \infty$ *are said Hôlder conjugates if* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *In particular* $p = 2$ *is its conjugate,(* 1 *and* ∞ *) are conjugates* .

Lemma 1.2.4 *(* ***Young's Inequality****). Let* $p, q \geq 1$, *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\forall a, b \geq 0, \quad ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.8}$$

proof: The function $\exp : t \to e^t$ **is convex (** $\mathbb{R} \to \mathbb{R}$**). Thus for all** $t, s \in \mathbb{R}$ **and** $\alpha \in [0, 1]$

$$e^{(\alpha t + (1-\alpha)s)} \leq \alpha e^t + (1-\alpha)e^s.$$

Let $a, b > 0$. **Take** $\alpha = \frac{1}{p}$ **(**$1 - \alpha = \frac{1}{q}$,**)** $t = p\ln(a)$ **and** $s = q\ln(b)$. **we get inequality** (1.8)**.**

Theorem 1.2.5 ***[29](Riesz-Hölder's inequality for integrals). Let*** Ω ***be a measurable set and*** $1 \leq p, q < \infty$ ***be such that*** $1/p + 1/q = 1$. ***If*** $f \in L_p(\Omega), g \in L_q(\Omega)$, ***then***

$$\|fg\|_{L_1(\Omega)} := \int_\Omega |fg| dx \leq \|f\|_{L_p(\Omega)} \|g\|_{L_q(\Omega)}, \tag{1.9}$$

and $fg \in L_1(\Omega)$.

Idea of proof. If either $\|f\|_{L_p} = 0$ **or** $\|g\|_{L_p} = 0$, **the result is trivial so we assume these two quantities** $\|f\|_{L_p(\Omega)}, \|g\|_{L_q(\Omega)}$ **are both finite and non-zero.**

[1] Otto Hôlder, 1859-1937, born in Stuttgart, active in Guttingen and Tbingen. He gave important contributions.

Si $1 \leq p, q < \infty$, **we apply lemma 1.2.4 with**

$$a = \frac{|f(x)|}{\|f\|_{L_p(\Omega)}}, \text{ and } \quad b = \frac{|g(x)|}{\|g\|_{L_q(\Omega)}},$$

For the complete proof see [17].

Remark 1.2.6

- ***If*** $\|f\|_{L_p(\Omega)}$ ***or*** $\|g\|_{L_q(\Omega)}$ ***is infinite,or equal to zero, inequality (1.9) is trivial.***

- ***A very special case. If*** $p = 2$, ***then*** $q = 2$ ***and the Hôlder's inequality (1.9) leads to***

$$\|fg\|_{L_1(\Omega)} := \int_\Omega |fg| dx \leq \|f\|_{L_2(\Omega)} \|g\|_{L_2(\Omega)}, \tag{1.10}$$

known as the Cauchy-Schwarz[2] inequality. It is a particular case of the Cauchy-Schwarz for semi-inner product spaces.

1.2.2 Minkowski's Inequality.

Theorem 1.2.7 ***[29](Inequality of Riesz-Minkowski for integrals). Let*** Ω ***be a measurable set, and let*** $1 \leq p \leq \infty$, $f \in L_p(\Omega)$ ***et*** $g \in L_p(\Omega)$. ***Then***

$$\|f + g\|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}. \tag{1.11}$$

[2]Cauchy (1821) first proved the inequality (Cauchy's inequality) for square summable sequences. This inequality was generalized to integrals by A. Schwarz (1885). Also known as the Cauchy-Bunyakovsky inequality (1859). 0tto Holder (1989) extended Cauchy's inequality for the general values of p and q by establishing,for sequences (a_n) and (b_n). The latter inequality is then generalized to the case of integrals by F. Riesz (1910).

$$\|f+g\|_{L_\infty(\Omega)} \leq \|f\|_{L_\infty(\Omega)} + \|g\|_{L_\infty(\Omega)}. \tag{1.12}$$

Equality holds if $Af = Bg\,\mu -$ ***a.e. for*** A ***and*** B ***of the same sign and not simultaneously zero.***

Minkovski's[3] inequality is the triangle inequality for the spaces $L_p(\Omega)$**.**

Idea of the proof. 1) If $p = 1$**:**

$$\|f+g\|_{L_1(\Omega)} = \int_\Omega |f+g|dx \leq \int_\Omega |f|dx + \int_\Omega |g|dx = \|f\|_{L_1(\Omega)} + \|g\|_{\mathcal{L}_1(\Omega)}.$$

2) If $1 < p < \infty$

$$\begin{aligned}\int_\Omega |f+g|^p dx &= \int_\Omega |f+g||f+g|^{p-1}dx \\ &\leq \int_\Omega |f||f+g|^{p-1}dx + \int_\Omega |g||f+g|^{p-1}dx.\end{aligned}$$

We apply Hölder's inequality:Note that $\frac{1}{p} + \frac{1}{q} = 1,$ **and** $q = \frac{p}{p-1}.$

1.3 Some Concepts in Fractional Calculus

1.3.1 Some special functions

Gamma and Beta functions.

In 1783, *Leonhard Euler* made his first comments on fractional order derivative. He worked on progressions of numbers and introduced first time the

[3] the inequality in Theorem 1.2.7. was first proved for finite sums of numbers by a German mathematician Hermann Minkowski (1896) and then generalized to the case of integrals of functions by F. Riesz (1910).

generalization of factorials to *Gamma* function. This function is generalization of a factorial in the following form:

$$\Gamma(n) = (n-1)!.$$

All through the work we utilize the functions $\Gamma(z)$ (see [27, 29]).

Definition 1.3.1 ***[29] The Euler-Gamma function is defined as***

$$\Gamma(\tau) = \int_0^{\infty} \mu^{\tau-1} e^{-\mu} d\mu, \tag{1.13}$$

where $\tau > 0$.

The Beta function, or the first order Euler function, is defined as:

Definition 1.3.2 ***[29]. Let*** $a, b \in \mathbb{R}$ ***be such that*** $a > 0$ ***and*** $b > 0$, ***which guarantee the existence of the integral. We define the Beta function, denoted by*** $B(a, b)$, ***from the following integral***

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt. \tag{1.14}$$

the beta-function is connected with the Gamma-function by the relation

Proposition 1.3.3 ***[29] Let*** $a; b \in \mathbb{R}$ ***such that*** $a, b > 0$. ***Then***

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

1.3.2 Fractional Integrals

The Riemann-Liouville-fractional integrals [29] of $f \in L_1 := L_1(\delta, \Delta),\ \delta, \Delta \in R$ **having an order** $\nu \in \mathbb{R}, \nu > 0)$ **are defined as follows**

$$(J^{\nu}_{\delta+}f)(\xi) = \frac{1}{\Gamma(\nu)} \int_{\delta}^{\xi} (\xi - \mu)^{\nu-1} f(\mu) d\mu, \quad \xi > \delta$$

$$(J^{\nu}_{\Delta-}f)(\xi) = \frac{1}{\Gamma(\nu)} \int_{\Delta}^{\xi} (\mu - \xi)^{\nu-1} f(\mu) d\mu, \quad \xi < \Delta.$$

Remark 1.3.4 • ***The integrals*** $J^{\nu}_{\Delta^-}f, J^{\nu}_{\delta+}f$ ***are defined for functions*** $f \in L_1 := L_1(\delta, \Delta),$ ***existing almost everywhere.***

Theorem 1.3.5 *[29](theorem 2.6 p 48).* ***The Riemann-Liouville-fractional integrals functions are bounded in*** $L_p([\delta, \Delta]), p \geq 1$ ***i.e. the following estimates***

$$\|J^{\nu}_{\delta+}f\|_{L_p(\delta,\Delta)} \leq C_{rl}\|f\|_{L_p([\delta,\Delta])}, \tag{1.15}$$

$$\|J^{\nu}_{\Delta-}f\|_{L_p([\delta,\Delta])} \leq C_{rl}\|f\|_{L_p([\delta,\Delta])}, \tag{1.16}$$

hold with the constant $C_{rl} = \frac{(\Delta-\delta)^{\nu}}{|\Gamma(\nu+1)|}.$

The inequalities (1.15) and (1.16) may be verified by simple operations using the generalized Minkowski's inequality.

Theorem 1.3.6 *[29] .* ***For any*** $f \in C([a, b], \mathbb{R})$ ***for*** $\alpha, \beta > 0.$ ***The Riemann-Liouville fractional integral satisfies the property***

$$J^{\alpha}J^{\beta}f(t) = J^{\beta}J^{\alpha}f(t) = J^{\alpha+\beta}f(t),$$

this result is called the semi-group property of fractional integration.

1.4 Convexity and Fractional Inequalities

The following lemma [15] allows us to prove Theorem 1.4.3.

Lemma 1.4.1 ***Let*** $f : I \subset \mathbb{R} \to \mathbb{R}$ ***be differentiable function on*** I° ***where*** $\delta, \Delta \in I$ ***with*** $\delta < \Delta$***. If*** $f' \in L_1[\delta, \Delta$***, the following equality holds:***

$$\frac{f(\delta) + f(\Delta)}{2} - \frac{1}{\Delta - \delta} \int_{\delta}^{\Delta} f(x)dx = \frac{\Delta - \delta}{2} \int_0^1 (1 - 2t) f'(t\delta + (1 - t)\Delta)dt. \quad (1.17)$$

Proof 1.4.2 ***We set***

$$J = \frac{\Delta - \delta}{2} \int_0^1 (1 - 2t) f'(t\delta + (1 - t)\Delta)dt.$$

By applying integration by parts, we get:

$$\begin{aligned} J &= \frac{\Delta - \delta}{2} \left[(1 - 2t) \frac{f(t\delta + (1 - t)\Delta)}{\delta - \Delta} |_0{}^1 + 2 \int_0^1 \frac{f(t\delta + (1 - t)\Delta)}{\delta - \Delta} dt \right] \\ &= \frac{\Delta - \delta}{2} \left[\frac{f(\delta) + f(\Delta)}{\Delta - \delta} + 2 \int_0^1 \frac{f(t\delta + (1 - t)\Delta)}{\delta - \Delta} dt \right] \\ &= \frac{f(\delta) + f(\Delta)}{2} + (\Delta - \delta) \int_0^1 \frac{f(t\delta + (1 - t)\Delta)}{\delta - \Delta} dt \end{aligned}$$

Using the change of the variable $x = t\delta + (1 - t)\Delta$***, we get:***

$$\begin{aligned} J &= \frac{f(\delta) + f(\Delta)}{2} + (\Delta - \delta) \int_{\Delta}^{\delta} \frac{f(x)}{(\delta - \Delta)^2} dx \\ &= \frac{f(\delta) + f(\Delta)}{2} - \frac{1}{\Delta - \delta} \int_{\delta}^{\Delta} f(x)dx. \end{aligned}$$

So, we get

$$\frac{f(\delta)+f(\Delta)}{2}-\frac{1}{\Delta-\delta}\int_{\delta}^{\Delta} f(x)dx=\frac{\Delta-\delta}{2}\int_0^1(1-2t)f'(t\delta+(1-t)\Delta)dt.$$

The following inequalities of the Hermite-Hadamard type were established for the above convex function.

Theorem 1.4.3 ***Let*** $f:[a,b]\to\mathbb{R}$ ***be differentiable function on*** $[a,b]$***. If*** $|f'|$ ***is convex on*** $[a,b]$***. Then***

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.18}$$

Proof 1.4.4 ***By using*** (1.17)***, we have:***

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &= \left|\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt\right|\\
&\leq \frac{b-a}{2}\int_0^1|1-2t||f'(ta+(1-t)b)|dt\\
&\leq \frac{b-a}{2}\int_0^1|1-2t|\left[t|f'(a)|+(1-t)|f'(b)|\right]dt\\
&\leq \frac{b-a}{2}\left[\int_0^1|1-2t|t|f'(a)|dt+\int_0^1|1-2t|(1-t)|f'(b)|dt\right]\\
&\leq \frac{b-a}{2}\left[\int_0^{\frac{1}{2}}(1-2t)t|f'(a)|dt+\int_{\frac{1}{2}}^1(2t-1)t|f'(a)|dt\right]\\
&+ \frac{b-a}{2}[\int_0^{\frac{1}{2}}(1-2t)(1-t)|f'(b)|dt\\
&+ \int_{\frac{1}{2}}^1(2t-1)(1-t)|f'(b)|dt]\\
&\leq \frac{b-a}{2}\left[\frac{1}{4}|f'(a)|+\frac{1}{4}|f'(b)|\right]\\
&\leq \frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right].\end{aligned}$$

One of the most important inequalities for convex functions, is the famous Hermite-Hadamard inequality: published by Hermite in 1883 and, independently, by Hadamard in 1893.

Theorem 1.4.5 ***(Hermite-Hadamard inequality). Let*** $\varphi : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ ***be a convex function, then the following inequality holds.***

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(\tau)d\tau \leq \frac{\varphi(a)+\varphi(b)}{2}. \tag{1.19}$$

In [30], Fejer gave the weighted version of the inequalities (1.19),so-called Hermite-Hadamard-Fejer inequalities, as follow: If $\varphi : [a,b] \to \mathbb{R}$ **is convex and the function** $\varUpsilon : [a,b] \to \mathbb{R}$ **is positive and symmetric with respect to** $(a+b)/2$, **then:**

$$\varphi\left(\frac{a+b}{2}\right)\int_a^b \varUpsilon(t)dt \leq \int_a^b \varUpsilon(t)\varphi(t)dt \leq \frac{\varphi(a)+\varphi(b)}{2}\int_a^b \varUpsilon(t)dt.$$

Remark 1.4.6 ***Taking*** $\varUpsilon(t) = 1$, ***we get the inequality of Hermite-Hadamard (1.19).***

Inequalities via h-convexity

Other extensions of the Hermite-Hadamard inequality (1.19) are established related to the h-convexity, we have

$$\frac{1}{2h(1/2)}\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(\tau)d\tau \leq [\varphi(a)+\varphi(b)]\int_0^1 h(t)dt. \tag{1.20}$$

with $h(1/2) > 0$ **and** h **is Riemann integrable on** $[0,1]$.

In [30] is given the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.4.7 ***Let*** $\alpha > 0,\ 0 \leq a < b$ ***and*** $\phi : [a, b] \to \mathbb{R}$ ***be a positive function ,*** $\phi \in L^1[a, b]$. ***If*** ϕ ***is a convex function on*** $[a, b]$, ***then the following inequalities for fractional integrals hold:***

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\phi(b) + J_{b-}^{\alpha}\phi(a)\right] \leq \frac{\phi(a)+\phi(b)}{2}. \tag{1.21}$$

Theorem 1.4.8 ***Let*** $\alpha > 0,\ 0 \leq a < b$ ***and*** $\phi : [a, b] \to \mathbb{R}$ ***be a positive function ,*** $\phi, h \in L^1[a, b]$. ***Assume that*** h ***is superadditive on*** $[a, b]$. ***If*** ϕ ***is an h- convex function on*** $[a, b]$, ***then***

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\phi(b) + J_{b-}^{\alpha}\phi(a)\right] \leq \frac{h(1)(\phi(a)+\phi(b))}{\alpha}. \tag{1.22}$$

Chapter 2

ESTIMATES FOR FRACTIONAL INTEGRALS OF RIEMANN-LIOUVILLE TYPE USING A CLASS OF FUNCTIONS

2.1 Introduction

Due the wide application of inequalities,integral inequalities for example in the study of existence and the uniqueness of the solutions of differential equations,integral equations,in optimization problems where the objective function is convex or h-convex and the constraints are given by fractional integral inequalities.It is natural to study integral inequalities involving frac-

tional calculus.

Fractional calculus generalizes derivative and integral operations to non-integer orders, providing a more flexible approach to modeling complex phenomena.

In recent years, fractal and Fractional Problems in Mathematics, especially fractional integral inequalities involving h-convex functions have garnered significant attention due to their broad applications across optimization, differential equations, signal processing, and related areas. Researchers have explored various inequalities to establish connections with existing theories and uncover new insights. Notable works such as [8, 32, 37, 31, 20, 38] have utilized Riemann-Liouville and Hadamard [22, 10, 12] integrals and their generalizations.

Almeida, Ricardo, et al. (2020) [1] investigated fractional integral inequalities for h-convex functions, providing applications to differential equations and integral equations.

Pachpatte, B. G. (2021) [25] contributed to understanding these inequalities by deriving explicit bounds and highlighting the importance of h-convexity.

Ahmad, Bashir, and Saleem Ullah (2021) [6] explored Hermite-Hadamard type inequalities for h-convex functions, demonstrating applications in special functions and integral transforms.

These inequalities are powerful tools for analyzing the properties of functions, normed vector spaces, and measure spaces. Their understanding and application are crucial to many fundamental results and theorems in various

areas of mathematics.

This work aims to provide a comprehensive understanding of fractional integral inequalities involving h-convex functions and their significance in various mathematical domains. It establishes new inequalities, explores their applications, and contributes to advancing theoretical frameworks.

2.2 Preliminaries

Definition 2.2.1 ***Let*** $\mathbb{I} \subset \mathbb{R}$ ***be an interval and*** $\phi : \mathbb{I} \to \mathbb{R},\ h : [0,1] \to (0,\infty)$ ***be non-negative functions. The function*** ϕ ***is said to be*** $h-$***convex if***

$$\phi(\rho c + (1-\rho)d) \leq h(\rho)\phi(c) + h(1-\rho)\phi(d) \tag{2.1}$$

holds for all $c, d \in \mathbb{I}$ ***and*** $\rho \in [0,1]$***.If (2.1) is reversed*** ϕ ***is said*** h***-concave.***

Definition 2.2.2 ***Let*** $\mathbb{I} \subset \mathbb{R}$ ***be an interval and*** $\phi : \mathbb{I} \to \mathbb{R},\ h : [0,1] \to (0,\infty)$ ***be non-negative functions. The function*** ϕ ***is said to be*** $h-$ ***J-convex if***

$$\phi\left(\frac{c+d}{2}\right) \leq h\left(\frac{1}{2}\right)[\phi(c) + \phi(d)]. \tag{2.2}$$

Remark 2.2.3 ***The class of convex functions is a special case of*** h***-convex functions, where*** $h(t) = t$ ***for all*** t***. Similarly, the class of concave functions is a special case of*** h***-concave functions with*** $h(t) = -t$***. By choosing different functions for*** h***, one can obtain various subclasses of*** h***. The*** s***-convex functions (in the second sence), Godunova-Levin functions and*** $P-$***functions, which are obtained by taking in (2.1)*** $h(t) = t^s(s \in (0.1)), h(t) =$

$1/t$ ***and*** $h(t) = 1$, ***respectively [16, 37, 35, 38].***

Example 2.2.4 ***A special case of*** $h-$ ***convex function*** $h(t) = \frac{\sqrt{t}}{\sqrt{1-t}}, t \in (0,1)$

$$\phi(tc + (1-t)d) \leq \frac{\sqrt{t}}{\sqrt{1-t}}\phi(c) + \frac{\sqrt{1-t}}{\sqrt{t}}\phi(d) \tag{2.3}$$

for $t = \frac{1}{2}$, ***we get the*** $h-$ ***J-convexity.***

Our objective in this work is to establish some estimates for a more general fractional integral than the Riemann-Liouville fractional integral using the h-convexity property of functions (see Theorem 2.3.6.and 2.3.19.) as well as of absolute values of ordinary derivative (see Theorem 2.3.12.).

2.3 Main results

Definition 2.3.1 ***Let*** $0 < \delta < \Delta < \infty, 1 \leq p < \infty, \mu > 0, \nu > 1$. ***Let*** $\boldsymbol{F}^{\mu,\nu}_{u,\omega}$ ***be the integral operator defined from*** $L_p([\delta,\Delta])$ ***to*** $L_p([\delta,\Delta])$ ***as follows***

$$\mathbf{F}^{\mu,\nu}_{u,\omega;\delta+}\phi(s) = \frac{\omega(s)}{\Gamma(\mu)}\int_{\delta}^{s}(s-t)^{\mu-1}\left[\ln\frac{s}{t}\right]^{\nu-1}\phi(t)\,u(t)dt, \tag{2.4}$$

and

$$\mathbf{F}^{\mu,\nu}_{u,\omega;\Delta-}\phi(s) = \frac{\omega(s)}{\Gamma(\mu)}\int_{s}^{\Delta}(t-s)^{\mu-1}\left[\ln\frac{t}{s}\right]^{\nu-1}\phi(t)\,u(t)dt. \tag{2.5}$$

Where u, ω ***are bounded,locally integrable and non-negative functions. Provided the integrals exist. We set*** $\mathbf{F}^{0,1}_{1,1;\delta+}\phi = \mathbf{F}^{0,1}_{1,1;\Delta-}\phi = \phi$.

Remark 2.3.2 • ***If*** $\nu = 1, \omega(s) = u(s) = 1,$ ***then the ntegrals*** $\mathbf{F}^{\mu,1}_{1,1} = J^{\mu}$ ***coincides with the classical Riemann-Liouville fractional integrals.***

- ***For*** $\mu > 0, \nu > 1,$ ***necessary and sufficient conditions for the boundedness of the integrals*** $\mathbf{F}^{\mu,\nu}_{u,\omega} f$ ***on*** $L_p(0,\infty)$***,are found (see [17],Theorem 3.1).***

- ***It follows from definition 2.3.1 that*** $\mathbf{F}^{\mu,\nu}_{u,\omega;.}\phi(x) = \omega(x)\mathbf{F}^{\mu,\nu}_{u,1;.}\phi(x).$

Theorem 2.3.3 ***Let*** $0 < \delta < \Delta < \infty, \mu > 0, \nu > 1.$ ***The integrals*** $\boldsymbol{F}^{\mu,\nu}_{u,\omega,\delta+}\phi$ ***and*** $\boldsymbol{F}^{\mu,\nu}_{u,\omega,\Delta-}\phi$ ***are well defined.***

- ***If*** $\omega, u \in L_\infty([\delta,\Delta]), \phi \in L_1([\delta,\Delta])$ ***then the integral functions are bounded from*** $L_1([\delta,\Delta])$ ***on*** $L_1([\delta,\Delta]).$

- ***If*** $\omega, u \in L_\infty([\delta,\Delta]), \phi \in L_\infty([\delta,\Delta])$ ***then the integral functions are bounded on*** $L_\infty().$

- ***If*** $\omega \in L_\infty([\delta,\Delta]), \phi u \in L_1([\delta,\Delta])$ ***then the integral functions are bounded from*** $L_{1,u}([\delta,\Delta])$ ***on*** $L_1([\delta,\Delta]).$

- ***If*** $\omega \in L_q([\delta,\Delta]), \mathbf{F}^{\mu,\nu}_{u,1;\Delta-}\phi \in L_p([\delta,\Delta])$ ***and*** p, q ***are conjugates, then the integral functions are bounded from*** $L_{1,u}([\delta,\Delta])$ ***on*** $L_1([\delta,\Delta]).$

Proof 2.3.4 ***We prove the item 1.***

Let $\varphi \in L_1([\delta,\Delta]),$ ***and*** u, ω ***essentially bounded on*** $[\delta,\Delta].$ ***We have***

$$|u| \leq \|u\|_\infty, |\omega| \leq \|\omega\|_\infty.$$

We estimate, $\|\boldsymbol{F}^{\mu,\nu}_{u,\omega,.}\phi\|$. ***Hence***

$$\begin{aligned}\left|\mathbf{F}^{\mu,\nu}_{u,\omega;\delta+}\phi(s)\right| &\leq \left|\frac{\omega(s)}{\Gamma(\mu)}\int_{\delta}^{s}(s-t)^{\mu-1}\left[\ln\frac{\Delta}{\delta}\right]^{\nu-1}\phi(t)\,u(t)dt\right| \\ &\leq \|\omega\|_{\infty}\|u\|_{\infty}\left[\ln\frac{\Delta}{\delta}\right]^{\nu-1}|J^{\mu}_{\delta+}\phi(s)|.\end{aligned}$$

It follows (see 1.15,and 1.16) that

$$\|\mathbf{F}^{\mu,\nu}_{u,\omega;\delta+}\phi\|_1 \leq \|\omega\|_{\infty}\|u\|_{\infty}\left[\ln\frac{\Delta}{\delta}\right]^{\nu-1}C_{rl}\|\phi\|_1.$$

The rest is similar.

Remark 2.3.5 ***The conditon on*** u,ω ***to be bounded is sufficient not necessary.***

Theorem 2.3.6 ***Let*** $\mu_1,\mu_2 \geq 1$ ***and*** $\nu_1,\nu_2 \geq 1$.***Let*** $\phi : [\delta;\Delta] \to \mathbb{R}$ ***be a non-negative*** h***-convex function, where*** h ***is Lebesgue integrable on*** $(0,1)$. ***Assume that*** u ***is non-decreasing on*** $[\delta,s]$ ***and non-increasing on*** $[s,\Delta]$***, for*** $s\in(\delta,\Delta)$. ***Then the following inequality***

$$\begin{aligned}&\frac{1}{u(s)\omega(s)}\left(\frac{\Gamma(\mu_1)\mathbf{F}^{\mu_1,\nu_1}_{u,\omega;\delta+}\phi(s)}{\left(\ln\dfrac{s}{\delta}\right)^{\nu_1-1}}+\frac{\Gamma(\mu_2)\mathbf{F}^{\mu_2,\nu_2}_{u,\omega;\Delta-}\phi(s)}{\left(\ln\dfrac{\Delta}{s}\right)^{\nu_2-1}}\right) \\ \leq\;& \phi(s)\left[(s-\delta)^{\mu_1}+(\Delta-s)^{\mu_2}\right]\int_0^1 h(1-z)dz \\ +\;& \left(\phi(\delta)(s-\delta)^{\mu_1}+\phi(\Delta)(\Delta-s)^{\mu_2}\right)\int_0^1 h(z)dz \qquad (2.6)\end{aligned}$$

holds.

Proof 2.3.7 ***Let*** $s \in (\delta, \Delta)$. ***Firstly, let us examine the function*** ϕ ***on the interval*** $[\delta, s]$. ***Therefore, for all*** $t \in [\delta, s]$, ***the following inequality***

$$u(t)\left[\ln\frac{s}{t}\right]^{\nu_1-1}(s-t)^{\mu_1-1} \leq u(s)\left[\ln\frac{s}{\delta}\right]^{\nu_1-1}(s-\delta)^{\mu_1-1} \tag{2.7}$$

holds.Due to the h***-convexity of*** ϕ, ***we write***

$$\phi(t) \leq h\left(\frac{s-t}{s-\delta}\right)\phi(\delta) + h\left(\frac{t-\delta}{s-\delta}\right)\phi(s). \tag{2.8}$$

Multiplying (2.7),(2.8) side to side and integrating the result over $[\delta, s]$, ***we get***

$$\begin{aligned}&\int_\delta^s u(t)\left[\ln\frac{s}{t}\right]^{\nu_1-1}(s-t)^{\mu_1-1}\phi(t)dt\\ &\leq u(s)(s-\delta)^{\mu_1}\left[\ln\frac{s}{\delta}\right]^{\nu_1-1}\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\delta)\int_0^1 h(z)dz\right\},\end{aligned} \tag{2.9}$$

that is

$$\begin{aligned}\Gamma(\mu_1)\mathbf{F}^{\mu_1,\nu_1}_{u,\omega;\delta+}\phi(s) &\leq u(s)\omega(s)\left[\ln\frac{s}{\delta}\right]^{\nu_1-1}(s-\delta)^{\mu_1}\\ &\times\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\delta)\int_0^1 h(z)dz\right\},\end{aligned} \tag{2.10}$$

thus

$$\frac{\Gamma(\mu_1)\mathbf{F}^{\mu_1,\nu_1}_{u,\omega;\delta+}\phi(s)}{u(s)\omega(s)\left[\ln\frac{s}{\delta}\right]^{\nu_1-1}} \leq (s-\delta)^{\mu_1}\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\delta)\int_0^1 h(z)dz\right\}. \tag{2.11}$$

Now let $\mu_2, \nu_2 \geq 1,$ ***then for*** $t \in [s, \Delta]$ ***the following inequalities***

$$u(t)\left[\ln\frac{t}{s}\right]^{\nu_2-1}(t-s)^{\mu_2-1} \leq u(s)\left[\ln\frac{\Delta}{s}\right]^{\nu_2-1}(\Delta-s)^{\mu_2-1} \tag{2.12}$$

and

$$\phi(t) \leq h\left(\frac{t-s}{\Delta-s}\right)\phi(\Delta) + h\left(\frac{\Delta-t}{\Delta-s}\right)\phi(s) \tag{2.13}$$

hold.And we proceed as in the first step.Thus it results that

$$\frac{\Gamma(\mu_2)\mathbf{F}^{\mu_2,\nu_2}_{u,\omega;\Delta-}\phi(s)}{u(s)\omega(s)\left[\ln\frac{\Delta}{s}\right]^{\nu_2-1}} \leq (\Delta-s)^{\mu_2}\left\{\phi(s)\int_0^1 h(1-z)dz + \phi(\Delta)\int_0^1 h(z)dz\right\}. \tag{2.14}$$

By adding (2.11) and (2.14), we get (2.6).

Corollary 2.3.8 ***By setting*** $\mu_1 = \mu_2 = \mu \geq 1$ ***and*** $\nu_1 = \nu_2 = \nu \geq 1$ ***in (2.6),we get***

$$\begin{aligned}
&\frac{\Gamma(\mu)}{u(s)\omega(s)}\left(\frac{\mathbf{F}^{\mu,\nu}_{u,\omega;\delta+}\phi(s)}{\left(\ln\frac{s}{\delta}\right)^{\nu-1}} + \frac{\mathbf{F}^{\mu,\nu}_{u,\omega;\Delta-}\phi(s)}{\left(\ln\frac{\Delta}{s}\right)^{\nu-1}}\right)\\
\leq\ & \phi(s)[(s-\delta)^{\mu} + (\Delta-s)^{\mu}]\int_0^1 h(1-z)dz\\
+\ & (\phi(\delta)(s-\delta)^{\mu} + \phi(\Delta)(\Delta-s)^{\mu})\int_0^1 h(z)dz.
\end{aligned} \tag{2.15}$$

Corollary 2.3.9 ***By choosing in (2.15)*** $u = 1,\ \omega = 1, h(x) = x$ ***and*** $\nu = 1$,***then***

$$\begin{aligned}
&\Gamma(\mu)\left(J^{\mu}_{\delta+}\phi(s) + J^{\mu}_{\Delta-}\phi(s)\right)\\
\leq\ & \phi(s)\frac{(s-\delta)^{\mu} + (\Delta-s)^{\mu}}{2} + \frac{\phi(\delta)(s-\delta)^{\mu} + \phi(\Delta)(\Delta-s)^{\mu}}{2}.
\end{aligned} \tag{2.16}$$

Corollary 2.3.10 ***If we choose*** $\mu = 1$ ***and taking*** $s = \dfrac{\delta + \Delta}{2}$ ***in (2.16) then,we have***

$$\frac{1}{\Delta - \delta} \int_{\delta}^{\Delta} \phi(t)dt \leq \frac{1}{2}\phi\left(\frac{\delta + \Delta}{2}\right) + \frac{\phi(\delta) + \phi(\Delta)}{2}. \tag{2.17}$$

Example 2.3.11 ***The following example shows the validity of the inequality established. Let*** $\phi : [\delta; \Delta] \to \mathbb{R}_+,\ \phi(t) = 1$ ***and*** $h(t) = t^k, k \leq 1, t > 0$. ***Let*** $\mu > 1, \nu = 2, u = 1, \omega = 1$. ***We verify easly that***

- ϕ ***is*** h***-convex.***

Hence from corollary 2.3.8.,we have the estimates

$$\frac{\Gamma(\mu)(F^{\mu,2}_{1,1;\delta+}1)(s)}{\ln \dfrac{s}{\delta}} \leq (s - \delta)^{\mu} \left\{ \int_0^1 (1 - z)^k dz + \int_0^1 z^k dz \right\}. \tag{2.18}$$

and

$$\frac{\Gamma(\mu)(F^{\mu,2}_{1,1;\Delta-}1)(s)}{\ln \dfrac{\Delta}{s}} \leq (\Delta - s)^{\mu} \left\{ \int_0^1 (1 - z)^k dz + \int_0^1 z^k dz \right\}. \tag{2.19}$$

or

$$\begin{aligned}\int_{\delta}^{s} (s - t)^{\mu - 1} \ln \frac{s}{t}\, dt &= \frac{(s - \delta)^{\mu}}{\mu} \ln \frac{s}{\delta} - \frac{1}{\mu} \int_{\delta}^{s} (s - t)^{\mu}\, t^{-1} dt \\ &\leq \frac{2}{k + 1}(s - \delta)^{\mu} \ln \frac{s}{\delta},\end{aligned} \tag{2.20}$$

$$\begin{aligned}\int_{s}^{\Delta} (t - s)^{\mu - 1} \ln \frac{t}{s}\, dt &= \frac{(\Delta - s)^{\mu}}{\mu} \ln \frac{\Delta}{s} - \frac{1}{\mu} \int_{s}^{\Delta} (t - s)^{\mu}\, t^{-1} dt \\ &\leq \frac{2}{k + 1}(\Delta - s)^{\mu} \ln \frac{\Delta}{s}.\end{aligned} \tag{2.21}$$

For $s = \dfrac{\delta+\Delta}{2}$ ***and*** $k=1$, ***we get***

$$\int_{\delta}^{\frac{\delta+\Delta}{2}} \left(\frac{\delta+\Delta}{2} - t\right)^{\mu-1} \ln\left(\frac{\delta+\Delta}{2t}\right) dt \leq \left(\frac{\Delta-\delta}{2}\right)^{\mu} \ln\frac{\delta+\Delta}{2\delta} \tag{2.22}$$

and

$$\int_{\frac{\delta+\Delta}{2}}^{\Delta} \left(t - \frac{\delta+\Delta}{2}\right)^{\mu-1} \ln\frac{2t}{\delta+\Delta}\, dt \leq \left(\frac{\Delta-\delta}{2}\right)^{\mu} \ln\frac{2\Delta}{\delta+\Delta}. \tag{2.23}$$

Theorem 2.3.12 ***Let*** $\mu_1, \mu_2, \nu_1, \nu_2 \geq 1$.***Let*** $\phi : [\delta;\Delta] \to \mathbb{R}$ ***be a non-negative differentiable function.Let*** u, ω ***be locally integrable, non-negative functions.Also suppose that*** u ***is absolutely continuous, non-decreasing on*** $[\delta, s]$ ***and non-increasing on*** $[s, \Delta]$, ***for*** $s \in (\delta, \Delta)$. ***If*** $|\phi'|$ ***is*** $h-$ ***convex, then***

$$\begin{aligned}
& \Big|\frac{\Gamma(\alpha_1+1)}{\omega(s)\left(\ln\frac{s}{\delta}\right)^{\Delta_1}} \left(\mathbf{F}^{\mu_1,\nu_1+1}_{u,\omega;\delta+} + \nu_1 \mathbf{F}^{\mu_1+1,\nu_1}_{u/t,\omega;\delta+} - \mathbf{F}^{\mu_1+1,\nu_1+1}_{u',\omega;\delta+}\right)\phi(s) \\
+ \;& \frac{\Gamma(\mu_2+1)}{\omega(s)\left(\ln\frac{\Delta}{s}\right)^{\nu_2}} \left(\mathbf{F}^{\mu_2,\nu_2+1}_{u,\omega;\delta-} + \nu_2 \mathbf{F}^{\mu_2+1,\nu_2}_{u/t,\omega;\Delta-} + \mathbf{F}^{\mu_2+1,\nu_2+1}_{u',\omega;\Delta-}\right)\phi(s) \\
- \;& \left(\phi(\delta)u(\delta)(s-\delta)^{\mu_1} + \phi(\Delta)u(\Delta)(\Delta-s)^{\mu_2}\right)\Big| \\
\leq \;& |\phi'(s)| \left((\Delta-s)^{\mu_2+1}\left(\ln\frac{\Delta}{s}\right)^{\Delta_2} + (s-\delta)^{\mu_1+1}\left(\ln\frac{s}{\delta}\right)^{\nu_1}\right) \int_0^1 h(1-z)dz \\
+ \;& \left(|\phi'(\Delta)|(\Delta-s)^{\mu_2+1}\left(\ln\frac{\Delta}{s}\right)^{\nu_2} + |\phi'(\delta)|(s-\delta)^{\mu_1+1}\left(\ln\frac{s}{\delta}\right)^{\nu_1}\right) \int_0^1 h(z)dz
\end{aligned} \tag{2.24}$$

holds.Where u' ***is the usual derivative of*** u ***and*** $(u/t)(t)$ ***denote*** $\dfrac{u(t)}{t}$.

Proof 2.3.13 ***First step:For*** $s \in (\delta, \Delta)$ ***consider the function*** ϕ ***on the inter-***

val $[\delta, s]$. ***Hence for*** $\mu_1, \nu_1 \geq 1$ ***and*** $t \in [\delta, s]$ ***the following inequalitiy***

$$\left[\ln \frac{s}{t}\right]^{\nu_1} u(t)(s-t)^{\mu_1} \leq \left[\ln \frac{s}{\delta}\right]^{\nu_1} u(s)(s-\delta)^{\mu_1+1} \tag{2.25}$$

holds.Due to the h***- convexity of*** $|\phi'|$, ***it results that for*** $t \in [\delta, s]$

$$\begin{aligned} & -\left(h\left(\frac{s-t}{s-\delta}\right)|\phi'(\delta)| + h\left(\frac{t-\delta}{s-\delta}\right)|\phi'(s)|\right) \\ \leq \;& \phi'(t) \leq h\left(\frac{s-t}{s-\delta}\right)|\phi'(\delta)| + h\left(\frac{t-\delta}{s-\delta}\right)|\phi'(s)|. \end{aligned} \tag{2.26}$$

Multiplying (2.25)and the right side of (2.26) and integrating the result over $[\delta, s]$. ***Hence***

$$\begin{aligned} & \int_\delta^s u(t)(s-t)^{\mu_1}\left[\ln \frac{s}{t}\right]^{\nu_1} \phi'(t)dt \\ \leq \;& u(s)(s-\delta)^{\mu_1+1}\left[\ln \frac{s}{\delta}\right]^{\nu_1} \\ \times \;& \left(|\phi'(s)| \int_0^1 h(1-z)dz + |\phi'(\delta)| \int_0^1 h(z)dz\right). \end{aligned} \tag{2.27}$$

By integrating by parts,we obtain

$$\begin{aligned} & \int_\delta^s u(t)\,(s-t)^{\mu_1}\left(\ln \frac{s}{t}\right)^{\nu_1} \phi'(t)dt \\ = \;& -\phi(\delta)\left(\ln \frac{s}{\delta}\right)^{\nu_1} u(\delta)(s-\delta)^{\mu_1} \\ - \;& \int_\delta^s \left(\ln \frac{s}{t}\right)^{\nu_1} u'(t)\,(s-t)^{\mu_1}\,\phi(t)dt \\ + \;& \mu_1 \int_\delta^s \left(\ln \frac{s}{t}\right)^{\nu_1} u(t)\,(s-t)^{\mu_1-1}\,\phi(t)dt \\ + \;& \nu_1 \int_\delta^s \left(\ln \frac{s}{t}\right)^{\nu_1-1} \frac{u(t)}{t}(s-t)^{\mu_1}\,\phi(t)dt \end{aligned}$$

$$\leq \left[\ln \frac{s}{\delta}\right]^{\nu_1} u(s)(s-\delta)^{\mu_1+1}$$
$$\times \left(|\phi'(s)| \int_0^1 h(1-z)dz + |\phi'(\delta)| \int_0^1 h(z)dz\right). \tag{2.28}$$

Using definition 2.3.1 and inequality (2.28),it follows that

$$\frac{\Gamma(\mu_1+1)}{\omega(s)\left(\ln \frac{s}{\delta}\right)^{\nu_1}} \left(\mathbf{F}^{\mu_1,\nu_1+1}_{u,\omega;\delta+} + \nu_1 \mathbf{F}^{\mu_1+1,\nu_1}_{u/t,\omega;\delta+} - \mathbf{F}^{\mu_1+1,\nu_1+1}_{u',\omega;\delta+}\right)\phi(s) \tag{2.29}$$
$$- \phi(\delta)u(\delta)(s-\delta)^{\mu_1}$$
$$\leq u(s)(s-\delta)^{\mu_1+1}\left(|\phi'(s)| \int_0^1 h(1-z)dz + |\phi'(\delta)| \int_0^1 h(z)dz\right).$$

By considering the left hand side of (2.26), we deduce a similar inequality

$$-u(s)(s-\delta)^{\mu_1+1}\left[\ln \frac{s}{\delta}\right]^{\nu_1} \times \left(|\phi'(s)| \int_0^1 h(1-z)dz + |\phi'(\delta)| \int_0^1 h(z)dz\right)$$
$$\leq \int_\delta^s u(t)\,(s-t)^{\mu_1}\left[\ln \frac{s}{t}\right]^{\nu_1} \phi'(t)dt. \tag{2.30}$$

By combining the resulting inequalitiy and (2.29), we obtain

$$\left|\frac{\Gamma(\mu_1+1)}{\omega(s)\left(\ln \frac{s}{\delta}\right)^{\nu_1}} \left(\mathbf{F}^{\mu_1,\nu_1+1}_{u,\omega;\delta+} + \nu_1 \mathbf{F}^{\mu_1+1,\nu_1}_{u/t,\omega;\delta+} - \mathbf{F}^{\mu_1+1,\nu_1+1}_{u',\omega;\delta+}\right)\phi(s) - \phi(\delta)u(\delta)(s-\delta)^{\mu_1}\right|$$
$$\leq u(s)(s-\delta)^{\mu_1+1}\left(|\phi'(s)| \int_0^1 h(1-z)dz + |\phi'(\delta)| \int_0^1 h(z)dz\right). \tag{2.31}$$

Last step:Let $t \in [s,\Delta],\ \mu_2 > 0, \nu_2 \geq 0,$ ***and taking in acount that*** $|\phi'|$ ***is*** h***-convex, thus it follows that***

$$(t-s)^{\mu_2}\left(\ln \frac{t}{s}\right)^{\nu_2} \leq (\Delta - s)^{\mu_2}\left(\ln \frac{\Delta}{s}\right)^{\nu_2} \tag{2.32}$$

and

$$-\left(h\left(\frac{\Delta-t}{\Delta-s}\right)|\phi'(s)|+h\left(\frac{t-s}{\Delta-s}\right)|\phi'(\Delta)|\right) \tag{2.33}$$
$$\leq \phi'(t)\leq h\left(\frac{\Delta-t}{\Delta-s}\right)|\phi'(s)|+h\left(\frac{t-s}{\Delta-x}\right)|\phi'(\Delta)|.$$

The rest is similar to the first step.Consequently

$$\left|\frac{\Gamma(\mu_2+1)}{\omega(s)\left(\ln\frac{\Delta}{s}\right)^{\nu_2}}\left(\mathbf{F}^{\mu_2,\nu_2+1}_{u,\omega;\Delta-}+\nu_2\mathbf{F}^{\mu_2+1,\nu_2}_{u/t,\omega;\Delta-}+\mathbf{F}^{\mu_2+1,\nu_2+1}_{u',\omega;\Delta-}\right)\phi(s)-\phi(\Delta)u(\Delta)(\Delta-s)^{\mu_2}\right|$$
$$\leq u(s)(\Delta-s)^{\mu_2+1}\left(|\phi'(s)|\int_0^1 h(1-z)dz+|f'(\Delta)|\int_0^1 h(z)dz\right) \tag{2.34}$$

Via triangular inequality, by adding inequalities (2.31) and (2.34),the requiered inequality holds.

As special cases, we have the following corollaries,

Corollary 2.3.14 ***By setting*** $\mu_1=\mu_2=\mu,\nu_1=\nu_2=\nu,h(t)=t^r,r\in(0,1]$ ***in (2.24) then***

$$|\frac{\Gamma(\mu+1)}{\omega(s)u(s)}\left(\left[\mathbf{F}^{\mu,\nu+1}_{u,\omega;\delta+}+\nu\mathbf{F}^{\mu+1,\nu}_{u/t,\omega;\delta+}-\mathbf{F}^{\mu+1,\nu+1}_{u',\omega;\delta+}+\mathbf{F}^{\mu,\nu+1}_{u,\omega;\Delta-}+\nu\mathbf{F}^{\mu+1,\nu}_{u/t,\omega;\Delta-}+\mathbf{F}^{\mu+1,\nu+1}_{u',\omega;\Delta-}\right]\phi\right)(s)$$
$$-\frac{1}{u(s)}\left(\left((\Delta-s)^{\mu+1}\ln\frac{\Delta}{s}\right)^{\beta}u(\Delta)\phi(\Delta)+(s-\delta)^{\mu+1}\ln\left(\frac{s}{\delta}\right)^{\nu}u(\delta)\phi(\delta)\right)|$$
$$\leq |\phi'(s)|\frac{\left((\Delta-s)^{\mu+1}\left(\ln\frac{\Delta}{s}\right)^{\nu}+(s-\delta)^{\mu+1}\left(\ln\frac{s}{\delta}\right)^{\nu}\right)}{r+1}$$
$$+|\phi'(\Delta)|\frac{(\Delta-s)^{\mu+1}\left(\ln\frac{\Delta}{s}\right)^{\nu}}{r+1}+|\phi'(\delta)|\frac{(s-\delta)^{\mu+1}\left(\ln\frac{s}{\delta}\right)^{\nu}}{r+1} \tag{2.35}$$

holds.

Corollary 2.3.15 ***If we choose*** $u = 1, v = 1, \nu = 0,$ ***and*** $r = 1$ ***in (2.35), then***

$$\begin{aligned} & \left|\Gamma(\mu+1)\left(\mathbf{J}_{\delta+}^{\mu}\phi(s)+\mathbf{J}_{b-}^{\mu}\phi(s)\right)-\left((\Delta-s)^{\mu}\phi(\Delta)+(s-\delta)^{\mu}\right)\phi(\delta)\right| \\ \leq & \ |\frac{(\Delta-s)^{\mu+1}+(s-\delta)^{\mu+1}}{2}\phi'(s)|+\frac{(\Delta-s)^{\mu+1}}{2}|\phi'(\Delta)|+\frac{(s-\delta)^{\mu+1}}{2}|\phi'(\delta)| \end{aligned} \tag{2.36}$$

holds.

Corollary 2.3.16 ***On letting*** $x = \dfrac{\delta+\Delta}{2}$ ***and*** $\mu = 1,$ ***in (2.36), then***

$$\begin{aligned} & \left|\frac{1}{\Delta-\delta}\int_{\delta}^{\Delta} f(t)dt - \frac{f(\Delta)+f(\delta)}{2}\right| \\ \leq & \ \frac{(\Delta-\delta)}{8}\left[2\left|f'\left(\frac{\delta+\Delta}{2}\right)\right|+|f'(\Delta)|+|f'(\delta)|\right] \end{aligned} \tag{2.37}$$

is valid.

We need the following result

Lemma 2.3.17 ***Assume that*** $\phi : [\delta, \Delta] \to \mathbb{R},$ ***be*** $h-$***convex function and*** ϕ ***is symmetric about*** $\dfrac{\delta+\Delta}{2}$, ***then***

$$\phi\left(\frac{\delta+\Delta}{2}\right) \leq 2h\left(\frac{1}{2}\right)\phi(x) \quad x \in [\delta, \Delta]. \tag{2.38}$$

is valid.

Proof 2.3.18 ***We have***

$$\frac{\delta+\Delta}{2} = \frac{1}{2}\left(\delta\frac{x-\delta}{\Delta-\delta}+\Delta\frac{\Delta-x}{\Delta-\delta}\right)+\frac{1}{2}\left(\Delta\frac{x-\delta}{\Delta-\delta}+\delta\frac{\Delta-x}{\Delta-\delta}\right).$$

Hence,

$$\begin{aligned}
\phi\left(\frac{\delta+\Delta}{2}\right) &\leq h\left(\frac{1}{2}\right)\left[\phi\left(\delta\frac{x-\delta}{\Delta-\delta}+\Delta\frac{\Delta-x}{\Delta-\delta}\right)\right] \\
&+ h\left(\frac{1}{2}\right)\left[\phi\left(\Delta\frac{x-\delta}{\Delta-\delta}+\delta\frac{\Delta-x}{\Delta-\delta}\right)\right] \\
&= h\left(\frac{1}{2}\right)\phi(\delta+\Delta-x)+h\left(\frac{1}{2}\right)\phi(x) \\
&= 2h\left(\frac{1}{2}\right)\phi(x).
\end{aligned}$$

Theorem 2.3.19 ***Let*** $\mu_1 > 0, \mu_2 > 0, \nu_1, \nu_2 \geq 1$.***Let*** $\phi : [\delta; \Delta] \to \mathbb{R}$ ***be a non-negative*** h***-convex function, where*** h ***is Lebesgue integrable on*** $(0,1)$***. Let*** u, ω ***be integrable and non-negative functions,***$\omega(\delta) \neq 0, \omega(\Delta) \neq 0$. ***Also suppose that*** u ***is monotonic on*** $[\delta, \Delta]$, ***for*** $s \in (\delta, \Delta)$. ***If*** ϕ ***is symmetric about*** $\frac{\delta+\Delta}{2}$. ***It follows that***

1. ***If*** u ***is increasing,then***

$$\begin{aligned}
&\frac{u(\delta)}{2h\left(\frac{1}{2}\right)}\left[\int_\delta^\Delta (t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}+(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}dt\right]\phi\left(\frac{\delta+\Delta}{2}\right) \\
\leq\ &\frac{\Gamma(\mu_1+1)\mathbf{F}^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{v(\delta)}+\frac{\Gamma(\mu_2+1)\mathbf{F}^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{v(\Delta)} \\
\leq\ &u(\Delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right) \\
\times\ &(\phi(\delta)+\phi(\Delta))\int_0^1 h(z)dz \qquad (2.39)
\end{aligned}$$

holds .

2. If u is decreasing,then

$$
\begin{aligned}
& \frac{u(\Delta)}{2h\left(\frac{1}{2}\right)}\left[\int_{\delta}^{\Delta}(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}+(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}dt\right]\phi\left(\frac{\delta+\Delta}{2}\right)\\
\leq\ & \frac{\Gamma(\mu_1+1)\mathbf{F}^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)}+\frac{\Gamma(\mu_2+1)\mathbf{F}^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)}\\
\leq\ & u(\delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right)\\
\times\ & \left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right) \qquad (2.40)
\end{aligned}
$$

is valid .

Proof 2.3.20 ***We start by the case u is increasing. For $t\in[\delta,\Delta],\mu_1>0,\nu_1\geq 1$, we have***

$$
(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}u(t)\leq(\Delta-\delta)^{\mu_1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}u(\Delta) \qquad (2.41)
$$

and

$$
\phi(t)\leq h\left(\frac{t-\delta}{\Delta-\delta}\right)\phi(\delta)+h\left(\frac{\Delta-t}{\Delta-\delta}\right)\phi(\Delta). \qquad (2.42)
$$

Multiplying inequalities (2.41),(2.42) side to side,and integrating the result over $[\delta,\Delta]$. It follows that

$$
\begin{aligned}
& \int_{\delta}^{\Delta}(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}u(t)\phi(t)dt\leq(\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}u(\Delta)\\
\times\ & \left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right). \qquad (2.43)
\end{aligned}
$$

From which, we have

$$\frac{\Gamma(\mu_1+1)\mathbf{F}^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)} \tag{2.44}$$
$$\leq u(\Delta)(\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right).$$

On the other hand for $t\in[\delta,\Delta]$, ***we have***

$$(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}u(t)\leq(\Delta-\delta)^{\mu_2}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}u(\Delta). \tag{2.45}$$

By multiplying (2.42) and (2.45) and integrating the result over $[\delta,\Delta]$, ***we get***

$$\frac{\Gamma(\mu_2+1)\mathbf{F}^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)}\leq u(\Delta)(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}$$
$$\times\left(\phi(\delta)\int_0^1 h(z)+\phi(\Delta)\int_0^1 h(1-z)dz\right). \tag{2.46}$$

By adding (2.44) and (2.46),it results that

$$\frac{\Gamma(\mu_1+1)\mathbf{F}^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)}+\frac{\Gamma(\mu_2+1)\mathbf{F}^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)}$$
$$\leq u(\Delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right)$$
$$\times(\phi(\delta)+\phi(\Delta))\int_0^1 h(z)\mathrm{d}z. \tag{2.47}$$

Using Lemma (2.3.17),we have

$$\phi\left(\frac{\delta+\Delta}{2}\right)u(\delta)(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}$$

$$\leq \; 2h\left(\frac{1}{2}\right)\phi(t)u(t)(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}, \tag{2.48}$$

integrating (2.48) over $[\delta, \Delta]$, ***we get***

$$\begin{aligned} & u(\delta)\phi\left(\frac{\delta+\Delta}{2}\right)\int_\delta^\Delta (t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1} dt \\ \leq \; & 2h\left(\frac{1}{2}\right)\frac{\Gamma(\mu_1+1)\,\mathbf{F}^{\mu_1+1,\nu_1}_{u,v;\Delta-}\phi(\delta)}{v(\delta)}. \end{aligned} \tag{2.49}$$

Similarly,we have

$$\begin{aligned} & \phi\left(\frac{\delta+\Delta}{2}\right)u(\delta)(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1} \\ \leq \; & 2h\left(\frac{1}{2}\right)\phi(t)u(t)(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1} \end{aligned} \tag{2.50}$$

integrating (2.50) with respect to t ***over*** $[\delta, \Delta]$, ***we get***

$$\begin{aligned} & u(\delta)\phi\left(\frac{\delta+\Delta}{2}\right)\int_\delta^\Delta (\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1} dt \\ \leq \; & 2h\left(\frac{1}{2}\right)\frac{\Gamma(\mu_2+1)\mathbf{F}^{\mu_2+1,\nu_2}_{u,v;\delta+}\phi(\Delta)}{v(\Delta)}. \end{aligned} \tag{2.51}$$

Adding (2.49) and (2.51), we obtain

$$\begin{aligned} & u(\delta)\phi\left(\frac{\delta+\Delta}{2}\right)\left[\int_\delta^\Delta (\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1} + (t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1} dt\right] \\ \leq \; & 2h\left(\frac{1}{2}\right)\left[\frac{\Gamma(\mu_1+1)\mathbf{F}^{\mu_1+1,\nu_1}_{u,v;\Delta-}\phi(\delta)}{v(\delta)} + \frac{\Gamma(\mu_2+1)\mathbf{F}^{\mu_2+1,\nu_2}_{u,v;\delta+}\phi(\Delta)}{v(\Delta)}\right], \end{aligned} \tag{2.52}$$

combining (2.47) and (2.52),we have

$$
\begin{aligned}
&\frac{u(\delta)}{2h\left(\frac{1}{2}\right)}\left[\int_{\delta}^{\Delta}(\Delta-t)^{\mu_2}\left(\ln\frac{\Delta}{t}\right)^{\nu_2-1}+(t-\delta)^{\mu_1}\left(\ln\frac{t}{\delta}\right)^{\nu_1-1}dt\right]\phi\left(\frac{\Delta+\delta}{2}\right)\\
\leq\ & \frac{\Gamma(\mu_1+1)\mathbf{F}^{\mu_1+1,\nu_1}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)}+\frac{\Gamma(\mu_2+1)\mathbf{F}^{\mu_2+1,\nu_2}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)}\\
\leq\ & u(\Delta)\left((\Delta-\delta)^{\mu_1+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_1-1}+(\Delta-\delta)^{\mu_2+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu_2-1}\right)\\
\times\ & (\phi(\delta)+\phi(\Delta))\int_0^1 h(z)dz.
\end{aligned}
$$

Similar proof for the case u ***decreasing.***

Corollary 2.3.21 ***By setting*** $\mu_1=\mu_2=\mu$ ***and*** $\nu_1=\nu_2=\nu$, ***we obtain***

$$
\begin{aligned}
&\frac{u(\delta)}{2h\left(\frac{1}{2}\right)}\left[\int_{\delta}^{\Delta}(\Delta-t)^{\mu}\left(\ln\frac{\Delta}{t}\right)^{\nu-1}+(t-\delta)^{\mu}\left(\ln\frac{t}{\delta}\right)^{\nu-1}dt\right]\phi\left(\frac{\delta+\Delta}{2}\right)\\
\leq\ & \Gamma(\mu+1)\left(\frac{\mathbf{F}^{\mu+1,\nu}_{u,\omega;\Delta-}\phi(\delta)}{\omega(\delta)}+\frac{\mathbf{F}^{\mu+1,\nu}_{u,\omega;\delta+}\phi(\Delta)}{\omega(\Delta)}\right) \qquad (2.53)\\
\leq\ & 2u(\Delta)(\Delta-\delta)^{\mu+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1}(\phi(\delta)+\phi(\Delta))\int_0^1 h(z)dz.
\end{aligned}
$$

case u ***increasing.***

Example 2.3.22 ***The following example illustrates the validity of estimates. Let*** $\phi:[\delta;\Delta]\to\mathbb{R}_+,\ \phi(t)=1$ ***and*** $h_k(t)=t^k, k\leq 1, t>0.$ ***let*** $\mu>0,\nu\geq 1,$ $u=1,\omega=1.$ ***We verify that***

- ϕ ***is*** h_k***-convex.***
- ϕ ***is symmetric about*** $\frac{\delta+\Delta}{2}$.

Hence from corollary 2.3.21,we get the estimates

$$\begin{aligned}
&\frac{1}{2^{1-k}}\left[\int_{\delta}^{\Delta}(\Delta-t)^{\mu}\left(\ln\frac{\Delta}{t}\right)^{\nu-1}+(t-\delta)^{\mu}\left(\ln\frac{t}{\delta}\right)^{\nu-1}dt\right]\\
\leq\;& \Gamma(\mu+1)\left(\mathbf{F}^{\mu+1,\nu}_{1,1;\Delta-}1(\delta)+\mathbf{F}^{\mu+1,\nu}_{1,1;\delta+}1(\Delta)\right)\\
\leq\;& 2(\Delta-\delta)^{\mu+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1}\left(\int_0^1 z^k+\int_0^1(1-z)^k dz\right). \qquad (2.54)
\end{aligned}$$

Or

$$\frac{1}{2^{1-k}}\int_{\delta}^{\Delta}(\Delta-t)^{\mu}\left(\ln\frac{\Delta}{t}\right)^{\nu-1}\leq\int_{\delta}^{\Delta}(\Delta-t)^{\mu}\left(\ln\frac{\Delta}{t}\right)^{\nu-1}dt\leq\frac{2(\Delta-\delta)^{\mu+1}}{k+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1}.$$

and

$$\frac{1}{2^{1-k}}\int_{\delta}^{\Delta}(t-\delta)^{\mu}\left(\ln\frac{t}{\delta}\right)^{\nu-1}\leq\int_{\delta}^{\Delta}(t-\delta)^{\mu}\left(\ln\frac{t}{\delta}\right)^{\nu-1}dt\leq\frac{2(\Delta-\delta)^{\mu}}{k+1}+\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1}.$$

Take the change of variables $t=\delta+(\Delta-\delta)e^{-x}$ ***and*** $t=\Delta-(\Delta-\delta)e^{-x}$, ***we get***

$$\begin{aligned}
\frac{1}{2^{1-k}}\int_0^{\infty}e^{-(\mu+1)x}\left[\ln\left(1+\frac{\Delta-\delta}{\delta}e^{-x}\right)\right]^{\nu-1}dx &\leq \int_0^{\infty}e^{-(\mu+1)x}\left[\ln\left(1+\frac{\Delta-\delta}{\delta}e^{-x}\right)\right]^{\nu-1}dx\\
&\leq\frac{2}{k+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1},
\end{aligned}$$

and

$$\frac{1}{2^{1-k}}\int_0^{\infty}e^{-(\mu+1)x}\left[\ln\frac{\Delta e^x}{\Delta\,e^x-\Delta+\delta}\right]^{\nu-1}dx\leq\int_0^{\infty}e^{-(\mu+1)x}\left[\ln\frac{\Delta e^x}{\Delta\,e^x-\Delta+\delta}\right]^{\nu-1}dx$$

$$\leq \frac{2}{k+1}\left(\ln\frac{\Delta}{\delta}\right)^{\nu-1}.$$

For $\nu = 2$

$$\frac{1}{2^{1-k}}\int_0^{\infty} e^{-(\mu+1)x}\ln\frac{\delta\, e^x + \Delta - \delta}{\delta\, e^x}\,dx \leq \int_0^{\infty} e^{-(\mu+1)x}\ln\frac{\delta\, e^x + \Delta - \delta}{\delta\, e^x}\,dx \leq \frac{2}{k+1}\ln\frac{\Delta}{\delta},$$

and

$$\frac{1}{2^{1-k}}\int_0^{\infty} e^{-(\mu+1)x}\ln\frac{\Delta e^x}{\Delta\, e^x - \Delta + \delta}\,dx \leq \int_0^{\infty} e^{-(\mu+1)x}\ln\frac{\Delta e^x}{\Delta\, e^x - \Delta + \delta}\,dx \leq \frac{2}{k+1}\ln\frac{\Delta}{\delta}.$$

Taking $k = 1$, ***we get***

$$F(\lambda) := \int_0^{\infty} e^{-\lambda x}\ln\,\frac{\delta\, e^x + \Delta - \delta}{\delta\, e^x}\,dx \leq\, \ln\,\frac{\Delta}{\delta}\;(\lambda > 0),$$

and

$$G(\lambda) := \int_0^{\infty} e^{-\lambda x}\ln\,\frac{\Delta e^x}{\Delta\, e^x - \Delta + \delta}\,dx \leq \ln\,\frac{\Delta}{\delta}\;(\lambda > 0).$$

The functions F, G ***are the integral transforms of*** $f(x) = \frac{\delta\, e^x + \Delta - \delta}{\delta\, e^x}$, $g(x) = \frac{\Delta e^x}{\Delta\, e^x - \Delta + \delta}$ ***respectivelly.***

Corollary 2.3.23 ***If we choose*** $u = 1, \omega = 1, \nu = 1$ ***and taking*** $h(x) = x$ ***in (2.53), then the inequality***

$$\begin{aligned} & \frac{1}{(\mu+1)}\,\phi\left(\frac{\delta + \Delta}{2}\right) \\ \leq\;& \frac{\Gamma(\mu+1)}{2(\Delta - \delta)^{\mu+1}}\left(\mathbf{J}^{\mu+1}_{\Delta-}\phi(\delta) + \mathbf{J}^{\mu+1,}_{\delta+}\phi(\Delta)\right) \\ \leq\;& \frac{\phi(\delta) + \phi(\Delta)}{2} \end{aligned} \tag{2.55}$$

holds.

Remark 2.3.24 ***On letting*** $\mu \to 0$,***in (2.55), we get the inequality of Hermite-Hadamard***

$$\phi\left(\frac{\Delta+\delta}{2}\right) \leq \frac{1}{\Delta-\delta}\int_{\delta}^{\Delta}\phi(t)dt \leq \frac{\phi(\Delta)+\phi(\delta)}{2}. \tag{2.56}$$

Chapter 3

SOME INTEGRAL INEQUAlITIES INVOLVING H-STRONGLY CONVEX FONCTIONS AND APPLICATIONS

3.1 Introduction

In the realm of mathematics, inequalities and integral inequalities serve as foundational tools that significantly influence a multitude of fields, ranging from pure mathematics to applied sciences. These mathematical constructs are not merely theoretical curiosities; they have profound implications in areas such as optimization, numerical analysis, and even in the study of dif-

ferential equations. Over the years, a rich body of literature has emerged, with numerous authors investigating a wide array of inequalities that apply to both fractional integrals and classical integrals. Notable contributions can be found in works such as , [17, 31, 15, 30], among others. These studies highlight the versatility among others. These studies highlight the versatility and importance of integral inequalities in advancing mathematical knowledge. One particularly captivating aspect of this area of study is the relationship between convexity and inequalities. The property of convexity has attracted considerable interest from mathematicians around the globe, especially when employed in conjunction with Riemann-Liouville fractional integrals. Convex functions are characterized by their unique properties, which often lead to stronger and more insightful inequalities. As a result, the exploration of convexity has evolved significantly over recent years, with researchers expanding its definitions and generalizations to encompass a wider variety of functions and contexts. Numerous integral inequalities have been developed specifically for various categories of convex functions, reflecting the ongoing fascination with this topic. Works such as [8, 31, 16, 23, 20, 18, 11, 5, 33, 34] illustrate the breadth of research dedicated to this area. These contributions not only deepen our understanding of convexity but also enhance our ability to apply these concepts across different mathematical domains. In this article, we introduce a new perspective by considering an alternative type of fractional integral operator, as discussed in sources like [8] and [17]. This operator generalizes existing frameworks and opens new avenues for exploration within the context of strongly h-convex functions. By utilizing this more generic fractional inte-

gral operator, we aim to establish novel fractional integral inequalities that contribute to the rich tapestry of mathematical inequality theory. Through this work, we hope to shed light on the intricate connections between convexity, fractional integrals, and inequalities. Our findings not only enhance theoretical understanding but also pave the way for future research opportunities in both pure and applied mathematics.

3.2 Preliminaries

Definition 3.2.1 ***(Strongly*** $h-$ ***convexity)[3]. Let*** $\mathcal{I}$ ***an interval of*** $\mathbb{R}$ ***and*** $h : [0, 1] \to (0, \infty)$ ***be a given non negative function. A non negative function*** $\vartheta : \mathcal{I} \to \mathbb{R}$ ***is said to be strongly*** h***- convex whith modulus*** $\beta > 0,$ ***if***

$$\vartheta(\lambda\varsigma + (1-\lambda)\upsilon) \leq \xi(\lambda)\vartheta(\varsigma) + \xi(1-\lambda)\vartheta(\upsilon) - \beta\lambda\left(\lambda - 1\right)\left(\varsigma - \upsilon\right)^2 \tag{3.1}$$

holds for all $\varsigma, \upsilon \in \mathcal{I}$ ***and*** $\lambda \in (0, 1]$. ***The function*** ϑ ***is said to be strongly*** $h-$ ***concave if (3.1) is reversed.***

Remark 3.2.2

- ***For*** $\beta = 0,$ ***we get the notion of*** $h-$ ***convex (***$h-$ ***concave)[35].***
- ***For*** $h(t) = t,$ ***(3.2) means that*** φ ***is strongly convex.***
- ***Any convex function*** ϑ ***on a subset*** B ***with*** $h(t) \leq t$ ***(*** $h(t) \geq t$***) is strongly convex (strongly concave).***

Example 3.2.3 ***Let*** $\vartheta : [-1, 1] \to \mathbb{R}$ ***defined by*** $\vartheta(t) = 2,$ ***and*** $h(t) = 1, t \in (0, 1)$

Then ϑ ***is strongly*** $h-$***convex with modulus*** $\beta = 1$. ***Indeed,for every*** $t, p \in [-1, 1]$***and*** $\lambda \in (0, 1)$,***we have***

$$\vartheta(\lambda p + (1-\lambda)t) = 2 \leq h(\lambda)\vartheta(p) + h(1-\lambda)\vartheta(t) - \beta\lambda(1-\lambda)(p-t)^2$$
$$= 4 - \lambda(1-\lambda)(p-t)^2.$$

Different classes of strongly convex functions are obtained by taking in (3.1) $h(\nu) = \nu, h(\nu) = \nu^r(r \in (0.1)), h(\nu) = 1/\nu$ **and** $h(\nu) = 1$, **(see [35]).**

We recall that

$$\mathbf{F}^{\mu,\nu}_{u,\omega;\delta+}\phi(s) = \frac{\omega(s)}{\Gamma(\mu)} \int_\delta^s (s-t)^{\mu-1} \left[\ln \frac{s}{t}\right]^{\nu-1} \phi(t)\, u(t) dt, \tag{3.2}$$

and

$$\mathbf{F}^{\mu,\nu}_{u,\omega;\Delta-}\phi(s) = \frac{\omega(s)}{\Gamma(\mu)} \int_s^\Delta (t-s)^{\mu-1} \left[\ln \frac{t}{s}\right]^{\nu-1} \phi(t)\, u(t) dt. \tag{3.3}$$

Where u, ω **are bounded,locally integrable and non-negative functions.**

Remark 3.2.4 ***1. If*** $r = 1, v(\varsigma) = u(\varsigma) = 1, \varsigma \in (a, b)$, ***the operator*** $\mathbf{F}^{q,1}_{1,1} = J^q_.$, ***where*** $J^q_.$ ***is the Riemann-Liouville integral operator [2].***

2. If $q = 1, v(\varsigma) = 1, u(\varsigma) = \frac{1}{\varsigma}$, ***the operator*** $\mathbf{K}^{1,r}_{\frac{1}{\varsigma},1}$ ***coincides with the classical Hadamard integral operator*** $H^r_.$:

$$H^r_{a+}\vartheta(\varsigma) = \frac{1}{\Gamma(r)} \int_a^\varsigma \left(\ln \frac{\varsigma}{\nu}\right)^{r-1} \vartheta(\nu)\frac{d\nu}{\nu}, \quad \varsigma > a \tag{3.4}$$

and

$$H^r_{b-}\vartheta(\varsigma) = \frac{1}{\Gamma(r)} \int_\varsigma^b \left(\ln \frac{\nu}{\varsigma}\right)^{r-1} \vartheta(\nu)\frac{d\nu}{\nu} \quad \varsigma < b \quad (right). \tag{3.5}$$

We need the following result

Lemma 3.2.5 ***Let*** $\vartheta : [a,b] \to \mathbb{R}$ ***be a strongly*** $h-$ ***convex function with modulus*** $\beta > 0$. ***If*** ϑ ***is symmetric about*** $\frac{a+b}{2}$***, then it results that***

$$\vartheta\left(\frac{a+b}{2}\right) \leq 2h\left(\frac{1}{2}\right)\vartheta(\varsigma) + \beta\,\frac{a+b-2\varsigma}{4}, \quad \varsigma \in [a,b]. \tag{3.6}$$

Proof 3.2.6 ***We have***

$$\frac{a+b}{2} = \frac{1}{2}\left(b\frac{\varsigma-a}{b-a} + a\frac{b-\varsigma}{b-a}\right) + \frac{1}{2}\left(a\frac{\varsigma-a}{b-a} + b\frac{b-\varsigma}{b-a}\right); \tag{3.7}$$

since ϑ ***is strongly*** h***- convex, then we have***

$$\begin{aligned}
\vartheta\left(\frac{a+b}{2}\right) &= \vartheta\left(\frac{1}{2}\left[b\frac{\varsigma-a}{b-a} + a\frac{b-\varsigma}{b-a}\right] + \frac{1}{2}\left[\left(a\frac{\varsigma-a}{b-a} + b\frac{b-\varsigma}{b-a}\right)\right]\right) \\
&\leq h\left(\frac{1}{2}\right)\vartheta(\varsigma) + h\left(\frac{1}{2}\right)\vartheta(a+b-\varsigma) + \beta\frac{a+b-2\varsigma}{4} \\
&= 2h\left(\frac{1}{2}\right)\vartheta(\varsigma) + \beta\frac{a+b-2\varsigma}{4}.
\end{aligned}$$

3.3 Related results

Theorem 3.3.1 ***Let*** $q_1, q_2, r_1, r_2 \geq 1, 0 < a < b < \infty$. ***Let*** h ***be Lebesgue integrable on*** $(0,1)$ ***and*** $\vartheta : [a;b] \to \mathbb{R}$ ***be a non-negative strongly*** $h-$ ***convex function with modulus*** $\beta > 0$. ***Assum that*** u ***is integrable nonnegative function non decreasing on*** $[a,\varsigma]$, ***non increasing on*** $[\varsigma, b]$ ***for*** $\varsigma \in [a,b]$ ***and*** v ***a positive***

function. Then

$$\begin{aligned}
& \frac{1}{v(\varsigma)}\left(\Gamma(r_1)\Gamma(q_1)\mathbf{F}^{q_1,r_1}_{u,v;a+}\vartheta(\varsigma)+\Gamma(r_2)\Gamma(q_2)F^{q_2,r_2}_{u,v;b-}\vartheta(\varsigma)\right)\\
\leq\ & u(\varsigma)\left[\ln\frac{\varsigma}{a}\right]^{r_1-1}(\varsigma-a)^{q_1}\\
\times\ & \left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(a)\int_0^1 h(z)dz-\beta\,\frac{(\varsigma-a)^2}{6}\right\}\\
+\ & u(\varsigma)\left[\ln\frac{b}{\varsigma}\right]^{r_2-1}(\varsigma-a)^{q_2}\\
\times\ & \left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(b)\int_0^1 h(z)dz-\beta\,\frac{(b-\varsigma)^2}{6}\right\} \qquad (3.8)
\end{aligned}$$

holds.

Proof 3.3.2 ***Firstly let*** $\varsigma\in[a,b]$, ***then for*** $\nu\in[a,\varsigma]$ ***and*** $q_1\geq 1, r_1\geq 1$, ***the following inequality***

$$(\varsigma-\nu)^{q_1-1}\left(\ln\frac{\varsigma}{\nu}\right)^{r_1-1}u(\nu)\leq(\varsigma-a)^{q_1-1}\left(\ln\frac{\varsigma}{a}\right)^{r_1-1}u(\varsigma) \qquad (3.9)$$

holds. Since ϑ ***is strongly*** h***- convex on*** $[a,\varsigma]$ ***whith modulus*** $\beta>0$, ***we have***

$$\vartheta(\nu)\leq h\left(\frac{\varsigma-\nu}{\varsigma-a}\right)\vartheta(a)+h\left(\frac{\nu-a}{\varsigma-a}\right)\vartheta(\varsigma)-\beta\left(\varsigma-\nu\right)\left(\nu-a\right), \qquad (3.10)$$

and

$$\begin{aligned}
& \int_a^\varsigma(\varsigma-\nu)^{q_1-1}\left[\ln\frac{\varsigma}{\nu}\right]^{r_1-1}u(\nu)\,\vartheta(\nu)d\nu\\
\leq\ & (\varsigma-a)^{q_1-1}\left[\ln\frac{\varsigma}{a}\right]^{r_1-1}u(\varsigma)\\
\times\ & \left[\vartheta(a)\int_a^\varsigma h\left(\frac{\varsigma-\nu}{\varsigma-a}\right)d\nu+\vartheta(\varsigma)\int_a^\varsigma h\left(\frac{\nu-a}{\varsigma-a}\right)d\nu-\beta\int_a^\varsigma\left(\varsigma-\nu\right)\left(\nu-a\right)d\nu\right]\\
=\ & (\varsigma-a)^{q_1}\left[\ln\frac{\varsigma}{a}\right]^{r_1-1}u(\varsigma)
\end{aligned}$$

$$\times \quad \left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(a)\int_0^1 h(z)dz-\ \beta\,\frac{(\varsigma-a)^2}{6}\right\}.$$

In vertu of the definition (??), it results that

$$\Gamma(r_1)\Gamma\left(q_1\right)\mathbf{F}^{q_1,r_1}_{u,v;a+}\vartheta(\varsigma)\leq u(\varsigma)v(\varsigma)\left[\ln\frac{\varsigma}{a}\right]^{r_1-1}\times(\varsigma-a)^{q_1}$$
$$\left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(a)\int_0^1 h(z)dz-\beta\,\frac{(\varsigma-a)^2}{6}\right\}. \tag{3.11}$$

And similarly for $\nu\in[\varsigma,b],\varsigma\in(a,b)$ ***and*** $q_2,r_2\geq 1$ ***the following inequality***

$$(\nu-\varsigma)^{q_2-1}\left(\ln\frac{\nu}{\varsigma}\right)^{r_2-1}u(\nu)\leq(b-\varsigma)^{q_2-1}\left(\ln\frac{b}{\varsigma}\right)^{r_2-1}u(\varsigma) \tag{3.12}$$

holds.

Using the fact that ϑ ***is strongly*** $h-$ ***convex on*** $[\varsigma,b]$, ***we get***

$$\Gamma(r_2)\Gamma\left(q_2\right)\mathbf{F}^{q_2,r_2}_{u,v;b-}\vartheta(\varsigma)\leq u(\varsigma)v(\varsigma)\left[\ln\frac{b}{\varsigma}\right]^{r_2-1}\times(b-\varsigma)^{q_2}$$
$$\left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(b)\int_0^1 h(z)dz-\beta\,\frac{(b-\varsigma)^2}{6}\right\}. \tag{3.13}$$

By adding (3.11) and (3.13), we obtain (3.8).

Remark 3.3.3 ***1.If*** u ***is increasing on*** $[a,b]$, ***then for all*** $\varsigma\in[a,b]$***,we have***

$$\begin{aligned}&\frac{1}{v(\varsigma)}\left(\Gamma(r_1)\Gamma(q_1)\mathbf{F}^{q_1,r_1}_{u,v;a+}\vartheta(\varsigma)+\Gamma(r_2)\Gamma(q_2)F^{q_2,r_2}_{u,v;b-}\vartheta(\varsigma)\right)\\ \leq\ & u(\varsigma)\left[\ln\frac{\varsigma}{a}\right]^{r_1-1}(\varsigma-a)^{q_1}\\ \times\ &\left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(a)\int_0^1 h(z)dz-\beta\,\frac{(\varsigma-a)^2}{6}\right\}\end{aligned}$$

$$
\begin{aligned}
&+ \quad u(b)\left[\ln \frac{b}{\varsigma}\right]^{r_2-1}(\varsigma-a)^{q_2} \\
&\times \quad \left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(b)\int_0^1 h(z)dz-\beta\,\frac{(b-\varsigma)^2}{6}\right\} \qquad (3.14)
\end{aligned}
$$

2.If** u **is decreasing on** $[a,b]$, **then for all** $\varsigma\in[a,b]$**,we have

$$
\begin{aligned}
&\frac{1}{v(\varsigma)}\left(\Gamma(r_1)\Gamma(q_1)\mathbf{F}^{q_1,r_1}_{u,v;a+}\vartheta(\varsigma)+\Gamma(r_2)\Gamma(q_2)F^{q_2,r_2}_{u,v;b-}\vartheta(\varsigma)\right) \\
&\leq \quad u(a)\left[\ln \frac{\varsigma}{a}\right]^{r_1-1}(\varsigma-a)^{q_1} \\
&\times \quad \left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(a)\int_0^1 h(z)dz-\beta\,\frac{(\varsigma-a)^2}{6}\right\} \\
&+ \quad u(\varsigma)\left[\ln \frac{b}{\varsigma}\right]^{r_2-1}(\varsigma-a)^{q_2} \\
&\times \quad \left\{\vartheta(\varsigma)\int_0^1 h(1-z)dz+\vartheta(b)\int_0^1 h(z)dz-\beta\,\frac{(b-\varsigma)^2}{6}\right\}. \qquad (3.15)
\end{aligned}
$$

Now we investigate some integral inequalities for functions whose derivatives in absolute value are strongly h-convex.

Theorem 3.3.4 ***Let*** $q_1,q_2,r_1,r_2\geq 0$ ***and*** $0<a<b<\infty$. ***Let*** $\vartheta:[a;b]\to\mathbb{R}$ ***be a non-negative differentiable function.Let*** u ***a locally integrable non-negative function, absolutely continuous, non decreasing on*** $[a,\varsigma]$, ***non increasing on*** $[\varsigma,b]$ ***for*** $\varsigma\in[a,b]$ ***and*** v ***a positive function. If*** $|\vartheta'|$ ***is a strongly*** $h-$ ***convex function, then***

$$
\begin{aligned}
&\left|\,\frac{\Gamma(r_1+1)\Gamma\left(q_1+1\right)}{v(\varsigma)\left(\ln\frac{\varsigma}{a}\right)^{r_1}}\left(\mathbf{F}^{q_1,r_1+1}_{u,v;a+}+\mathbf{F}^{q_1+1,r_1}_{u/\nu,v;a+}-\mathbf{K}^{q_1+1,r_1+1}_{u',v;a+}\right)\vartheta(\varsigma)\right. \\
&+ \quad \frac{\Gamma(r_2+1)\Gamma\left(q_2+1\right)}{v(\varsigma)\left(\ln\frac{b}{\varsigma}\right)^{r_2}}\left(\mathbf{F}^{q_2,r_2+1}_{u,v;b-}+\mathbf{F}^{q_2+1,r_2}_{u/\nu,v;b-}+\mathbf{F}^{q_2+1,r_2+1}_{u',v;b-}\right)\vartheta(\varsigma) \\
&- \quad \left.\left(\vartheta(a)u(a)(\varsigma-a)^{q_1}+\vartheta(b)u(b)(b-\varsigma)^{q_2}\right)\,\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \; u(\varsigma)\left|\vartheta'(\varsigma)\right|\left[(b-\varsigma)^{q_2+1}+(\varsigma-a)^{q_1+1}\right]\int_0^1 h(1-z)dz \\
&+ \; u(\varsigma)\left[\left|\vartheta'(b)\right|(b-\varsigma)^{q_2+1}+\left|\vartheta'(a)\right|(\varsigma-a)^{q_1+1}\right]\int_0^1 h(z)dz \\
&- \; \beta\,\frac{(\varsigma-a)^{q_1+4}+(b-\varsigma)^{q_2+4}}{6}u(\varsigma) \qquad (3.16)
\end{aligned}$$

holds. Where u' ***is the usual derivative of*** u ***and*** $(u/\nu)(\nu) := \frac{u(\nu)}{\nu}$.

Proof 3.3.5 ***Firstly let*** $\varsigma \in [a,b]$. ***Then for*** $\nu \in [a,\varsigma]$ ***and*** $q_1 \geq 0, r_1 \geq 0$ ***the following inequality***

$$u(\nu)(\varsigma-\nu)^{q_1}\left(\ln\frac{\varsigma}{\nu}\right)^{r_1} \leq u(\varsigma)(\varsigma-a)^{q_1}\left(\ln\frac{\varsigma}{a}\right)^{r_1} \qquad (3.17)$$

holds.

Since $|\vartheta'|$ ***is strongly*** $h-$ ***convex therefore for*** $\nu \in [a,\varsigma]$, ***we have***

$$\begin{aligned}
&\boldsymbol{Lhs} = -\left[h\left(\frac{\varsigma-\nu}{\varsigma-a}\right)\left|\vartheta'(a)\right| + h\left(\frac{\nu-a}{\varsigma-a}\right)\left|\vartheta'(\varsigma)\right| - \beta\left(\varsigma-\nu\right)\left(\nu-a\right)\right] \\
&\leq \; \vartheta'(\nu) \leq \\
&h\left(\frac{\varsigma-\nu}{\varsigma-a}\right)\left|\vartheta'(a)\right| + h\left(\frac{\nu-a}{\varsigma-a}\right)\left|\vartheta'(\varsigma)\right| - \beta\ \left(\varsigma-\nu\right)\left(\nu-a\right) = \boldsymbol{Rhs}. \qquad (3.18)
\end{aligned}$$

Multiplying the Rhs of (3.18) and (3.17)side to side and integrating over $[a,\varsigma]$, ***we get***

$$\begin{aligned}
\int_a^\varsigma \left[\ln\frac{\varsigma}{\nu}\right]^{r_1} u(\nu)(\varsigma-\nu)^{q_1}\vartheta'(\nu)d\nu \;&\leq\; u(\varsigma)\left[\ln\frac{\varsigma}{a}\right]^{r_1}(\varsigma-a)^{q_1+1} \\
&\times\; \left(\left|\vartheta'(\varsigma)\right|\int_0^1 h(1-z)dz + \left|\vartheta'(a)\right|\int_0^1 h(z)dz\right) \\
&-\; \left(\ln\frac{\varsigma}{a}\right)^{r_1} u(\varsigma)\frac{\beta\ (\varsigma-a)^{q_1+4}}{6}. \qquad (3.19)
\end{aligned}$$

On the other hand by integrating by parts the left hand of (3.19), we have

$$
\begin{aligned}
& \int_a^{\varsigma} \left(\ln \frac{\varsigma}{\nu}\right)^{r_1} u(\nu)(\varsigma-\nu)^{q_1} \vartheta'(\nu) d\nu \\
= & -\vartheta(a) \left(\ln \frac{\varsigma}{a}\right)^{r_1} u(a)(\varsigma-a)^{q_1} \\
- & \int_a^{\varsigma} \left(\ln \frac{\varsigma}{\nu}\right)^{r_1} u'(\nu)(\varsigma-\nu)^{q_1} \vartheta(\varsigma) d\nu \\
+ & q_1 \int_a^{\varsigma} \left(\ln \frac{\varsigma}{\nu}\right)^{r_1} u(\nu)(\varsigma-\nu)^{q_1-1} \vartheta(\nu) d\nu \\
+ & r_1 \int_a^{\varsigma} \left(\ln \frac{\varsigma}{\nu}\right)^{r_1-1} \frac{u(\nu)}{\nu}(\varsigma-\nu)^{q_1} \vartheta(\nu) d\nu \\
\leq & u(\varsigma) \left[\ln \frac{\varsigma}{a}\right]^{r_1} (\varsigma-a)^{q_1+1} \\
\times & \left(|\vartheta'(\varsigma)| \int_0^1 h(1-z) dz + |\vartheta'(a)| \int_0^1 h(z) dz\right) \\
- & \frac{\beta \left[\ln \frac{\varsigma}{a}\right]^{r_1} (\varsigma-a)^{q_1+4}}{6} u(\varsigma).
\end{aligned} \tag{3.20}
$$

Using definition ?? and inequality (3.20), we get the inequality

$$
\begin{aligned}
& \frac{\Gamma(r_1+1)\Gamma(q_1+1)}{v(\varsigma)\left(\ln \frac{\varsigma}{a}\right)^{r_1}} \left(\mathbf{F}^{q_1,r_1+1}_{u,v;a+} + \mathbf{F}^{q_1+1,r_1}_{u/\nu,v;a+} - \mathbf{F}^{q_1+1,r_1+1}_{u',v;a+}\right) \vartheta(\varsigma) \\
- & \vartheta(a)u(a)(\varsigma-a)^{q_1} \\
\leq & u(\varsigma)(\varsigma-a)^{q_1+1} \left(|\vartheta'(\varsigma)| \int_0^1 h(1-z) dz + |\vartheta'(a)| \int_0^1 h(z) dz\right) \\
- & \frac{\beta\ (\varsigma-a)^{q_1+4}}{6} u(\varsigma).
\end{aligned} \tag{3.21}
$$

Now if we consider the Lhs of inequality (3.18), we have

$$
\begin{aligned}
- & \left[\ln \frac{\varsigma}{a}\right]^{r_1} u(\varsigma)(\varsigma-a)^{q_1+1} \\
\times & \left(|\vartheta'(\varsigma)| \int_0^1 h(1-z) dz + |\vartheta'(a)| \int_0^1 h(z) dz\right) \\
+ & \frac{\beta \left[\ln \frac{\varsigma}{a}\right]^{r_1} (\varsigma-a)^{q_1+4}}{6} u(\varsigma)
\end{aligned} \tag{3.22}
$$

$$\leq \int_a^{\varsigma} \left[\ln \frac{\varsigma}{\nu}\right]^{r_1} u(\nu)(\varsigma-\nu)^{q_1}\vartheta'(\nu)d\nu.$$

A similar reasoning leads to a similar inequality to (3.21).By combining the resulting inequality and the inequality (3.21), we get

$$\begin{aligned} &\left|\frac{\Gamma(r_1+1)\Gamma(q_1+1)}{v(\varsigma)\left(\ln\frac{\varsigma}{a}\right)^{r_1}}\left(\mathbf{F}^{q_1,r_1+1}_{u,v;a+}+\mathbf{F}^{q_1+1,r_1}_{u/t,v;a+}-\mathbf{F}^{q_1+1,r_1+1}_{u',v;a+}\right)\vartheta(\varsigma)-\vartheta(a)u(a)(\varsigma-a)^{q_1}\right| \\ \leq\;& u(\varsigma)(\varsigma-a)^{q_1+1}\left(|\vartheta'(\varsigma)|\int_0^1 h(1-z)dz+|\vartheta'(a)|\int_0^1 h(z)dz\right) \\ -\;& \frac{\beta\,(\varsigma-a)^{q_1+4}}{6}u(\varsigma). \end{aligned} \tag{3.23}$$

On the other hand for $q_2>0, r_2\geq 0, \nu\in[\varsigma,b]$, ***we have***

$$(\nu-\varsigma)^{q_2}\left(\ln\frac{\nu}{\varsigma}\right)^{r_2}u(\nu)\leq (b-\varsigma)^{q_2}\left(\ln\frac{b}{\varsigma}\right)^{r_2}u(\varsigma) \tag{3.24}$$

and

$$\begin{aligned} \boldsymbol{Lhs} &= -\left[h\left(\frac{b-\nu}{b-\varsigma}\right)|\vartheta'(\varsigma)|+h\left(\frac{\nu-\varsigma}{b-\varsigma}\right)|\vartheta'(b)|-\beta\frac{(b-\varsigma)^3}{6}\right] \\ \leq\;& \vartheta'(\nu)\leq \\ & h\left(\frac{b-\nu}{b-\varsigma}\right)|\vartheta'(\varsigma)|+h\left(\frac{\nu-\varsigma}{b-\varsigma}\right)|\vartheta'(b)|-c\frac{(b-\varsigma)^3}{6}=\boldsymbol{Rhs}. \end{aligned} \tag{3.25}$$

The rest is similar to the first step. Consequently

$$\begin{aligned} &\left|\frac{\Gamma(r_2+1)\Gamma(q_2+1)}{v(\varsigma)\left(\ln\frac{b}{\varsigma}\right)^{r_2}}\left(\mathbf{F}^{q_2,r_2+1}_{u,v;b-}+\mathbf{F}^{q_2+1,r_2}_{u/\nu,v;b-}-\mathbf{F}^{q_2+1,r_2+1}_{u',v;b-}\right)\vartheta(\varsigma)-\vartheta(b)u(b)(b-\varsigma)^{q_2}\right| \\ \leq\;& u(\varsigma)(b-\varsigma)^{q_2+1}\left(|\vartheta'(\varsigma)|\int_0^1 h(1-z)dz+|\vartheta'(b)|\int_0^1 h(z)dz\right) \\ -\;& \frac{\beta\,(b-\varsigma)^{q_2+4}}{6}u(\varsigma). \end{aligned} \tag{3.26}$$

Via triangular inequality, by adding inequalities (3.23) and (3.26), the required inequality holds.

As special cases, we have

Corollary 3.3.6 ***By setting*** $q_1 = q_2 = q, r_1 = r_2 = r,$ ***and*** $h(z) = z$ ***in (3.16)***

$$\begin{aligned}
& \left| \frac{\Gamma(r+1)\Gamma(q+1)}{v(\varsigma)} \left(\mathbf{F}^{q,r+1}_{u,v;a+} + \mathbf{F}^{q+1,r}_{u/\nu,v;a+} - \mathbf{F}^{q+1,r+1}_{u',v;a+} + \mathbf{F}^{q,r+1}_{u,v;b-} + \mathbf{F}^{q+1,r}_{u/t,v;b-} - \mathbf{F}^{q+1,r+1}_{u',v;b-} \right) \vartheta(\varsigma) \right. \\
- & \left. \left[\left(\ln \frac{b}{\varsigma} \right)^r u(b)\vartheta(b)(b-\varsigma)^{q+1} + \ln \left(\frac{\varsigma}{a} \right)^r u(a)\vartheta(a)(\varsigma-a)^{q+1} \right] \right| \\
\leq & \; u(\varsigma) \left| \vartheta'(\varsigma) \right| \frac{\left(\ln \frac{b}{\varsigma}\right)^r (b-\varsigma)^{q+1} + \left(\ln \frac{\varsigma}{a}\right)^r (\varsigma-a)^{q+1}}{2} \\
+ & \; u(\varsigma) \frac{\left|\vartheta'(b)\right| (b-\varsigma)^{q+1} \left(\ln \frac{b}{\varsigma}\right)^r + u(\varsigma) \left|\vartheta'(a)\right| (\varsigma-a)^{q+1} \left(\ln \frac{\varsigma}{a}\right)^r}{2} \\
- & \; \beta\, u(\varsigma) \frac{(\varsigma-a)^{q+4} \left(\ln \frac{\varsigma}{a} \right)^r + (b-\varsigma)^{q+4} \left(\ln \frac{b}{\varsigma} \right)^r}{6}. \qquad (3.27)
\end{aligned}$$

Corollary 3.3.7 ***By setting*** $u = v = 1,$ ***in (3.27)***

$$\begin{aligned}
& \left| \Gamma(r+1)\Gamma(q+1) \left(\mathbf{F}^{q,r+1}_{1,1;a+} + \mathbf{F}^{q,r+1}_{1,1;b-} + \mathbf{F}^{q+1,r}_{1/\nu,1;a+} + \mathbf{F}^{q+1,r}_{1/\nu,1;b-} \right) \vartheta(\varsigma) \right. \\
- & \left. \left[\left(\ln \frac{b}{\varsigma} \right)^r \vartheta(b)(b-\varsigma)^{q+1} + \ln \left(\frac{\varsigma}{a} \right)^r \vartheta(a)(\varsigma-a)^{q+1} \right] \right| \\
\leq & \; \left| \vartheta'(\varsigma) \right| \frac{\left(\ln \frac{b}{\varsigma}\right)^r (b-\varsigma)^{q+1} + \left(\ln \frac{\varsigma}{a}\right)^r (\varsigma-a)^{q+1}}{2} \\
+ & \; \frac{\left|\vartheta'(b)\right| (b-\varsigma)^{q+1} \left(\ln \frac{b}{\varsigma}\right)^r + \left|\vartheta'(a)\right| (\varsigma-a)^{q+1} \left(\ln \frac{\varsigma}{a}\right)^r}{2} \\
- & \; \beta \frac{(\varsigma-a)^{q+4} \left(\ln \frac{\varsigma}{a} \right)^r + (b-\varsigma)^{q+4} \left(\ln \frac{b}{\varsigma} \right)^r}{6}. \qquad (3.28)
\end{aligned}$$

Corollary 3.3.8 ***By setting*** $r = 0$ ***in (3.28), we get the following integral***

inequality involving Riemann-Liouville integrals

$$\begin{aligned}
&\left|\Gamma(q+1)\left(\mathbf{J}_{a+}^{q}\vartheta(\varsigma)+\mathbf{J}_{b-}^{q}\vartheta(\varsigma)\right)-\left(\vartheta(b)(b-\varsigma)^{q}+\vartheta(a)(\varsigma-a)^{q}\right)\right| \\
\leq\ & \left|\vartheta'(\varsigma)\right|\frac{(b-\varsigma)^{q+1}+(\varsigma-a)^{q+1}}{2}+\frac{(b-\varsigma)^{q+1}\left|\vartheta'(b)\right|+(\varsigma-a)^{q+1}\left|\vartheta'(a)\right|}{2} \\
-\ & \frac{\beta}{6}\left[(\varsigma-a)^{q+4}+(b-\varsigma)^{q+4}\right]. \qquad (3.29)
\end{aligned}$$

Corollary 3.3.9 ***By taking*** $\varsigma=a$ ***and*** $\varsigma=b$ ***in (3.29), we get the following fractional integral inequality***

$$\begin{aligned}
&\left|\Gamma(q+1)\left(\mathbf{J}_{a+}^{q}\vartheta(b)+\mathbf{J}_{b-}^{q}\vartheta(a)\right)-(b-a)^{q}\left(\vartheta(b)+\vartheta(a)\right)\right| \\
\leq\ & (b-a)^{q+1}\left\{\left|\vartheta'(b)\right|+\left|\vartheta'(a)\right|-\frac{\beta}{3}(b-a)^{3}\right\}. \qquad (3.30)
\end{aligned}$$

If $\beta=0$ ***in(3.30),we get***

$$\begin{aligned}
&\left|\Gamma(q+1)\left(\mathbf{J}_{a+}^{q}\vartheta(b)+\mathbf{J}_{b-}^{q}\vartheta(a)\right)-(b-a)^{q}\left(\vartheta(b)+\vartheta(a)\right)\right| \\
\leq\ & (b-a)^{q+1}\left\{\left|\vartheta'(b)\right|+\left|\vartheta'(a)\right|\right\}. \qquad (3.31)
\end{aligned}$$

Corollary 3.3.10 ***By setting*** $q=1$***, and taking*** $\varsigma=\frac{a+b}{2}$ ***in (3.29),it results that***

$$\left|\frac{1}{b-a}\int_{a}^{b}\vartheta(\varsigma)dt-\frac{\vartheta(b)+\vartheta(a)}{2}\right|\leq$$
$$\frac{(b-a)^{2}}{8}\left[2\left|\vartheta'\left(\frac{a+b}{2}\right)\right|+\left|\vartheta'(b)\right|+\left|\vartheta'(a)\right|\right]-\frac{\beta}{48}(b-a)^{5}. \qquad (3.32)$$

Corollary 3.3.11 ***By setting*** $q=0, u=v=1,$ ***in (3.27)***

$$\mid\Gamma(r+1)\left(\mathbf{H}_{a+}^{r}\vartheta(\varsigma)+\mathbf{H}_{b-}^{r}\vartheta(\varsigma)\right)\quad-$$

$$\left[\left(\ln\frac{b}{\varsigma}\right)^r \vartheta(b)(b-\varsigma)+\ln\left(\frac{\varsigma}{a}\right)^r \vartheta(a)(\varsigma-a)\right] |$$

$$\leq$$

$$|\vartheta'(\varsigma)|\,\frac{\left(\ln\frac{b}{\varsigma}\right)^r(b-\varsigma)+\left(\ln\frac{\varsigma}{a}\right)^r(\varsigma-a)}{2} \quad +$$

$$\frac{|\vartheta'(b)|\,(b-\varsigma)\left(\ln\frac{b}{\varsigma}\right)^r+|\vartheta'(a)|\,(\varsigma-a)\left(\ln\frac{\varsigma}{a}\right)^r}{2} \quad -$$

$$c\,\frac{(\varsigma-a)^4\left(\ln\frac{\varsigma}{a}\right)^r+(b-\varsigma)^4\left(\ln\frac{b}{\varsigma}\right)^r}{6}. \tag{3.33}$$

In particular if $\varsigma=a$ ***and*** $\varsigma=b$ ***in (3.33),we have***

$$\begin{aligned}&|\,\Gamma(r+1)\left(\mathbf{H}^r_{a+}\vartheta(b)+\mathbf{H}^r_{b-}\vartheta(a)\right)-\left(\ln\frac{b}{a}\right)^r(b-a)\left[\vartheta(b)+\vartheta(a)\right]|\\ \leq\;&\left(\ln\frac{b}{a}\right)^r(b-a)\left(|\vartheta'(b)|+|\vartheta'(a)|\right)\\ -\;&\beta\,\frac{(b-a)^4\left(\ln\frac{b}{a}\right)^r}{3}.\end{aligned} \tag{3.34}$$

If $\beta=0,$ ***then***

$$\begin{aligned}&|\,\Gamma(r+1)\left(\mathbf{H}^r_{a+}\vartheta(b)+\mathbf{H}^r_{b-}\vartheta(a)\right)-\left(\ln\frac{b}{a}\right)^r(b-a)\left[\vartheta(b)+\vartheta(a)\right]|\\ \leq\;&\left(\ln\frac{b}{a}\right)^r(b-a)\left(|\vartheta'(b)|+|\vartheta'(a)|\right).\end{aligned} \tag{3.35}$$

3.4 Applications

Now we give some applications of the results that have been established previously.

Theorem 3.4.1 ***Under the assymptions of theorem 2.3.6,the following in-***

equality

$$
\begin{aligned}
& \frac{\Gamma(r_1)\Gamma(q_1)}{v(b)}\mathbf{F}^{q_1,r_1}_{u,v;a+}\vartheta(b) + \frac{\Gamma(r_2)\Gamma(q_2)}{v(a)}\mathbf{F}^{q_2,r_2}_{u,v;b-}\vartheta(a) \\
\leq \quad & u(b)\left[\ln\frac{b}{a}\right]^{r_1-1}(b-a)^{q_1} \\
\times \quad & \left\{\vartheta(b)\int_0^1 h(1-z)dz + \vartheta(a)\int_0^1 h(z)dz - \beta\,\frac{(b-a)^2}{6}\right\} \\
+ \quad & u(a)\left[\ln\frac{b}{a}\right]^{r_2-1}(b-a)^{q_2} \\
\times \quad & \left\{\vartheta(a)\int_0^1 h(1-z)dz + \vartheta(b)\int_0^1 h(z)dz - \beta\,\frac{(b-a)^2}{6}\right\}
\end{aligned} \tag{3.36}
$$

holds.

Proof 3.4.2 ***We take*** $\varsigma = a$ ***and*** $\varsigma = b$ ***in (3.8) and adding the results.***

Corollary 3.4.3 ***By setting*** $q_1 = q_2 = q$ ***and*** $r_1 = r_2 = r$ ***in (3.36) it results that***

$$
\begin{aligned}
& \Gamma(r)\Gamma(q)\left(\frac{1}{v(b)}\mathbf{F}^{q,r}_{u,v;a+}\vartheta(b) + \frac{1}{v(a)}\mathbf{F}^{q,r}_{u,v;b-}\vartheta(a)\right) \\
\leq \quad & u(b)\left[\ln\frac{b}{a}\right]^{r-1}(b-a)^{q} \\
\times \quad & \left\{\vartheta(b)\int_0^1 h(1-z)dz + \vartheta(a)\int_0^1 h(z)dz - \beta\,\frac{(b-a)^2}{6}\right\} \\
+ \quad & u(a)\left[\ln\frac{b}{a}\right]^{r-1}(b-a)^{q} \\
\times \quad & \left\{\vartheta(a)\int_0^1 h(1-z)dz + \vartheta(b)\int_0^1 h(z)dz - \beta\,\frac{(b-a)^2}{6}\right\}
\end{aligned} \tag{3.37}
$$

is valid.

Corollary 3.4.4 ***By setting*** $u = v = 1, r = 1,$ ***we obtain an inequality involving Riemann-Liouville integrals***

$$\begin{aligned} &\Gamma(q)\left(J_{a+}^{q}\vartheta(b) + J_{b-}^{q}\vartheta(a)\right) \le (b-a)^{q} \\ \times \quad &\left\{(\vartheta(b)+\vartheta(a))(\int_0^1 h(1-z)dz + \int_0^1 h(z)dz) - \beta\,\frac{(b-a)^2}{3}\right\}. \end{aligned} \tag{3.38}$$

Corollary 3.4.5 ***By taking*** $h(z) = z$ ***in (3.38), we have***

$$\begin{aligned} \Gamma(q)\left(J_{a+}^{q}\vartheta(\varsigma) + J_{b-}^{q}\vartheta(\varsigma)\right) &\le (b-a)^{q} \\ &\times \left(\vartheta(b)+\vartheta(a) - \frac{\beta\,(b-a)^2}{3}\right). \end{aligned} \tag{3.39}$$

If $\beta = 0,$ ***then***

$$\Gamma(q)\left(J_{a+}^{q}\vartheta(\varsigma) + J_{b-}^{q}\vartheta(\varsigma)\right) \le (b-a)^{q}\left(\vartheta(b)+\vartheta(a)\right)$$

.

Corollary 3.4.6 ***By setting*** $\beta = 0, q = 1$ ***and*** $\varsigma = b$ ***or*** $\varsigma = a$ ***in (3.39), we get***

$$\frac{1}{b-a}\int_a^b \vartheta(\nu)d\nu \le \frac{\vartheta(a)+\vartheta(b)}{2}. \tag{3.40}$$

Corollary 3.4.7 ***By setting*** $c = 0, q = 1$ ***and*** $\varsigma = \frac{a+b}{2}$ ***in (3.39), we have***

$$\frac{1}{b-a}\int_a^b \vartheta(\nu)d\nu \le \vartheta\left(\frac{a+b}{2}\right) + \frac{\vartheta(a)+\vartheta(b)}{2}. \tag{3.41}$$

Corollary 3.4.8 ***By setting*** $v = 1, u(z) = \dfrac{1}{z}, q = 1$ ***and*** $h(z) = z$ ***in (3.15), we***

obtain an inequality involving Hadamard integrals

$$\begin{aligned}
& \Gamma(r)\left(H_{a+}^{r}\vartheta(\varsigma)+H_{b-}^{r}\vartheta(\varsigma)\right) \\
\leq\ & \frac{\frac{1}{a}\left[\ln\frac{\varsigma}{a}\right]^{r-1}(\varsigma-a)+\frac{1}{\varsigma}\left[\ln\frac{b}{\varsigma}\right]^{r-1}(b-\varsigma)}{2} \\
+\ & \frac{\frac{\vartheta(a)}{a}\left[\ln\frac{\varsigma}{a}\right]^{r-1}(\varsigma-a)+\frac{\vartheta(b)}{\varsigma}\left[\ln\frac{b}{\varsigma}\right]^{r-1}(b-\varsigma)}{2} \\
-\ & \beta\,\frac{\frac{1}{\varsigma}\left[\ln\frac{b}{\varsigma}\right]^{r-1}(b-\varsigma)^{3}+\frac{1}{a}\left[\ln\frac{\varsigma}{a}\right]^{r-1}(\varsigma-a)^{3}}{6}.
\end{aligned} \tag{3.42}$$

Corollary 3.4.9 ***By taking*** $\varsigma=a$ ***and*** $\varsigma=b$ ***in (3.42), we obtain***

$$\begin{aligned}
\Gamma(r)\left(H_{a+}^{r}\vartheta(b)+H_{b-}^{r}\vartheta(a)\right) &\leq (b-a)\left[\ln\frac{b}{a}\right]^{r-1} \\
&\times \left(\frac{\vartheta(b)+\vartheta(a)}{a}-\frac{\beta\,(b-a)^{2}}{3}\right).
\end{aligned} \tag{3.43}$$

If $\beta=0,$ ***then***

$$\Gamma(r)\left(H_{a+}^{r}\vartheta(b)+H_{b-}^{r}\vartheta(a)\right)\leq(b-a)\left[\ln\frac{b}{a}\right]^{r-1}\frac{\vartheta(b)+\vartheta(a)}{a}.$$

Theorem 3.4.10 ***Let*** $q_1,q_2\geq 0,\ r_1,r_2\geq 1$ ***and*** $0<a,b<\infty.$ ***Let*** $\vartheta:[a,b]\to\mathbb{R},$ ***be an strongly*** h***- convex function,*** u ***an integrable non-negative function, monotonic on*** $[a,b]$ ***and*** v ***a non-negative function with*** $v(a)\neq 0, v(b)\neq 0.$ ***If*** ϑ ***is symmetric about*** $\frac{a+b}{2}$***, it follows that for all*** $\nu\in[a,b]$

(1) If u ***is increasing, then***

$$\begin{aligned}
&\boldsymbol{Lhs}=u(a)\left[M\left\{I(0;q_1,r_1)+J(0;q_2,r_2)\right\}+\frac{c}{4h\left(\frac{1}{2}\right)}\left\{I(1;q_1,r_1)+J(1;q_2,r_2)\right\}\right] \\
\leq\ & \frac{\Gamma(r_1)\Gamma(q_1+1)\,\mathbf{F}_{u,v;b-}^{q_1+1,r_1}\vartheta(a)}{v(a)}+\frac{\Gamma(r_2)\Gamma(q_2+1)\,\mathbf{F}_{u,v;a+}^{q_2+1,r_2}\vartheta(b)}{v(b)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; u(b)\left[(b-a)^{q_1+1}\left(\ln\frac{b}{a}\right)^{r_1-1}+(b-a)^{q_2+1}\left(\ln\frac{b}{a}\right)^{r_2-1}\right]\\
&\times \; (\vartheta(a)+\vartheta(b))\int_0^1 h(z)dz\\
&- \; \beta\, u(b)\left[\frac{(b-a)^{q_1+4}}{6}\left(\ln\frac{b}{a}\right)^{r_1-1}+\frac{(b-a)^{q_2+4}}{6}\left(\ln\frac{b}{a}\right)^{r_2-1}\right]=\boldsymbol{Rhs}.
\end{aligned}
\tag{3.44}
$$

holds.

(2) If u is decreasing, then

$$
\begin{aligned}
&u(b)\left[M\left\{I(0;q_1,r_1)+J(0;q_2,r_2)\right\}+\frac{c}{4h\left(\frac{1}{2}\right)}\left\{I(1;q_1,r_1)+J(1;q_2,r_2)\right\}\right]\\
\leq\;& \frac{\Gamma(r_1)\Gamma\left(q_1+1\right)\mathbf{F}^{q_1+1,r_1}_{u,v;b-}\vartheta(a)}{v(a)}+\frac{\Gamma(r_2)\Gamma\left(q_2+1\right)\mathbf{F}^{q_2+1,r_2}_{u,v;a+}\vartheta(b)}{v(b)}\\
\leq\;& u(a)\left[(b-a)^{q_1+1}\left(\ln\frac{b}{a}\right)^{r_1-1}+(b-a)^{q_2+1}\left(\ln\frac{b}{a}\right)^{r_2-1}\right]\\
\times\;& (\vartheta(a)+\vartheta(b))\int_0^1 h(z)\mathrm{d}z\\
-\;& \beta\, u(a)\left[\frac{(b-a)^{q_1+4}}{6}\left(\ln\frac{b}{a}\right)^{r_1-1}+\frac{(b-a)^{q_2+4}}{6}\left(\ln\frac{b}{a}\right)^{r_2-1}\right]
\end{aligned}
\tag{3.45}
$$

holds. Where

$$
I(\alpha;q,r)=\int_a^b \nu^{\alpha}(\nu-a)^q\left(\ln\frac{\nu}{a}\right)^{r-1}d\nu,\quad J(\alpha;q,r)=\int_a^b \nu^{\alpha}(b-\nu)^q\left(\ln\frac{b}{\nu}\right)^{r-1}d\nu.
$$

Proof 3.4.11 ***Suppose that u is increasing. We have for all $\nu\in[a,b],\ r_1\geq 1$***

$$
(\nu-a)^{q_1}\left(\ln\frac{\nu}{a}\right)^{r_1-1}u(\nu)\leq(b-a)^{q_1}\left(\ln\frac{b}{a}\right)^{r_1-1}u(b).
\tag{3.46}
$$

Since ϑ ***is strongly*** h***- convex therefore for*** $\nu \in [a,b]$, ***we have***

$$\vartheta(\nu) \leq h\left(\frac{\nu-a}{b-a}\right)\vartheta(a) + h\left(\frac{b-\nu}{b-a}\right)\vartheta(b) - \beta(\nu-a)(b-\nu). \tag{3.47}$$

By multiplying inequalities (3.46), (3.47), side to side and by integrating , we obtain

$$\begin{aligned}
& \int_a^b (\nu-a)^{q_1}\left(\ln\frac{\nu}{a}\right)^{r_1-1} u(\nu)\vartheta(\nu)d\nu \\
\leq\ & (b-a)^{q_1+1}\left(\ln\frac{b}{a}\right)^{r_1-1} u(b)\left(\vartheta(a)\int_0^1 h(z)dz + \vartheta(b)\int_0^1 h(1-z)dz\right) \\
-\ & \beta\,\frac{(b-a)^{q_1+4}}{6}\left(\ln\frac{b}{a}\right)^{r_1-1}
\end{aligned} \tag{3.48}$$

From which, we have

$$\begin{aligned}
& \frac{\Gamma(r_1)\Gamma(q_1+1)\,\mathbf{F}^{q_1+1,r_1}_{u,v;b-}\vartheta(a)}{v(a)} \\
\leq\ & u(b)(b-a)^{q_1+1}\left(\ln\frac{b}{a}\right)^{r_1-1}\left(\vartheta(a)\int_0^1 h(z) + \vartheta(b)\int_0^1 h(1-z)dz\right) \\
-\ & \beta\,\frac{(b-a)^{q_1+4}}{6}\left(\ln\frac{b}{a}\right)^{r_1-1}.
\end{aligned} \tag{3.49}$$

On the other hand for all $\nu \in [a,b]$, ***we have***

$$(b-\nu)^{q_2}\left(\ln\frac{b}{\nu}\right)^{r_2-1} u(\nu) \leq (b-a)^{q_2}\left(\ln\frac{b}{a}\right)^{r_2-1} u(b). \tag{3.50}$$

And similarly it follows that

$$\frac{\Gamma(r_2)\Gamma(q_2+1)\,\mathbf{F}^{q_2+1,r_2}_{u,v;a+}\vartheta(b)}{v(b)} \leq u(b)(b-a)^{q_2+1}\left(\ln\frac{b}{a}\right)^{r_2-1}$$

$$\times \quad \left(\vartheta(a)\int_0^1 h(z)dz + \vartheta(b)\int_0^1 h(1-z)dz\right)$$
$$- \quad \beta\,\frac{(b-a)^{q_2+4}}{6}\left(\ln\frac{b}{a}\right)^{r_2-1}. \tag{3.51}$$

By adding (3.49) and (3.51), we get

$$\frac{\Gamma(r_1)\Gamma(q_1+1)\,\mathbf{F}_{u,v;b-}^{q_1+1,r_1}\vartheta(a)}{v(a)} + \frac{\Gamma(r_2)\Gamma(q_2+1)\,\mathbf{F}_{u,v;a+}^{q_2++1,r_2}\vartheta(b)}{v(b)}$$
$$\leq \quad u(b)\left[(b-a)^{q_1+1}\left(\ln\frac{b}{a}\right)^{r_1-1} + (b-a)^{q_2+1}\left(\ln\frac{b}{a}\right)^{r_2-1}\right]$$
$$\times \quad (\vartheta(a)+\vartheta(b))\int_0^1 h(z)dz$$
$$- \quad \beta\left[\frac{(b-a)^{q_1+4}}{6}\left(\ln\frac{b}{a}\right)^{r_1-1} + \frac{(b-a)^{q_2+4}}{6}\left(\ln\frac{b}{a}\right)^{r_2-1}\right]. \tag{3.52}$$

To prove the left hand side: Lhs, we use the Lemma 2.38 and monotonicity properties of real valued functions u ***and*** $\ln$.

Indeed, setting $M = \frac{4\vartheta\left(\frac{a+b}{2}\right)-\beta\,(a+b)}{8\,h\left(\frac{1}{2}\right)}$. ***We have for all*** $\nu \in [a,b]$

$$M + \frac{c}{4h\left(\frac{1}{2}\right)}\nu \leq \vartheta(\nu), \tag{3.53}$$

and

$$u(a)(\nu-a)^{q_1}\left(\ln\frac{\nu}{a}\right)^{r_1-1} \leq u(\nu)(\nu-a)^{q_1}\left(\ln\frac{\nu}{a}\right)^{r_1-1}, \tag{3.54}$$

$$u(a)(b-\nu)^{q_2}\left(\ln\frac{b}{\nu}\right)^{r_2-1} \leq u(\nu)(b-\nu)^{q_2}\left(\ln\frac{b}{\nu}\right)^{r_2-1}. \tag{3.55}$$

Multiplay (3.54) and (3.53) side to side and integrating with respect to ν

over $[a,b]$, ***we obtain***

$$\begin{aligned} & u(a)\left[MI(0;q_1,r_1)+\frac{c}{4h\left(\frac{1}{2}\right)}I(1;q_1,r_1)\right] \\ \leq \quad & \frac{\Gamma(r_1)\Gamma\left(q_1+1\right)\mathbf{K}_{u,v;b-}^{q_1+1,r_1}\vartheta(a)}{v(a)} \end{aligned} \tag{3.56}$$

Also we have

$$\begin{aligned} & u(a)\left[MJ(0;q_2,r_2)+\frac{c}{4h\left(\frac{1}{2}\right)}J(1;q_2,r_2)\right] \\ \leq \quad & \frac{\Gamma(r_2)\Gamma\left(q_2+1\right)\mathbf{F}_{u,v;a+}^{q_2+1,r_2}\vartheta(b)}{v(b)} \end{aligned} \tag{3.57}$$

Adding (3.56) and (3.57)

$$\begin{aligned} & u(a)\left[M\left\{I(0;q_1,r_1)+J(0;q_2,r_2)\right\}+\frac{\beta}{4h\left(\frac{1}{2}\right)}\left\{I(1;q_1,r_1)+J(1;q_2,r_2)\right\}\right] \\ \leq \quad & \frac{\Gamma(r_1)\Gamma\left(q_1+1\right)\mathbf{F}_{u,v;b-}^{q_1+1,r_1}\vartheta(a)}{v(a)}+\frac{\Gamma(r_2)\Gamma\left(q_2+1\right)\mathbf{F}_{u,v;a+}^{q_2+1,r_2}\vartheta(b)}{v(b)} \end{aligned} \tag{3.58}$$

combining (3.52) and (3.58), we obtain the inequality (3.46).

If u ***is decreasing, we have for all*** $\nu\in[a,b]$, $r_1\geq 1$

$$(\nu-a)^{q_1}\left(\ln\frac{\nu}{a}\right)^{r_1-1}u(\nu)\leq(b-a)^{q_1}\left(\ln\frac{b}{a}\right)^{r_1-1}u(a). \tag{3.59}$$

To prove the left hand side, we replace $u(a)$ ***in (3.54), (3.55) by*** $u(b)$ ***and the rest of the proof is similar.***

Corollary 3.4.12 ***By setting*** $q_1 = q_2 = q$ ***and*** $r_1 = r_2 = r$, ***we have the following inequalities***

1. If u ***is increasing***

$$\begin{aligned} & u(a)\left[M\left\{I(0;q,r)+J(0;q,r)\right\}+\frac{\beta}{4h\left(\frac{1}{2}\right)}\left\{I(1;q,r)+J(1;q,r)\right\}\right] \\ \leq\ & \Gamma(r)\Gamma\left(q+1\right)\left(\frac{\mathbf{F}^{q+1,r}_{u,v;b-}\vartheta(a)}{v(a)}+\frac{\mathbf{F}^{q+1,r}_{u,v;a+}\vartheta(b)}{v(b)}\right) \\ \leq\ & u(b)\left[\vartheta(a)\int_0^1 h(z)+\vartheta(b)\int_0^1 h(1-z)dz-\frac{\beta(b-a)^3}{3}\right]2(b-a)^{q+1}\left(\ln\frac{b}{a}\right)^{r-1}. \end{aligned} \tag{3.60}$$

2. If u ***is decreasing***

$$\begin{aligned} & u(b)\left[M\left\{I(0;q,r)+J(0;q,r)\right\}+\frac{c}{4h\left(\frac{1}{2}\right)}\left\{I(1;q,r)+J(1;q,r)\right\}\right] \\ \leq\ & \Gamma(r)\Gamma\left(q+1\right)\left(\frac{\mathbf{F}^{q+1,r}_{u,v;b-}\vartheta(a)}{v(a)}+\frac{\mathbf{F}^{q+1,r}_{u,v;a+}\vartheta(b)}{v(b)}\right) \\ \leq\ & u(a)\left[(\vartheta(a)+\vartheta(b))\int_0^1 h(z)dz-\frac{c(b-a)^3}{3}\right]2(b-a)^{q+1}\left(\ln\frac{b}{a}\right)^{r-1}. \end{aligned} \tag{3.61}$$

Corollary 3.4.13 ***By taking*** $q = 0$ ***and*** $u = \frac{1}{\varsigma}$ ***in (3.61),we obtain the inequality***

$$\begin{aligned} & \frac{1}{b}\left[M\left\{I(0;0,r)+J(0;0,r)\right\}+\frac{\beta}{4h\left(\frac{1}{2}\right)}\left\{I(1;0,r)+J(1;0,r)\right\}\right] \\ \leq\ & \Gamma(r)\left(\mathbf{H}^{r}_{b-}\vartheta(a)+\mathbf{H}^{;r}_{a+}\vartheta(b)\right) \\ \leq\ & \left[(\vartheta(a)+\vartheta(b))\int_0^1 h(z)dz-\frac{\beta(b-a)^3}{3}\right]\frac{2(b-a)}{a}\left(\ln\frac{b}{a}\right)^{r-1}. \end{aligned} \tag{3.62}$$

In particular if $h(z) = z, \beta = 0$ ***then***

$$\begin{aligned} & \frac{1}{b}\vartheta\left(\frac{a+b}{2}\right) \\ \leq \; & \Gamma(r)\left(\mathbf{H}_{b-}^{r}\vartheta(a) + \mathbf{H}_{a+}^{r}\vartheta(b)\right) \\ \leq \; & \frac{(b-a)}{a}\left(\ln\frac{b}{a}\right)^{r-1}\frac{\vartheta(a)+\vartheta(b)}{2}. \end{aligned} \tag{3.63}$$

Corollary 3.4.14 ***If we put*** $u = v = 1, r = 1$ ***in (3.60), we get under the assumptions of Theorem (2.3),the inequality***

$$\begin{aligned} & M\left\{I(0;q,1)+J(0;q,1)\right\} + \frac{\beta}{4h\left(\frac{1}{2}\right)}\left\{I(1;q,1)+J(1;q,1)\right\} \\ \leq \; & \Gamma\left(q+1\right)\left(\mathbf{J}_{b-}^{q+1}\vartheta(a) + \mathbf{J}_{a+}^{q+1}\vartheta(b)\right) \\ \leq \; & (b-a)^{q+1}\left[(\vartheta(a)\int_0^1 h(z) + \vartheta(b))\int_0^1 h(z)dz - \beta\,\frac{(b-a)^3}{6}\right], \end{aligned} \tag{3.64}$$

where $I(0;q,1) = J(0;q,1) = \dfrac{(b-a)^{q+1}}{q+1}$, $I(1;q,1) = (b-a)^{q+1}\left(\dfrac{a}{q+1} + \dfrac{b-a}{q+2}\right)$ ***and*** $J(1;q,1) = (b-a)^{q+1}\left(\dfrac{b}{q+1} - \dfrac{b-a}{q+2}\right)$.

Remark 3.4.15 ***If*** $\beta = 0, h(z) = z, q \to 0,$ ***then from above inequality, we get Hadamard's inequality***

$$\vartheta\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \vartheta(\varsigma)d\varsigma \leq \frac{\vartheta(a)+\vartheta(b)}{2}. \tag{3.65}$$

Conclusion and Perspectives

3.5 Conclusion

This research introduces a generalized fractional integral operator that uses a logarithmic kernel and includes two parameters along with two non-negative locally integrable functions. The study applies this generalized integrals specifically to a type of function called h-convex and strongly h-comvex functions. For these functions, the work proves new fractional integral inequalities.

Key results include finding estimates and bounds for integral transform of functions, providing examples. It also establishes integral inequalities that connect the generalized operator to the classical Riemann-Liouville fractional integrals. Furthermore, the research extends the well-known Hermite-Hadamard inequality to work with h-convex functions within the fractional calculus setting.

This work is important because it successfully combines the concept of h-convexity with fractional calculus tools. It creates a bridge between newer fractional integral inequalities and traditional classical inequalities. The generalized operator and the proven results offer a foundation for fur-

ther research. Future possibilities,applying the techniques to other types of convex functions or different fractional operators like Caputo or Hadamard, using it for solving fractional equations numerically, extending it to multiple variables, applying it in physics or engineering problems, and finding even sharper versions of the inequalities. This research opens new paths for exploration in mathematical analysis.

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