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THÈSE

Préparée par

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Spécialité : Mathématiques
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Thème

Contribution à l'étude de quelques classes d'équations différentielles fractionnaires d'ordre variable

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Devant le Jury composé de :

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ملخص

الغرض من هذه الأطروحة هو دراسة وجود الحلول ووحدانيتها لبعض مسائل القيم الإبتدائية و مسائل القيم الحدية للمعادلات التفاضلية غير خطية ذات المشتق الكسري ذو الرتبة المتغيرة لريمان ليوفيل جميع النتائج تم الحصول عليها باستعمال نظريات النقطة الصامدة وتقنية القياس اللا متراص و, منهجيتنا واضحة حيث تستند إلى عامل كسري جديد أكثر ملاءمة ويوضح قابلية حل المسألة الرئيسية في ظل فرضيات أقل تقييدا. على عكس التقنيات المتبعة في الأدبيات، والتي كانت تستند إلى استخدام مفهوم المجالات المعممة والدوال الثابتة بالجزء

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Résumé

L'objectif de cette thèse est d'examiner l'existence et l'unicité de solutions pour quelques types de problèmes non linéaires à valeurs initiales et aux limites d'équations différentielles fractionnaires de dérivée au sens de Riemann-Liouville d'ordre variable. Les résultats de cette étude sont établis par des théorèmes de point fixe et l'approche de mesure de non-compactité, notre technique est simple et basée sur un nouvel opérateur fractionnaire qui est plus approprié et démontre la solvabilité du problème principal sous des présomptions moins restrictives. Contrairement aux techniques utilisées dans la littérature, qui étaient basées sur l'utilisation du concept d'intervalles généralisés et l'idée de fonctions constantes par morceaux.
Mots clés : *Équations différentielles fractionnaires d'ordre variable, Problème à valeurs initiales et aux limites, Théorèmes du point fixe, Mesure de non-compactité de Kuratowski, Équations différentielles fonctionnelles à retard fini.*
Classifcations : *26A33, 34A08, 34A37, 34A60.*

Abstract

The purpose of this thesis is to examine the existence and uniqueness of solutions for a few types of nonlinear initial and boundary value problems involving Riemann-Liouville fractional differential equations of variable order. This study's results are all supported by fixed point theorems and the measure of noncompactness approach, our technique is straightforward and based on a novel fractional operator that is more appropriate and demonstrates the solvability of the main problems under less restrictive presumptions. Contrary to the techniques taken in the literature, which were based on the usage of the concept of generalized intervals and the idea of piecewise constant functions.

Key words: *Fractional differential equations of variable order, Initial and Boundary value problem, Fixed point theorems, Kuratowski measure of noncompactness, Stability .*

Classifcations: *26A33, 34A08, 34A37, 34A60.*

Publications and Communications

International Publications

1. **A. Hallouz**, G. Stamov, M. S. Souid and I. Stamova , New Results Achieved for Fractional Differential Equations with R-Liouville Derivatives of Nonlinear Variable Order, *Axioms*, **2023:895**, 9-12, (2023).

2. M. S. Souid, **A. Hallouz**, G. Hatira,On The Finite Delayed Fractional Differential Equation Via R-Liouville Derivative of Non-linear Variable-Order (Submitted).

3. **A. Hallouz**, M. S. Souid and J. Alzabut, New solvability and stability results for variable-order fractional initial value problem, *The Journal of Analysis*, **2024** (2024).

4. M. S. Souid, **A. Hallouz**, G. Hatira, On The Finite Delayed Fractional Differential Equation Via R-Liouville Derivative of Non-linear Variable-Order (Submitted).

5. M. S. Souid, **A. Hallouz**, Border Value Problem For R-Liouville Differential Equations Of Nonlinear Variable Order (Submitted).

International Communications

1 Existence uniqueness and stability of solutions of the boundary value problem for non linear fractional differential equation of variable order. 6 ème Workshop International sur les Mathématiques Appliquées et la Modélisation « WIMAM'2022 » 26-27 Octobre 2022 ont été prolongées au 15 Octobre 2022, Guelma-Algeria.

2 Well-Posedness and Asymtotic Behavior of The Klein-Gordon Equation With Dynamic Boundaries Dissipation of Fractional Derivative Type. The Second International Workshop on Applied Mathematics 2nd-IWAM'2023, 5-7 December, 2023, Constantine, ALGERIA.

National Communications

1 ON A BOUNDARY VALUE PROBLEM WITH HADAMARD DERIVATIVE NATA' 2022 Nonlinear Analysis, Theory and Applications at the University of Chlef from 27 to 29 November 2022

2 wave equation with boundary fractional damping. The Second Online National Conference on Pure and Applied Mathematics "CNMPA 2023" April 27, Tebessa, Algeria

Contents

List of Abbreviations

AAT :Arzela Ascoli Theorem.

BS :Banach Space.

CF :Continuous Functions.

CC :Completely Continuous.

CCBNE :Convex, Closed, Bounded and Nonempty.

CS :Convex Subset.

CO :Continuous.

EC :Equicontinuous.

FP :Fixed Point.

RC :Relatively Compact.

RLDVO :Riemann Liouville Derivative of Variable Order.

RLFDVO :Riemann Liouville Fractional Derivative of Variable Order.

RLDCO :Riemann Liouville Derivative of Constant Order.

WRLDVO :Weighted Riemann Liouville Derivative of Variable Order.

WRLFDVO :Weighted Riemann Liouville Fractional Derivative of Variable Order.

WRLDCO :Riemann Liouville Derivative of Constant Order.

WRLFIVO :Weighted Riemann Liouville Fractional Integral of Variable Order.

WRLICO :$\mathcal{W}$eighted $\mathcal{R}$iemann $\mathcal{L}$iouville $\mathcal{I}$ntegral of $\mathcal{C}$onstant $\mathcal{O}$rder.

RLFIVO :$\mathcal{R}$iemann $\mathcal{L}$iouville $\mathcal{I}$ntegral of $\mathcal{V}$ariable $\mathcal{O}$rder.

RLICO :$\mathcal{R}$iemann $\mathcal{L}$iouville $\mathcal{I}$ntegral of $\mathcal{C}$onstant $\mathcal{O}$rder.

SFPT :$\mathcal{S}$chauder $\mathcal{F}$ixed $\mathcal{P}$oint $\mathcal{T}$heorem.

UB :$\mathcal{U}$niformly $\mathcal{B}$ounded.

UFP :$\mathcal{U}$nique $\mathcal{F}$ixed $\mathcal{P}$oint.

US :$\mathcal{U}$nique $\mathcal{S}$olution.

ALS :$\mathcal{A}$t $\mathcal{L}$east a $\mathcal{S}$olution.

DCT :$\mathcal{D}$ominated $\mathcal{C}$onvergevce $\mathcal{T}$heorem.

General Introduction

Fractional operators of variable order, representing a more complex class, are derivatives and integrals whose orders depend on certain variables. Therefore, these variable-order fractional derivatives and integrals generalize the constant-order fractional operators.

In 1993, Samko and Ross [44] pioneered fractional integration and differentiation where the order $\ltimes$ is time-dependent, given by $\ltimes(t)$, instead of a constant. This work extended the definitions of R-Liouville and Fourier fractional operators [41, 44, 45]. Various definitions for variable-order fractional derivatives and integrals exist in the literature, including those by R-Liouville, Caputo, Hadamard, and C-Hadamard; see [44, 45].

Subsequently, several works have focused on variable-order fractional operators, their applications, and interpretations; see, for example, [3, 30, 19]. In particular, Samko's variable-order fractional calculus has proven to be highly useful in mechanics and the theory of viscous flows [30, 19, 33, 37, 35]. Many physical processes indeed exhibit fractional-order behavior that may vary with time or space [30]. The paper [19] is dedicated to studying a variable-order fractional differential equation that characterizes certain problems in viscoelasticity theory. In [20], the authors analyze the dynamics and control of a nonlinear variable viscoelastic oscillator, proposing two controllers for variable-order differential equations to track an arbitrary reference function. The study [33] investigates the drag force acting on a particle in an oscillatory flow of a viscous fluid, where the drag force is determined using variable-order fractional calculus, with the derivative order varying according to the flow dynamics. In [35], a variable-order differential equation is developed for a particle in a quiescent viscous liquid. For more on the application of variable-order fractional operators in modeling dynamic systems, we refer the reader to the recent review article [37].

In recent years, there has been an increasing application of fractional operators and variable-order fractional differential equations in engineering; see [54, 49] and the references therein for examples and details. This extensive application highlights the urgent need for systematic studies on the existence, uniqueness, and stability of solutions to initial and boundary value problems for these equations. Research in this domain is still at an early stage, with only a few published papers so far, typically addressing relatively simple problems with limited methodologies, such as those in [46, 51].

Complex competitive interactions are frequently observed in natural systems, such as ecological models that incorporate species food chains linked by trophic interactions, nutrient diffusion or spread among different states, and competition between healthy and pathogenic cells. These biological systems often demonstrate long-range temporal memory or spatial interactions, with the strength of these interactions fluctuating across space and time. Consequently, variable-order fractional operators provide an effective approach for capturing the dynamics of these interactions as they evolve over both spatial and temporal dimensions.

In this context, Ghanbari et al. [21] developed a model for the competitive dynamics in a nutrient-phytoplankton-zooplankton interaction system using variable-order operators. Their findings indicate that the variable-order model alters the system's memory effects, with the temporal memory of interactions influenced by both the relative populations within the nutrient-phytoplankton-zooplankton system and the specific variations in order. Additionally, a variable-order growth model was applied in [2] to analyze the population histories of various countries, demonstrating that this approach achieved a significantly higher accuracy than traditional constant-order models.

Another example of competitive dynamics can be found in interactions among populations affected by three tuberculosis strains—drug-sensitive, emerging multi-drug-resistant, and extensively drug-resistant—as well as those unaffected by tuberculosis. Sweilam et al. [52] modeled this complex interaction numerically using variable-order fractional differential equations with Grunwald-Letnikov operators.

Additionally, using data from the World Health Organization, researchers have estimated the treatment rates necessary to control the spread of tuberculosis in Egypt. Another biological application of variable-order fractional operators is seen in models of competitive dynamics between healthy and tumorous bone cells. Neto et al. [31, 32] showed that variable-order fractional differential equations can effectively replicate the outcomes of traditional integer-order models for bone cell and tumor interactions, with fewer parameters. This variable-order approach introduces a non-local framework with memory effects, where the order varies with both time and spatial location. The variable order is influenced by tumor dynamics, allowing the effects of tumors to be integrated into the original healthy bone model.

The authors emphasize that comparing the variable-order models with actual experimental data will offer meaningful insights into tumor growth dynamics and provide a basis for the development of efficient, targeted tumor therapies. A notable application of variable-order fractional calculus is in modeling the influence of Twitter on the spread of alcoholism [1]. Time delays were incorporated into the variable-order operators of existing constant-order models, demonstrating that the variable-order framework more accurately represented the propagation of alcoholism compared to traditional constant-order models. In all the studies reviewed, variable-order fractional calculus has been utilized to describe complex competitive dynamics between biological entities, with the variable-order approach successfully capturing transitions between different dynamic regimes of the biological systems involved. Variable-order fractional calculus has also been employed in the study of random-order models [48, 50, 55]. However, to date, the research on random-order operators and their applications remains limited. A significant challenge is the absence of a formal mathematical definition for these operators and their associated properties. Despite this, the concept of random-order fractional calculus holds considerable promise and may have important applications in modeling stochastic and chaotic dynamics, such as those observed in financial systems, turbulent flow, and noise/vibration control. These models could potentially serve as a foundation for the development of precise risk analysis and control methodologies.

Fractional calculus has gained significant attention and importance due to its wide applicability in various fields of research and engineering, including physics, chemistry, and dynamical systems control.

In recent years, there has been growing interest in the existence and uniqueness of solutions to boundary value problems for fractional differential equations. While the literature on solutions to boundary value problems with fractional (constant) order is abundant, there is comparatively limited research focused on the existence of solutions to boundary value problems involving variable-order fractional differential equations.

Nonlinear functional analysis techniques, such as certain fixed-point theorems, have been essential in establishing the existence of solutions to fractional differential equations of constant order. In these contexts, differential equations can often be reformulated as integral equations, leveraging key properties of constant-order differential and integral calculus. However, in general, the calculus of variable-order derivatives does not possess these foundational properties. A critical example is the semigroup property of the fractional integral, which plays a central role in studying the existence of solutions to fractional-order differential equations. Based on some results of some experts, we know that the variable order fractional integral does not have semigroup property, i.e. for general functions $\ltimes(t), \beta(t)$, $I_{a^+}^{\ltimes(t)} I_{a^+}^{\beta(t)} h(t) \neq I_{a^+}^{\ltimes(t)+\beta(t)} h(t)$ *and* $D_{a^+}^{\ltimes(t)} D_{a^+}^{\beta(t)} h(t) \neq D_{a^+}^{\ltimes(t)+\beta(t)} h(t)$.) then the transform between the variable order fractional integral and derivative is not clear. Thus, it brings us extreme difficulties, we can't get these properties for the variable order fractional operators (integral and derivative). Without these properties for variable order fractional derivative and integral, we can hardly consider the existence of solutions of differential equations for variable order derivative by means of nonlinear functional analysis. Thus, one can not transform a differential equation of variable-order into an equivalent integral equation without these propositions. It is a difficulty for us in dealing with the initial value or boundary value problems of differential equations of variable-order. It is necessary and significant for us to conquer the difficulty and obtain the solution to a differential equations of variable-order.

Recently, specifically in the last three years 2021, 2022, 2023 Souid et al they published more than 30 papers in theory of fractional differential equations of variable-order, in which we studied the existence, uniqueness, and stability of solutions to various problems, we used the fixed point theory for this; for example see [7, 9, 10, 11, 12, 14, 37, 16, 25, 17, 40, 18, 39, 38, 36, 8, 13, 47, 53].

In the aforementioned articles the authors used either piece-wise constant functions or the Picard iterative method, whereas our method used in this thesis is totally different from theirs, it is based on a perturbation of an operator resulting from an integration of the fractional derivative operator. As far as we know this method is new and does not appear in any previous work.

An overview of our thesis structure, which consists of **4 chapters** outlining the contributions, is provided below. The **first chapter** introduce the terminology, notations, and introductory information that will be utilized throughout this thesis.

The main content of Chapter 2, is an affirmative response to the existence question for two different initial value problems (IVP for short) involving RLD of linear variable order given as follows.

1.

$$\begin{cases} \left(\mathcal{D}_{0^+}^{\sigma(.)}y\right)(t) &= \mathcal{A}_1(t, y(t)), \quad t \in := (0, \digamma], \quad 0 < \digamma < \infty, \quad (1) \\ y(0) = 0, & \quad (2) \end{cases}$$

where $\mathcal{D}_{0^+}^{\sigma(.)}$ set forth RLDVO $\sigma(t)$, $\sigma : \overline{\mathcal{D}_1} \to \mathbb{R}$ is a CFs and $0 < \sigma_* \le \sigma(t) \le \sigma^* < 1$, $\mathcal{A}_1$ is a function pending specification.

2.

$$\begin{cases} \left(\mathcal{D}_{0^+}^{\sigma(.)}\xi\right)(t) = \mathcal{A}_2(t, \xi_t), & t \in \mathcal{D}_1^2 := (0, N], \quad 0 < N < \infty, \quad (1) \\ \xi(t) = \theta(t), & t \in \mathcal{D}_2^2 := [-r, 0], \quad (2) \end{cases}$$

where $\mathcal{D}_{0^+}^{\sigma(.)}$ set forth RLDVO $\sigma(t)$, $r > 0$ and $\sigma(t)$ satisfies $0 < \sigma_* \leq \sigma(t) \leq \sigma^* < 1$, $\mathcal{A}_2 : \mathcal{D}_1^2 \times L^1(\mathcal{D}_2^2, \mathbb{R}) \to \mathbb{R}$ is a function pending specification and $\theta \in L^1\left(\mathcal{D}_2^2\right)$ with $\theta(0) = 0$. For any function ξ defined on $[-r, N]$ and any $t \in \mathcal{D}_1^2$, we denote by ξ_t the elements of $L^1([-r, 0], \mathbb{R})$ defined by

$$\xi_t(\leftthreetimes) = \xi(t + \leftthreetimes), \;\; \leftthreetimes \in [-r, 0].$$

Here $\xi_t(.)$ quantifies the history of the state from time $t - r$ up to the present time t. We denote $\mathcal{D}^2 = \mathcal{D}_1^2 \cup \mathcal{D}_2^2 = [-r, N]$.

The main content of Chapter 3, is to provide an answer in the positive to the existence query for three distinct Initial and boundary value problems (abbreviated IVP/BVP) involving RLD with non-linear variable order given as follows.

1.
$$\begin{cases} \left(\mathcal{D}_{0^+}^{\mu(.,y(.))} y\right)(t) = \mathcal{A}_3(t, y(t)), \;\; t \in \mathcal{D}_3 := (0, \digamma], \;\; 0 < \digamma < \infty, & (A) \\ \\ y(0) = 0, & (B) \end{cases}$$

where $\mathcal{D}_{0^+}^{\mu(.,y(.))}$ set forth the RLFDVO $\mu(t, y(t))$, $\mathcal{A}_3$ is a function pending specification and μ satisfies $0 < \mu_* \leq \mu(t, y(t)) \leq \mu^* < 1$.

2.
$$\begin{cases} \left(\mathcal{D}_{0^+}^{\mu(.,u(.))} u\right)(t) = \mathcal{A}_4(t, u_t), \;\; t \in \mathcal{D}_1^4 := (0, \digamma], \;\; 0 < \digamma < \infty, & (A) \\ \\ u(t) = \zeta(t), \qquad\qquad t \in \mathcal{D}_2^4 := [-r, 0], & (B) \end{cases}$$

where $\mathcal{D}_{0^+}^{\mu(.,u(.))}$ set forth the RLFDVO $\mu(t, u(t))$, μ satisfies $0 < \mu_* \leq \mu(t, u(t)) \leq \mu^* < 1$, $\mathcal{A}_4 : \mathcal{D}_1^4 \times L^1(\mathcal{D}_2^4) \to \mathbb{R}$ is a function pending specification and $\zeta \in L^1\left(\mathcal{D}_2^4\right)$ with $\zeta(0) = 0$. For any function u defined on $[-r, \digamma]$ and any $t \in \mathcal{D}_1^4$, we denote by u_t the elements of $L^1([-r, 0])$ defined by

$$u_t(\leftthreetimes) = u(t + \leftthreetimes), \;\; \leftthreetimes \in [-r, 0].$$

Here $u_t(.)$ quantifies the history of the state from time $t - r$ up to the present time t. We denote $\mathcal{D}^4 = \mathcal{D}_1^4 \cup \mathcal{D}_2^4 = [-r, \digamma]$.

3.
$$\begin{cases} \left(\mathcal{D}_{0^+}^{\mu(.,\omega(.))} \omega\right)(t) = \mathcal{A}_5(t, \omega(t)), \;\; t \in \mathcal{D}_5 := (0, D), \;\; 0 < D < \infty, & (A) \\ \\ \omega(0) = \omega(D) = 0, , & (B) \end{cases}$$

where $\mathcal{D}_{0^+}^{\mu(.,\omega(.))}$ set forth the RLFDVO $\mu(t,\omega(t))$, $\mathcal{A}_5$ is a function pending specification, $1 < \mu_* \leq \mu(t,\omega(t)) \leq \mu^* < 2$.

The main content of Chapter 4, is to provide an answer in the positive to the existence query for finite delayed weighted fractional problem (abbreviated FDFP) involving RLD with non-linear variable order given as follows.

$$\begin{cases} \left({}_{0^+}\mathcal{D}_w^{\mu(.)}u\right)(t) = \mathcal{A}_6(t,u_t), & t \in \mathcal{D}_1^6 := (0,\digamma], \quad 0 < \digamma < \infty, \quad (A) \\ \\ u(t) = \zeta(t), & t \in \mathcal{D}_2^6 := [-r,0], \quad (B) \end{cases}$$

Where ${}_{0^+}\mathcal{D}_w^{\mu(.)}$ set forth the WRLFDVO$\mu(t)$, $r > 0$, μ satisfies $0 < \mu_* \leq \mu(t) \leq \mu^* < 1$, $\mathcal{A}_6 : \mathcal{D}_1^6 \times L_w^1(\mathcal{D}_2^6) \to \mathbb{R}$ is a function pending specification and $\zeta \in L_w^1(\mathcal{D}_2^6)$ with $\zeta(0) = 0$. For any function u defined on $[-r,\digamma]$ and any $t \in \mathcal{D}_1^6$, we denote by u_t the elements of $L_w^1([-r,0])$ defined by

$$u_t(\curlywedge) = u(t+\curlywedge), \quad \curlywedge \in [-r,0].$$

Here $u_t(.)$ quantifies the history of the state from time $t-r$ up to the present time t. We denote $\mathcal{D}_6 = \mathcal{D}_1^6 \cup \mathcal{D}_2^6 = [-r,\digamma]$.

Chapter 1

Preliminary

In this chapter, we introduce the core definitions that will be utilized throughout the thesis.

1.1 Mathematical Notations and Fundamental Definitions

Consider $\mathcal{A} = [0, \digamma]$. By $C(\mathcal{A}, \mathbb{R})$, we denote the BS of CF $x : \mathcal{A} \to \mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in \mathcal{A}\}.$$

The symbol $C_\varsigma(\mathcal{A}, \mathbb{R})$ denotes the BS of functions $x : (0, \digamma] \to \mathbb{R}$ such that

$$0 < \varsigma < 1 \;\; and \;\; t^\varsigma x(t) \in C(\mathcal{A}, \mathbb{R}),$$

equipped with the norm

$$\|x\|_\varsigma = Sup\{t^\varsigma |x(t)| : t \in \mathcal{A}\}.$$

The symbol $L^p(\mathcal{A}, \mathbb{R})$, $1 \leq p < \infty$ denotes the BS of functions $x : \mathcal{A} \to \mathbb{R}$ which are Lebesgue measurable such that

$$p \geq 1 \;\; and \;\; \int_0^\digamma |x(\lambda)|^p d\lambda < \infty.$$

$L^p(\mathcal{A}, \mathbb{R})$ is associated with the norm

$$\|x\|_p =: \Big(\int_0^\digamma |x(\lambda)|^p d\lambda\Big)^{\frac{1}{p}}.$$

Lemma 1.1.1 *Let $0 < a, b < 1$ and $\widetilde{t} < 0$, then $f(\widetilde{t}) = a^{\widetilde{t}} - b^{\widetilde{t}}$ is a strictly monotonic function.*

Proof 1.1.1 *A simple calculation leads us to $f'(\widetilde{t}) = a^{\widetilde{t}} \ln a - b^{\widetilde{t}} \ln b$,*
***Case** $a < b$*
$\ln a < \ln b$ *(Since ln is increasing) and then $f'(\widetilde{t}) < 0$.*
***Case** $b < a$*
$\ln b < \ln a$ *(Since ln is increasing) and then $f'(\widetilde{t}) > 0$.*
Combining these two cases concludes our claim.

Remark 1.1.1 *The following observations are made to facilitate our research in the next section:*

1. *If* $0 < \digamma \leq 1$, *then* $\begin{cases} \digamma^{\mu(\lambda)-1} & \leq \digamma^{\mu_*-1}, \\ \digamma^{-\mu(\lambda)} & \leq \digamma^{-\mu^*}. \end{cases}$

2. *If* $1 < \digamma$, *then* $\begin{cases} \digamma^{\mu(\lambda)-1} & \leq 1, \\ \digamma^{-\mu(\lambda)} & \leq 1. \end{cases}$

$$\textit{Set}\ \ \Sigma^* = \max\{1, \digamma^{\mu_*-1}, \digamma^{-\mu^*}\}.$$

3. *The function $\Gamma(\mu(t)) \in L^1(\mathcal{A})$, hence we can set:*

$$\digamma_\mu = ess.\sup\left\{\left|\frac{1}{\Gamma(\mu(t))}\right| |t \in \mathcal{A}\right\}.$$

4. *If* $0 < \digamma \leq 1$, *then* $\begin{cases} \digamma^{\mu(\lambda,u(\lambda))-1} & \leq \digamma^{\mu_*-1}, \\ \digamma^{-\mu(\lambda,u(\lambda))} & \leq \digamma^{-\mu^*}. \end{cases}$

5. *If* $1 < \digamma$, *then* $\begin{cases} \digamma^{\mu(\lambda,u(\lambda))-1} & \leq 1, \\ \digamma^{-\mu(\lambda,u(\lambda))} & \leq 1. \end{cases}$

$$\textit{Set}\ \ \Sigma^* = \max\{1, \digamma^{\mu_*-1}, \digamma^{-\mu^*}\}.$$

6. *The function $\Gamma(\mu(t, u(t))) \in L^1(\mathcal{A})$, hence we can set:*

$$\digamma_\varphi = ess.\sup\left\{\left|\frac{1}{\Gamma(\mu(t, \varphi(t))}\right| |t \in \mathcal{A}\right\}.$$

1.2 Fractional Calculus.

This section is about some definitions and properties of fractional calculus of constant-order and variable-order

1.2.1 Fractional calculus of constant-order

Definition 1.2.1 *([29]) The left RLICO of the function $\phi \in L^1([a,b],\mathbb{R}_+)$ of order $\ltimes \in \mathbb{R}_+$ is defined by*

$$\mathcal{I}_{a+}^{\ltimes}\phi(t) = \frac{1}{\Gamma(\ltimes)}\int_a^t (t-\leftthreetimes)^{\ltimes-1}\phi(\leftthreetimes)d\leftthreetimes,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2.2 *([29]) The left RLDCO of order $\ltimes > 0$ of function $\phi \in L^1([a,b],\mathbb{R}_+)$, is given by*

$$(\mathcal{D}_{a+}^{\ltimes}\phi)(t) = \frac{1}{\Gamma(n-\ltimes)}\Big(\frac{d}{dt}\Big)^n \int_a^t (t-\leftthreetimes)^{n-\ltimes-1}\phi(\leftthreetimes)d\leftthreetimes,$$

here $n = [\ltimes]+1$.

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1.2.1 *([29]) Let $\ltimes > 0$, $a > 0$, $\phi \in L^1(a,b)$, $\mathcal{D}_{a+}^{\ltimes}\phi \in L^1(a,b)$. Then, the differential equation*

$$\mathcal{D}_{a+}^{\ltimes}\phi = 0,$$

has unique solution

$$\phi(s) = \phi_1(s-\beta)^{\ltimes-1} + \phi_2(s-\beta)^{\ltimes-2} + ... + \phi_n(s-\beta)^{\ltimes-n},$$

where $n = [\ltimes]+1$, $\phi_\ell \in \mathbb{R}$, $\ell = 1,2,...,n$.

Lemma 1.2.2 *([29]) Let $\ltimes > 0$, $a > 0$, $\phi \in L^1(a,b)$, $\mathcal{D}_{a+}^{\ltimes}\phi \in L^1(a,b)$. Then,*

$$\mathcal{I}_{a+}^{\ltimes}\left[\mathcal{D}_{a+}^{\ltimes}\phi(s)\right] = \phi(s) + \phi_1(s-\beta)^{\ltimes-1} + \phi_2(s-\beta)^{\ltimes-2} + ... + \phi_n(t-a)^{\ltimes-n},$$

where $n = [\ltimes]+1$, $\phi_\ell \in \mathbb{R}$, $\ell = 1,2,...,n$.

Lemma 1.2.3 *([29]) Let $\ltimes > 0$, $a > 0$, $\phi \in L^1(a,b)$, $\mathcal{D}_{a+}^{\ltimes}\phi \in L^1(a,b)$. Then,*

$$\mathcal{D}_{a+}^{\ltimes}\left[\mathcal{I}_{a+}^{\ltimes}\phi(s)\right] = \phi(s).$$

Lemma 1.2.4 *([29]) Let $\ltimes,\ \mu > 0$, $a > 0$, $\phi \in L^1(a,b)$. Then,*

$$\mathcal{I}_{a^+}^{\ltimes}\left[\mathcal{I}_{a^+}^{\mu}\phi(s)\right] = \mathcal{I}_{a^+}^{\mu}\left[\mathcal{I}_{a^+}^{\ltimes}\phi(s)\right] = \mathcal{I}_{a^+}^{\ltimes+\mu}\phi(s).$$

1.2.2 Fractional calculus of variable-order

We consider the CF $a(t) : \mathcal{A} \to (0, a^*]$ and $b(t) : \mathcal{A} \to (0, b^*]$. Then,

Definition 1.2.3 [44] *The left RLFIVO for a function $\phi \in C_\varsigma(\mathcal{A}, \mathbb{R})$ is defined by*

$$\left(\mathcal{I}_{a_1^+}^{a(.)}\phi\right)(t) = \int_{a_1}^{t} \frac{(t-\leftthreetimes)^{a(\leftthreetimes)-1}}{\Gamma(a(\leftthreetimes))}\phi(\leftthreetimes)d\leftthreetimes, \;\; t > a_1. \tag{1.1}$$

If $a(t)$ is a constant ($a(t) = \ltimes$), then (1.1) becomes

$$\mathcal{I}_{a_1^+}^{\ltimes}\phi(t) = \frac{1}{\Gamma(\ltimes)}\int_{a_1}^{t}(t-\leftthreetimes)^{\ltimes-1}\phi(\leftthreetimes)d\leftthreetimes, \;\; t > a_1. \tag{1.2}$$

Definition 1.2.4 *[44] The left RLFDVO for function $\phi \in C_{1-\varsigma}(\mathcal{A}, \mathbb{R})$ is defined by*

$$\left(\mathcal{D}_{a_1^+}^{b(.)}\phi\right)(t) = \frac{d}{dt}\left(\mathcal{I}_{a_1^+}^{1-b(.)}\phi\right)(t) = \frac{d}{dt}\int_{a_1}^{t}\frac{(t-\leftthreetimes)^{-b(\leftthreetimes)}}{\Gamma(1-b(\leftthreetimes))}\phi(\leftthreetimes)d\leftthreetimes, \;\; t > a_1. \tag{1.3}$$

If $b(t)$ is a constant ($b(t) = \mu$), then (1.3) has the form

$$\mathcal{D}_{a_1^+}^{\mu}\phi(t) = \frac{d}{dt}\mathcal{I}_{a_1^+}^{1-\mu}\phi(t) = \frac{d}{dt}\int_{a_1}^{t}\frac{(t-\leftthreetimes)^{-\mu}}{\Gamma(1-\mu)}\phi(\leftthreetimes)d\leftthreetimes, \;\; t > a_1. \tag{1.4}$$

Remark 1.2.1 *As remarked, when $a(\leftthreetimes) = C^{st}$ and $b(\leftthreetimes) = C^{st}$, then RLFDVO and RLFIVO is nothing then the standard RLDCO and RLICO, respectively; see [44] for more details.*

Remark 1.2.2 *For general functions $a(t)$ and $b(t)$, the semigroup property is lost, i.e:*

$$\mathcal{I}_{a_1^+}^{a(.)}\left[\left(\mathcal{I}_{a_1^+}^{b(.)}\phi\right)(t)\right] \neq \left(\mathcal{I}_{a_1^+}^{a(.)+b(.)}\phi\right)(t).$$

explore further in [56].

The following lemmas deals with the properties of RLICO.

Lemma 1.2.5 *([29])If $\varsigma \in \mathbb{R}$, then RLICO is bounded in $C_\varsigma(\mathcal{A}, \mathbb{R})$ and we have for $\phi \in C_\varsigma(\mathcal{A}, \mathbb{R})$*

$$\|\mathcal{I}_{0^+}^{\ltimes}\phi\|_\varsigma \leq \frac{\digamma^{\ltimes}\Gamma(1-\varsigma)}{\Gamma(1+\ltimes-\varsigma)}\|\phi\|_\varsigma, \;\; \ltimes > 0. \tag{1.5}$$

Lemma 1.2.6 *([29])If $\varsigma \in \mathbb{R}$, then RLICO is bounded in $L^p(A, \mathbb{R})$ and we have for $\phi \in L^p(A, \mathbb{R})$*

$$\|\mathcal{I}_{0^+}^{\ltimes}\phi\|_p \leq \frac{\digamma^{\ltimes}}{\ltimes\Gamma(\ltimes)}\|\phi\|_p, \;\; \ltimes > 0. \tag{1.6}$$

Based on Lemma 1.2.5 and Lemma 1.2.6, we obtain

Lemma 1.2.7 *([22]) Let $a : \mathcal{A} \to (0,1]$ be a CF, such that $0 < a_* \leq a(t) \leq a^* < 1$, then $\left(\mathcal{I}_{0^+}^{a(.)}\phi\right) \in C_\varsigma(\mathcal{A},\mathbb{R})$ for $\phi \in C_\varsigma(\mathcal{A},\mathbb{R})$. Moreover, we have*

$$\left\|\left(\mathcal{I}_{0^+}^{a(.)}\phi\right)\right\|_\varsigma \leq \frac{\digamma\Gamma(1-\varsigma)\Gamma(a_*)\digamma_\Gamma \Sigma^*}{\Gamma(1+a_*-\varsigma)}\|\phi\|_\varsigma. \tag{1.7}$$

Proof 1.2.1 *Let $\phi \in C_\varsigma(\mathcal{A},\mathbb{R})$. From Eq.(1.1), we have*

$$\begin{aligned}
\left|\left(\mathcal{I}_{0_1^+}^{a(.)}\phi\right)(t)\right| &\leq \int_0^t (t-\lambda)^{a(\lambda)-1}|\phi(\lambda)|d\lambda \\
&\leq \digamma_\Gamma \int_0^t \digamma^{a(\lambda)-1}\left(\frac{t-\lambda}{\digamma}\right)^{a(\lambda)-1}|\phi(\lambda)|d\lambda \\
&\leq \frac{\digamma_\Gamma \Sigma^*}{\digamma^{a^*-1}}\int_0^t (t-\lambda)^{a_*-1}|\phi(\lambda)|d\lambda \\
&\leq \frac{\Gamma(a^*)\digamma_\Gamma \Sigma^*}{\digamma^{a_*-1}}\mathcal{I}_{0_1^+}^{a_*}|\phi(t)|.
\end{aligned} \tag{1.8}$$

This implies that

$$\left\|\left(\mathcal{I}_{0^+}^{a(.)}\phi\right)\right\|_\varsigma \leq \frac{\Gamma(a_*)\digamma_\Gamma \Sigma^*}{\digamma^{a_*-1}}\|\mathcal{I}_{0^+}^{a_*}|\phi|\|_\varsigma.$$

From Eq.(1.5), we end up with

$$\left\|\left(\mathcal{I}_{0^+}^{a(.)}\phi\right)\right\|_\varsigma \leq \frac{\digamma\Gamma(1-\varsigma)\Gamma(a_*)\digamma_\Gamma \Sigma^*}{\Gamma(1+a_*-\varsigma)}\|\phi\|_\varsigma.$$

This is the desired inequality.

Lemma 1.2.8 *[22] Let $a : \mathcal{A} \to (0,1]$ be a CF, such that $0 < a_* \leq a(t) \leq a^* < 1$, then $\left(\mathcal{I}_{0^+}^{a(.)}\phi\right) \in L^p(\mathcal{A},\mathbb{R})$ for $\phi \in L^p(\mathcal{A},\mathbb{R})$. Moreover, we have*

$$\left\|\left(\mathcal{I}_{0^+}^{a(.)}\phi\right)\right\|_p \leq \frac{\digamma\digamma_\Gamma \Sigma^*}{a_*}\|\phi\|_p. \tag{1.9}$$

Proof 1.2.2 *Let $\phi \in L^p(\mathcal{A},\mathbb{R})$. By help of Eq.(1.27), we get*

$$\left\|\left(\mathcal{I}_{0^+}^{a(.)}\phi\right)\right\|_p \leq \frac{\Gamma(a_*)\digamma_\Gamma \Sigma^*}{\digamma^{a_*-1}}\|\mathcal{I}_{0^+}^{a_*}\phi\|_p.$$

From Eq.(1.6), we have

$$\left\|\left(\mathcal{I}_{0^+}^{a(.)}\phi\right)\right\|_p \leq \frac{\digamma\digamma_\Gamma \Sigma^*}{a_*}\|\phi\|_p.$$

Definition 1.2.5 *([42, 54]) The left RLFIVO μ, $\mu = \mu(t,\phi(t))$ for a function $\phi \in C(\mathcal{A},\mathbb{R})$ is*

$$\left(\mathcal{I}_{a_1^+}^{\mu(.,\phi(.))}\phi\right)(t) = \int_{a_1}^t \frac{(t-\lambda)^{\mu(\lambda,\phi(\lambda))-1}}{\Gamma(\mu(\lambda,\phi(\lambda)))}\phi(\lambda)d\lambda, \quad t > a_1, \tag{1.10}$$

where $\Gamma(.)$ *denotes the standard Gamma function and* $a_1 \in \mathcal{D}_3$*. If* $\mu(t,\phi(t))$ *is a constant* μ*, then (1.10) will be reduced to*

$$\mathcal{I}^{\mu}_{a_1^+}\phi(t) = \frac{1}{\Gamma(\mu)}\int_{a_1}^{t}(t-\leftthreetimes)^{\mu-1}\phi(\leftthreetimes)d\leftthreetimes, \quad t > a_1, \tag{1.11}$$

which is the classical RLICO [41, 43, 3].

We give here the definitions and properties of R-Liouvil's integral an derivative of implicit order

Definition 1.2.6 *([42, 54]) The left RLDVO* $\ltimes$*,* $\ltimes = \ltimes(t,\phi(t))$ *for a function* $\phi \in C(\mathcal{A},\mathbb{R})$ *is*

$$\left(\mathcal{D}^{\mu(.,\phi(.))}_{a_1^+}\phi\right)(t) = \frac{d}{dt}\left(\mathcal{I}^{1-\mu(.,\phi(.))}_{a_1^+}\phi\right)(t) = \frac{d}{dt}\int_{a_1}^{t}\frac{(t-\leftthreetimes)^{-\ltimes(\leftthreetimes,\phi(\leftthreetimes))}}{\Gamma(1-\ltimes(\leftthreetimes,\phi(\leftthreetimes)))}\phi(\leftthreetimes)d\leftthreetimes, \quad t > a_1. \tag{1.12}$$

If $\ltimes(t,y(t))$ *is a constant* $\ltimes$*, then (1.3) will be*

$$\mathcal{D}^{\ltimes}_{a_1^+}\phi(t) = \frac{d}{dt}\mathcal{I}^{1-\ltimes}_{a_1^+}\phi(t) = \frac{d}{dt}\int_{a_1}^{t}\frac{(t-\leftthreetimes)^{-\ltimes}}{\Gamma(1-\ltimes)}\phi(\leftthreetimes)d\leftthreetimes, \quad t > a_1, \tag{1.13}$$

which is the classical RLDCO $\ltimes$ *[41, 43].*

For further explorations on RLICO and RLDCO we can see [41, 43] and for RLFDVO see [4, 42].

Remark 1.2.3 *For general functions* $\mu(t,y(t))$ *and* $\ltimes(t,y(t))$*, the semigroup property is lost, i.e:*

$$\mathcal{I}^{\mu(.,\phi(.))}_{a_1^+}\left[\left(\mathcal{I}^{\ltimes(.,\phi(.))}_{a_1^+}\phi\right)(t)\right] \neq \left(\mathcal{I}^{\mu(.,\phi(.))+\ltimes(.,\phi(.))}_{a_1^+}\phi\right)(t).$$

explore further in [57, 58].

The following lemmas deals with the properties of RLFIVO.

Lemma 1.2.9 *[22] If* $\mu : \mathcal{A}\times\mathbb{R} \to$ *(0, 1] is a continuous function, such that* $0 < \mu_* \leq \mu(t,\phi(t)) \leq \mu^* < 1$*, then* $\mathcal{I}^{\mu(.,\phi(.))}_{0^+}\phi \in C_\varsigma(\mathcal{A},\mathbb{R})$ *for* $\phi \in C_\varsigma(\mathcal{A},\mathbb{R})$*. Moreover, we have:*

(i)

$$\|\mathcal{I}^{\mu(.,\phi(.))}_{0^+}\phi\|_\varsigma \leq \frac{\digamma\,\Gamma(1-\varsigma)\Gamma(\mu_*)\digamma_\phi\varSigma^*}{\Gamma(1+\mu_*-\varsigma)}\|\phi\|_\varsigma. \tag{1.14}$$

(ii) For $\phi_1,\phi_2 \in C_\varsigma(\mathcal{A},\mathbb{R})$*, we have*

$$\|\mathcal{I}^{\mu(.,\phi_1(.))}_{0^+}\phi_1 - \mathcal{I}^{\mu(.,\phi_2(.))}_{0^+}\phi_2\|_\varsigma \leq \frac{4\digamma B\Gamma(\mu_*)\varSigma^*\Gamma(1-\varsigma)}{\Gamma(1+\mu_*-\varsigma)}\|\phi_1-\phi_2\|_\varsigma, \tag{1.15}$$

where $B = \max\{\digamma_{\phi_2},\digamma_{\phi_1}\}$*.*

Proof 1.2.3 *(i) Let $\phi \in C_\varsigma(\mathcal{A}, \mathbb{R})$. From Eq. (1.10) we have*

$$\begin{aligned} \left|\left(\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi\right)(t)\right| &\leq F_\phi \int_0^t (t-\lambda)^{\mu(\lambda,\phi(\lambda))-1}|\phi(\lambda)|d\lambda \\ &\leq F_\phi \int_0^t F^{\mu(\lambda,\phi(\lambda))-1}\left(\frac{t-\lambda}{F}\right)^{\mu(\lambda,\phi(\lambda))-1}|\phi(\lambda)|d\lambda \\ &\leq \frac{F_\phi \Sigma^*}{F^{\mu_*-1}}\int_0^t (t-\lambda)^{\mu_*-1}|\phi(\lambda)|d\lambda \\ &\leq \frac{\Gamma(\mu_*)F_\phi \Sigma^*}{F^{\mu_*-1}}\mathcal{I}_{0^+}^{\mu_*}|\phi(t)|. \end{aligned} \tag{1.16}$$

The above estimate implies

$$\|\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi\|_\varsigma \leq \frac{\Gamma(\mu_*)F_\phi \Sigma^*}{F^{\mu_*-1}}\|\mathcal{I}_{0^+}^{\mu_*}|\phi|\|_\varsigma.$$

We apply (1.5) to obtain

$$\|\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi\|_\varsigma \leq \frac{F\Gamma(1-\varsigma)\Gamma(\mu_*)F_\phi \Sigma^*}{\Gamma(1+\mu_*-\varsigma)}\|\phi\|_\varsigma.$$

(ii) For $\phi_1, \phi_2 \in C_\varsigma(\mathcal{A}, \mathbb{R})$, we have

$$\begin{aligned} \left|\left(\mathcal{I}_{0^+}^{\mu(.,\phi_1(.))}\phi_1\right)(t) - \left(\mathcal{I}_{0^+}^{\mu(.,\phi_2(.))}\phi_2\right)(t)\right| &= \left|\int_0^t \frac{(t-\lambda)^{\mu(\lambda,\phi_1(\lambda))-1}}{\Gamma(\mu(\lambda,\phi_1(\lambda)))}\phi_1(\lambda)d\lambda - \frac{(t-\lambda)^{\mu(\lambda,\phi_2(\lambda))-1}}{\Gamma(\mu(\lambda,\phi_2(\lambda)))}\phi_2(\lambda)d\lambda\right| \\ &\leq 2B\Sigma^* \int_0^t \left(\left(\frac{t-\lambda}{F}\right)^{\mu(\lambda,\phi_1(\lambda))-1} + \left(\frac{t-\lambda}{F}\right)^{\mu(\lambda,\phi_2(\lambda))-1}\right) \\ &\quad \times|\phi_1(\lambda)-\phi_2(\lambda)|d\lambda \\ &\leq \frac{4B\Sigma^*}{F^{\mu_*-1}}\int_0^t (t-\lambda)^{\mu_*-1}|\phi_1(\lambda)-\phi_2(\lambda)|d\lambda \\ &\leq \frac{4B\Gamma(\mu_*)\Sigma^*}{F^{\mu_*-1}}\mathcal{I}_{0^+}^{\mu_*}|\phi_1(t)-\phi_2(t)|. \end{aligned} \tag{1.17}$$

From Eq.(1.5) we get

$$\begin{aligned} \|\mathcal{I}_{0^+}^{\mu(.,\phi_1(.))}\phi_1 - \mathcal{I}_{0^+}^{\mu(.,\phi_2(.))}\phi_2\|_\varsigma &\leq \frac{4B\Gamma(\mu_*)\Sigma^*}{F^{\mu_*-1}}\|\mathcal{I}_{0^+}^{\mu_*}|\phi_1-\phi_2|\|_\varsigma \\ &\leq \frac{4FB\Gamma(\mu_*)\Sigma^*\Gamma(1-\varsigma)}{\Gamma(1+\mu_*-\varsigma)}\|\phi_1-\phi_2\|_\varsigma \end{aligned}.$$

which conclude the proof.

Lemma 1.2.10 *[22] If $\mu : \mathcal{A}\times\mathbb{R} \to (0, 1]$ is a CF, such that $0 < \mu_* \leq \mu(t,\phi(t)) \leq \mu^* < 1$, then $\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi \in L^p(\mathcal{A},\mathbb{R})$ for $\phi \in L^p(\mathcal{A},\mathbb{R})$. Moreover, we have: (i)*

$$\|\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi\|_p \leq \frac{\digamma\digamma_\phi\Sigma^*}{\mu_*}\|\phi\|_p. \tag{1.18}$$

(ii) For $\phi_1, \phi_2 \in L^p(\mathcal{A},\mathbb{R})$ we have

$$\|\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi_1 - \mathcal{I}_{0^+}^{\mu(.,\phi_2(.))}\phi_2\|_p \leq \frac{4\digamma B\Sigma^*}{\mu_*}\|\phi_1-\phi_2\|_p. \tag{1.19}$$

Proof 1.2.4 *(i) Using (9), we obtain*

$$\|\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi\|_p \leq \frac{\Gamma(\mu_*)\digamma_\phi\Sigma^*}{\digamma^{\mu_*-1}}\|\mathcal{I}_{0^+}^{\mu_*}|\phi|\|_p.$$

Now, we apply Eq.(1.6) to get

$$\|\mathcal{I}_{0^+}^{\mu(.,\phi(.))}\phi\|_p \leq \frac{\digamma\digamma_\phi\Sigma^*}{\mu_*}\|\phi\|_p.$$

(ii) From Eq.(1.17) we obtain

$$\begin{aligned}\left|\left(\mathcal{I}_{0^+}^{\mu(.,\phi_1(.))}\phi_1\right)(t) - \left(\mathcal{I}_{0^+}^{\mu(.,\phi_2(.))}\phi_2\right)(t)\right|^p &\leq \left[\frac{4B\Gamma(\mu_*)\Sigma^*}{\digamma^{\mu_*-1}}\mathcal{I}_{0^+}^{\mu_*}|(\phi_1-\phi_2)(\curlywedge)|\right]^p \\ &\leq \left[\frac{4B\Gamma(\mu_*)\Sigma^*}{\digamma^{\mu_*-1}}\right]^p\left(\mathcal{I}_{0^+}^{\mu_*}|\phi_1-\phi_2|(t)\right)^p.\end{aligned} \tag{1.20}$$

Using Eq.(1.6) integrating both sides of (1.20) on $\mathcal{A}$ and take $\frac{1}{p}$-root on both sides, we get

$$\|\mathcal{I}_{0^+}^{\mu(.,\phi_1(.))}\phi_1 - \mathcal{I}_{0^+}^{\mu(.,\phi_2(.))}\phi_2\|_p \leq \frac{4\digamma B\Sigma^*}{\mu_*}\|\phi_1-\phi_2\|_p.$$

The proof of (12) is completed.

1.3 Weighted Fractional calculus of variable-order

This section is about some definitions and properties of weighted fractional calculus of constant-order and weighted fractional calculus of variable-order

Definition 1.3.1 *([15]) The left WRLFIVO μ, where $\mu = \mu(t)$ for a function $\varphi(t)$ is*

$$\left({}_{\sigma_1^+}\mathcal{I}_w^{\mu(.)}\varphi\right)(t) = \frac{1}{w(t)\Gamma(\mu(t))}\int_{\sigma_1}^t(\hbar(t)-\hbar(\curlywedge))^{\mu(\curlywedge)-1}w(\curlywedge)\varphi(\curlywedge)\hbar'(\curlywedge)d\curlywedge,\ \ t>\sigma_1, \tag{1.21}$$

where $\Gamma(.)$ is the standard Gamma function and $a_1 \in \mathcal{A}$. If $\mu(t)$ is a constant μ, then (1.21) will be reduced to

$$\left({}_{\sigma_1^+}\mathcal{I}_w^{\mu}\varphi\right)(t) = \frac{1}{w(t)\Gamma(\mu)}\int_{\sigma_1^+}^t(\hbar(t)-\hbar(\curlywedge))^{\mu-1}w(\curlywedge)\varphi(\curlywedge)\hbar'(\curlywedge)d\curlywedge,\ \ t>\sigma_1, \tag{1.22}$$

which is the classical WRLICO introduced in [24], when w and $\hbar$ are the identity see [41, 43].

Definition 1.3.2 *([15]) The left WRLFDVO Λ, where $\Lambda = \Lambda(t)$ for a function $\varphi(t)$ is*

$$\left({}_{\sigma_1^+}\mathcal{D}_w^{\Lambda(.)}\varphi\right)(t) = \frac{d}{dt}\left({}_{\sigma_1^+}\mathcal{I}_w^{1-\Lambda(.)}\varphi\right)(t) = \frac{d}{dt}\left(\frac{1}{w(t)\Gamma(1-\Lambda(t))}\int_{\sigma_1}^t (\hbar(t)-\hbar(\leftthreetimes))^{-\Lambda(\leftthreetimes)} w(\leftthreetimes)\varphi(\leftthreetimes)\hbar'(\leftthreetimes)d\leftthreetimes\right). \tag{1.23}$$

If $\Lambda(t)$ is a constant Λ, then (1.23) will be

$$\left({}_{\sigma_1^+}\mathcal{D}_w^{\Lambda}\varphi\right)(t) = \frac{d}{dt}\left({}_{\sigma_1^+}\mathcal{I}_w^{1-\Lambda}\right)\varphi(t) = \frac{d}{dt}\left(\frac{1}{w(t)\Gamma(1-\Lambda)}\int_{\sigma_1^+}^t (\hbar(t)-\hbar(\leftthreetimes))^{-\Lambda} w(\leftthreetimes)\varphi(\leftthreetimes)\hbar'(\leftthreetimes)d\leftthreetimes\right), \quad t > \sigma_1, \tag{1.24}$$

which is the classical WRLDCO for Λ introduced in [24], when w and $\hbar$ are the identity see [41, 43].

For further explorations on RLICO and WRLICO we can see [41, 43, 24], see also for WRLFDVO and RLFDVO [4, 42, 15].

Remark 1.3.1 *It is well known that the semigroup property does not hold in the case of the functions $\mu(t)$, $\Lambda(t)$, i.e.,*

$$\left({}_{\sigma_1^+}\mathcal{I}_w^{\mu(.)}\right)\left[{}_{\sigma_1^+}\mathcal{I}_w^{\Lambda(.)}\varphi\right](t) \neq \left({}_{\sigma_1^+}\mathcal{I}_w^{\mu(.)+\Lambda(.)}\varphi\right)(t),$$

for further explorations when w and $\hbar$ is the identity see [57, 58].

The following lemmas deals with the properties of WRLICO. Let $a, b \in \mathbb{R}$.

Lemma 1.3.1 *([43]) Let $\varphi \in\in L_w^p([a,b])$, then the WRLICO defined by Eq.(1.22) is bounded in $L_w^p([a,b])$, furthermore*

$$\|{}_{0^+}\mathcal{I}_w^{\ltimes}\varphi\|_{p,w} \leq \frac{(\hbar(b)-\hbar(a))^{\ltimes}}{\ltimes\Gamma(\ltimes)}\|\varphi\|_{p,w}, \quad \ltimes > 0. \tag{1.25}$$

On the base of lemmas 1.3.1 a similar inequality for the WRLFIVO.

Lemma 1.3.2 *Let $\varphi \in L_w^p([a,b])$, then $\mathcal{I}_{0^+}^{\mu(.)}\varphi \in L_w^p([a,b])$. Moreover, we have:*

$$\|{}_{\sigma_1^+}\mathcal{I}_w^{\mu(.)}\varphi\|_{p,w} \leq \frac{(\hbar(b)-\hbar(a))\,\digamma_\mu\theta^*}{\mu_*}\|\varphi\|_{p,w}. \tag{1.26}$$

Proof 1.3.1 *Let $\varphi \in L_w^p([a,b])$. From Eq. (1.1), and taking into account Remark 1.1.1 and* *assumptions on $\hbar$ and w we have*

$$\begin{aligned}\left|\left({}_{\sigma_1^+}\mathcal{I}_w^{\mu(\cdot)}\varphi\right)(t)\right| &\leq \frac{F_\mu}{|w(t)|}\int_0^t (\hbar(t)-\hbar(\lambda))^{\mu(\lambda)-1}|w(\lambda)\varphi(\lambda)|\hbar'(\lambda)d\lambda \\ &\leq \frac{F_\mu}{|w(t)|}\int_0^t (\hbar(b)-\hbar(a))^{\mu(\lambda)-1}\left(\frac{\hbar(t)-\hbar(\lambda)}{\hbar(b)-\hbar(a)}\right)^{\mu(\lambda)-1}|w(\lambda)\varphi(\lambda)|\hbar'(\lambda)d\lambda \\ &\leq \frac{F_\mu\theta^*}{|w(t)|\,(\hbar(b)-\hbar(a))^{\mu_*-1}}\int_0^t(\hbar(t)-\hbar(\lambda))^{\mu_*-1}|w(\lambda)\varphi(\lambda)|\hbar'(\lambda)d\lambda \\ &\leq \frac{\Gamma(\mu_*)F_\mu\theta^*}{|w(t)|\,(\hbar(b)-\hbar(a))^{\mu_*-1}}\left({}_{\sigma_1^+}\mathcal{I}_w^{\mu_*}\right)|\varphi(t)|.\end{aligned} \tag{1.27}$$

The above estimate implies

$$\|{}_{\sigma_1^+}\mathcal{I}_w^{\mu(\cdot)}\varphi\|_{p,w} \leq \frac{\Gamma(\mu_*)F_\mu\theta^*}{(\hbar(b)-\hbar(a))^{\mu_*-1}}\|{}_{\sigma_1^+}\mathcal{I}_w^{\mu_*}|\varphi|\|_{p,w}.$$

We apply (1.25) to obtain

$$\|{}_{\sigma_1^+}\mathcal{I}_w^{\mu(\cdot)}\varphi\|_{p,w} \leq \frac{(\hbar(b)-\hbar(a))\,F_\mu\theta^*}{\mu_*}\|\varphi\|_{p,w},$$

and this completes the proof.

1.3.1 Measure of Non-Compactness

This subsection discusses some necessary background information about the (**KMNC**).

Definition 1.3.3 *([5]) Let $\mathcal{X}$ be a BS and $\Omega_{\mathcal{X}} \subset \mathcal{X}$. The (**KMNC**) is a mapping Φ : $P(\mathcal{X}) \to [0,\infty]$ which is constructed as follows:*

$$\Phi(\Omega_{\mathcal{X}}) = \inf\left\{\rho > 0 : \Omega_{\mathcal{X}} \subseteq \cup_{\ell=1}^n \mho_\ell,\ diam(\mho_\ell) \leq \rho\right\},$$

where

$$diam(\mho_\ell) = \sup\left\{\|\varphi - \xi\| : \ \varphi,\ \xi \in \mho_\ell\right\}.$$

We can infer the following properties of (**KMNC**):

Proposition 1.3.1 *([5, 6]). Let X be a BS, $\mho$, $\mho_1$, $\mho_2$ are BS of X, then*

1. *$\Phi(\mho) = 0 \Longleftrightarrow \mho$ is RC.*

2. $\Phi(\emptyset)=0$.

3. $\Phi\left(\mho\right)=\Phi\left(\overline{\mho}\right)=\Phi\left(conv\mho\right)$.

4. $\mho_1\subset\mho_2\Longrightarrow\Phi\left(\mho_1\right)\leq\Phi\left(\mho_2\right)$.

5. $\Phi\left(\mho_1+\mho_2\right)\leq\Phi\left(\mho_1\right)+\Phi\left(\mho_2\right)$.

6. $\Phi\left(\lambda\mho\right)=|\lambda|\Phi\left(\mho\right),\lambda\in\mathbb{R}$.

7. $\Phi\left(\mho_1\cup\mho_2\right)=\max\left\{\Phi\left(\mho_1\right),\Phi\left(\mho_2\right)\right\}$.

8. $\Phi\left(\mho_1\cap\mho_2\right)=\min\left\{\Phi\left(\mho_1\right),\Phi\left(\mho_2\right)\right\}$.

9. $\Phi\left(\mho+\varphi_0\right)=\Phi\left(\mho\right)$ *for any* $\varphi_0\in X$.

Definition 1.3.4 *[5]Let $J\subset\mathbb{R}$ a bounded interval and $X\subset L^1(J)$ bounded, we define the (MNC) of X on $L^1(J)$ by*

$$\Phi(X)=\lim_{\delta\to 0}\left\{\sup\left\{\sup\left(\int_J\varphi(t+h)-\varphi(t)dt\right),|h|<\delta\right\},\varphi\in X\right\}. \tag{1.28}$$

Theorem 1.3.1 *[27, 23]Let J a bounded interval and $f\in Ł^1(J)$, then we have*

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\phi(\curlywedge)d\mu=\phi(t)\ \ a.e. \tag{1.29}$$

1.4 Some fixed point theorems

In this section we are going to list some FP which will serve as a tool in the coming two chapters.

Theorem 1.4.1 *(**DFPT**) [5] Let $\widetilde{\Lambda}$ be CCBNE of a BS X and $\Psi:\widetilde{\Lambda}\longrightarrow\widetilde{\Lambda}$ is a CO operator satisfying*

$$\Phi(\Psi(S))\leq k\Phi(S),\ \ for\ any\ (S\neq\emptyset)\subset\widetilde{\Lambda},\ \ k\in[0,1),$$

i.e., Ψ is $k-$ set contractions. Then Ψ has at least a FP in $\widetilde{\Lambda}$.

Theorem 1.4.2 *(SFPT) ([29] [28]) Let E be a BS and Q be a CCBNE of E and $\Psi:Q\longrightarrow Q$ is a CC map. Then Ψ has at least a FP in Q.*

Lemma 1.4.1 *([26]) Let Θ be a CCBNE of a Banach space and $\Psi:\Theta\longrightarrow\Theta$ be a mapping such that for some $n\in\mathbb{N}$, Ψ^n is a contraction. Then Ψ has a UFP in Θ.*

Chapter 2

Initial and Boundary Value Problems with Non-Linear Variable Order Derivative

2.1 New Results Achieved for Fractional Differential Equations with R-Liouville Derivatives of Nonlinear Variable Order

[1] We deal with the existence of solutions of the obtained solution for the Initial value problem (IVP for short)

$$\begin{cases} \left(\mathcal{D}_{0^+}^{\sigma(.)}\varphi\right)(t) = \mathcal{A}_1(t,\varphi(t)), \;\; t \in \mathcal{D}_1 := (0, F], \;\; 0 < F < \infty, & (1) \\ \varphi(0) = 0, & (2) \end{cases} \qquad \text{(IVPNFDEVO)}$$

where $\mathcal{D}_{0^+}^{\sigma(t)}$ set forth RLDVO $\sigma(.)$, $\mathcal{A}_1 : \mathcal{D}_1 \times \mathbb{R} \to \mathbb{R}$ is a CF and $0 < \sigma_* \leq \sigma(t) \leq \sigma^* < 1$.

2.1.1 Existence of solutions

Definition 2.1.1 *A function $y \in C_{1-\sigma^*}(\mathcal{D}_1, \mathbb{R})$ or $y \in L^p(\mathcal{D}_1, \mathbb{R})$ is said to be a solution for (IVPNFDEVO) if and only if it verifies (IVPNFDEVO(1)) and (IVPNFDEVO(2)), simultaneously.*

[1] **A. Hallouz**, G.Stamov, M.S.Souid and I.Stamova , New Results Achieved for Fractional Differential Equations with R-Liouville Derivatives of Nonlinear Variable Order, *Axioms*, **2023:895**, 12-9, (2023).

Before proceeding further, we expose the characterization of equation (IVPNFDEVO(1)) by an integral equation.

Lemma 2.1.1 *Let φ be an element of $C_{1-\sigma^*}(\mathcal{D}_1, \mathbb{R})$ or $L^p(\mathcal{D}_1, \mathbb{R})$. Then, (IVPNFDEVO(1)) is equivalent to*

$$\left(\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right)(t) = \int_0^t \frac{(t-\leftthreetimes)^{-\sigma(\leftthreetimes)}}{\Gamma(1-\sigma(\leftthreetimes))}\varphi(\leftthreetimes)d\leftthreetimes = \int_0^t \mathcal{A}_1(\leftthreetimes, \varphi(\leftthreetimes))d\leftthreetimes, t \in J. \tag{2.1}$$

Proof 2.1.1 *Let $\varphi \in C_\varsigma(\mathcal{D}_1, \mathbb{R})$ or $\varphi \in L^p(\mathcal{D}_1, \mathbb{R})$. Then, equation (IVPNFDEVO(1)) is*

$$\left(\mathcal{D}_{0^+}^{\sigma(.)}\varphi\right)(t) = \frac{d}{dt}\int_0^t \frac{(t-\leftthreetimes)^{-\sigma(\leftthreetimes)}}{\Gamma(1-\sigma(\leftthreetimes))}\varphi(\leftthreetimes)d\leftthreetimes = \mathcal{A}_1(t, \varphi(t)). \tag{2.2}$$

Integrating both sides of (2.2) over $[0, t]$ we get

$$\int_0^t \frac{(t-\leftthreetimes)^{-\sigma(\leftthreetimes)}}{\Gamma(2-\sigma(\leftthreetimes))}\varphi(\leftthreetimes)d\leftthreetimes = c_0 + \int_0^t \mathcal{A}_1(\leftthreetimes, \varphi(\leftthreetimes))d\leftthreetimes\,. \tag{2.3}$$

Evaluating (2.3) at 0 implies that $c_0 = 0$, which is the desired claim. Conversely, differentiating both sides of (2.1) to reach

$$\frac{d}{dt}\int_0^t \frac{(t-\leftthreetimes)^{-\sigma(\leftthreetimes)}}{\Gamma(1-\sigma(\leftthreetimes))}\varphi(\leftthreetimes)d\leftthreetimes = \mathcal{A}_1(t, \varphi(t)). \tag{2.4}$$

From this we get (IVPNFDEVO(1)) and this concludes the proof.

To proceed, we outline some essential assumptions for the analysis.

(H1) $\sigma : \mathcal{D}_1 \to (0, \sigma_*]$ is a CF.

(H2) $\mathcal{A}_1 : \mathcal{D}_1 \times \mathbb{R} \to \mathbb{R}$ is a CF in the first variable such that

$$|\mathcal{A}_1(t, \varphi_1) - \mathcal{A}_1(t, \varphi_2)| \leq k|\varphi_1 - \varphi_2|, \forall\ \varphi_1, \varphi_2 \in \mathbb{R},\ t \in \mathcal{D}_1,\ k > 0.$$

2.1.2 Existence Result in $C_\varsigma(\mathcal{D}_1, \mathbb{R})$

Theorem 2.1.1 *Assume (H1)-(H2) are satisfied. Then (IVPNFDEVO) has a US.*

Proof 2.1.2 *Let $\varsigma = 1 - \sigma^*$ and Θ be the set of element in $C_\varsigma(\mathcal{D}_1, \mathbb{R})$ such that $\varphi(0) = 0$, and consider the following operator*

$$\mathcal{O}_1 : \Theta \to \Theta,$$

where

$$(\mathcal{O}_1\varphi)(t) = \varphi(t) + \left(\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right)(t) - \int_0^t \mathcal{A}_1(\curlywedge, \varphi(\curlywedge))d\curlywedge.$$

It follows that

$$\left|\int_0^t \mathcal{A}_1(\curlywedge, \varphi_1(\curlywedge)) - \mathcal{A}_1(\curlywedge, \varphi_2(\curlywedge))d\curlywedge\right| \leq kt^{\sigma^*-1}||\varphi_1 - \varphi_2||_\varsigma,$$

and

$$\begin{aligned} |(\mathcal{O}_1\varphi_1)(t) - (\mathcal{O}_1\varphi_2)(t)| &\leq |\varphi_1(t) - \varphi_2(t)| + \left|\left(\mathcal{I}_{0^+}^{1-\sigma(.)}(\varphi_1 - \varphi_2)\right)(t)\right| \\ &+ \left|\int_0^t \mathcal{A}_1(\curlywedge, \varphi_1(\curlywedge)) - \mathcal{A}_1(\curlywedge, \varphi_2(\curlywedge))d\curlywedge\right| \\ &\leq t^{\sigma^*-1}||\varphi_1 - \varphi_2||_\varsigma + \left|\left(\mathcal{I}_{0^+}^{1-\sigma(.)}(\varphi_1 - \varphi_2)\right)(t)\right| \\ &+ kt^{\sigma^*-1}||\varphi_1 - \varphi_2||_\varsigma. \end{aligned} \tag{2.5}$$

For all $t \in \mathcal{D}_1$*, we have*

$$||\mathcal{O}_1\varphi_1 - \mathcal{O}_1\varphi_2||_\varsigma \leq ||\varphi_1 - \varphi_2||_\varsigma + ||\mathcal{I}_{0^+}^{1-\sigma(.)}(\varphi_1 - \varphi_2)||_\varsigma + k||\varphi_1 - \varphi_2||_\varsigma.$$

Using Eq.(1.7), we get

$$||\mathcal{O}_1\varphi_1 - \mathcal{O}_1\varphi_2||_\varsigma \leq ||\varphi_1 - \varphi_2||_\varsigma + \digamma B(\sigma^*, \sigma^*)\digamma_\Gamma \varSigma^*||\varphi_1 - \varphi_2||_\varsigma + k||\varphi_1 - \varphi_2||_\varsigma,$$

where B *is the beta function. Setting* $\varpi_1 = (1 + \digamma, B(\sigma^*, \sigma^*)\digamma_\Gamma \varSigma^* + k)$*, we have*

$$||\mathcal{O}_1 x - \mathcal{O}_1 y||_\varsigma \leq \varpi_1||\varphi_1 - \varphi_2||_\varsigma.$$

By induction, we can prove that

$$||\mathcal{O}_1\varphi_1 - \mathcal{O}_1\varphi_2||_\varsigma \leq \frac{\varpi_1^n}{n!}||\varphi_1 - \varphi_2||_\varsigma,$$

where $\mathcal{O}_1^n = \mathcal{O}_1 \circ \mathcal{O}_1 \circ \ldots \circ \mathcal{O}_1$*—n times. Since* $\frac{\varpi_1^n}{n!}$ *is the general term of the convergent exponential series* e^{ϖ_1}*, it tends to zero as* n *tends to infinity, and so for* n *sufficiently large we have*

$$\frac{\varpi_1^n}{n!} < 1.$$

Lemma (1.4.1) asserts that the operator $\mathcal{O}_1$ *has a UFP. This implies that*

$$\left(\mathcal{I}_{0^+}^{1-\sigma(.)}y\right)(t) = \int_0^t \mathcal{A}_1(\curlywedge, y(\curlywedge))d\curlywedge, \tag{2.6}$$

with $y(0) = 0$*. From Lemma(2.1.1), we get*

$$\mathcal{D}_{0^+}^{\sigma(t)}y(t) = \mathcal{A}_1(t, y(t)) \ \ with \ \ y(0) = 0. \tag{2.7}$$

This concludes the proof.

2.1.3 Existence Result in $L^p(\mathcal{D}_1, \mathbb{R})$

Theorem 2.1.2 *Assume (H1)-(H2) are satisfied. Then (IVPNFDEVO) has a US.*

Proof 2.1.3 *Let Θ be the set of element in $L^1(\mathcal{D}_1, \mathbb{R})$ such that $\varphi(0) = 0$, and consider the following operator*

$$\mathcal{O}_1 : \Theta \to \Theta,$$

where

$$(\mathcal{O}_1\varphi)(t) = \varphi(t) + \left(\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right)(t) - \int_0^t \mathcal{A}_1(\curlywedge, \varphi(\curlywedge))d\curlywedge .$$

We have

$$\left|\int_0^t \mathcal{A}_1(\curlywedge, \varphi_1(\curlywedge)) - \mathcal{A}_1(\curlywedge, \varphi_2(\curlywedge))d\curlywedge\right| \leq kF^{\frac{1}{p}}||\varphi_1 - \varphi_2||_p,$$

and

$$\begin{aligned} |(\mathcal{O}_1\varphi_1)(t) - (\mathcal{O}_1\varphi_2)(t)|^p &\leq 2^p(|\varphi_1(t) - \varphi_2(t)|^p + \left|\left(\mathcal{I}_{0^+}^{1-\sigma(.)}(\varphi_1 - \varphi_2)\right)(t)\right|^p \\ &+ |\int_0^t \mathcal{A}_1(\curlywedge, \varphi_1(\curlywedge)) - \mathcal{A}_1(\curlywedge, \varphi_2(\curlywedge))d\curlywedge|^p) \\ &\leq 2^p(|\varphi_1(t) - \varphi_2(t)|^p + \left|\left(\mathcal{I}_{0^+}^{1-\sigma(.)}(\varphi_1 - \varphi_2)\right)(t)\right|^p \\ &+ k^pF||\varphi_1 - \varphi_2||_p^p). \end{aligned} \tag{2.8}$$

Integrating Eq.(2.8) over $[0, F]$, we get

$$||\mathcal{O}_1\varphi_1 - \mathcal{O}_1\varphi_2||_p^p \leq 2^p\left(||\varphi_1 - \varphi_2||_p^p + ||\mathcal{I}_{0^+}^{1-\sigma(.)}(\varphi_1 - \varphi_2)||_p^p + F^2k^p||\varphi_1 - \varphi_2||_p^p\right).$$

Using Eq.(1.9), we get

$$\begin{aligned} ||\mathcal{O}_1\varphi_1 - \mathcal{O}_1\varphi_2||_p^p &\leq 2^p\left(||\varphi_1 - \varphi_2||_p^p + \left(\frac{FF_\Gamma\Sigma^*}{\sigma^*}\right)^p||\varphi_1 - \varphi_2||_p^p + F^2k^p||\varphi_1 - \varphi_2||_p^p\right) \\ &\leq 2^p\left(1 + \left(\frac{FF_\Gamma\Sigma^*}{\sigma^*}\right)^p + F^2k^p\right)||\varphi_1 - \varphi_2||_p^p. \end{aligned}$$

Setting $\varpi_2 = 2\left(1 + \left(\frac{FF_\Gamma\Sigma^*}{\sigma^*}\right)^p + F^2k^p\right)^{\frac{1}{p}}$, we have

$$||\mathcal{O}_1\varphi_1 - \mathcal{O}_1\varphi_2||_p \leq \varpi_2||\varphi_1 - \varphi_2||_p.$$

By induction we can prove that

$$||\mathcal{O}_1\varphi_1 - \mathcal{O}_1\varphi_2||_p \leq \frac{\varpi_2^n}{n!}||\varphi_1 - \varphi_2||_p,$$

where $\mathcal{O}_1^n = \mathcal{O}_1 \circ \mathcal{O}_1 \circ \ldots \circ \mathcal{O}_1 —$ *n times. Since* $\frac{\varpi_2^n}{n!}$ *is the general term of the convergent exponential series* e^{ζ}*, it tends to zero as n tends to infinity, and so for n sufficiently large we have.*

$$\frac{\varpi_2^n}{n!} < 1.$$

Lemma 1.4.1 implies that the operator $\mathcal{O}_1$ *has a UFP. This implies that*

$$\left(\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right)(t) = \int_0^t \mathcal{A}_1(\curlywedge, \varphi(\curlywedge))d\curlywedge, \tag{2.9}$$

with $\varphi(0) = 0$*. From Lemma 2.1.1, we get*

$$\left(\mathcal{D}_{0^+}^{\sigma(.)}\varphi\right)(t) = \mathcal{A}_1(t, \varphi(t)) \ \ with \ \ \varphi(0) = 0. \tag{2.10}$$

2.1.4 Applications

Example 1: Consider the following fractional initial–value problem

$$\begin{cases} \left(\mathcal{D}_{0^+}^{\sigma(.)}y\right)(t) = \mathcal{A}_1(t, y(t)), \quad t \in \mathcal{D}_1 := (0, 1] & (1) \\ y(0) = 0, & (2) \end{cases} \tag{IVPNFDEVO2}$$

where $\sigma(t) = \frac{-4t}{5} + \frac{9}{10}$ and $\mathcal{A}_1(t, y) = (t + 1)^{\frac{1}{2}} + \frac{1}{4}|y|$. Clearly, a is a CF on $[0, 1]$, and

$$0.1 = \frac{-4}{5} + \frac{9}{10} < \sigma(t) < \frac{9}{10} = 0.9 < 1.$$

We have $\mathcal{A}_1(t, x)$ is a CF on $\mathcal{D}_1 \times \mathbb{R}$, and

$$|\mathcal{A}_1(t, x) - \mathcal{A}_1(t, y)| \leq \frac{1}{4}||x| - |y|| \leq \frac{1}{4}|x - y|.$$

Therefore, the two conditions (H1)–(H2) are fulfilled by Theorem 2.1.2 and thus problem (IVPNFDEVO2) has a US.

2.2 On The Finite Delayed Fractional Differential Equation Via R-Liouville Derivative of Non-linear Variable-Order

[2] We will study the existence of solutions for the initial value problem (IVP for short)

$$\begin{cases} \left(\mathcal{D}_{0^+}^{\sigma(.)}\xi\right)(t) = \mathcal{A}_2(t, \xi_t), & t \in \mathcal{D}_1^2 := (0, N], \ \ 0 < N < \infty, & (1) \\ \xi(t) = \theta(t), & t \in \mathcal{D}_2^2 := [-r, 0], & (2) \end{cases} \tag{FDPLFDEVO}$$

[2] M.S. Souid, **A. Hallouz**, G.Hatira,On The Finite Delayed Fractional Differential Equation Via R-Liouville Derivative of Non-linear Variable-Order

where $\mathcal{D}_{0^+}^{\sigma(.)}$ set forth RLDVO $\sigma(.)$, $r > 0$ and $\sigma(t)$ satisfies $0 < \sigma_* \leq \sigma(t) \leq \sigma^* < 1$. Consider $\mathcal{E}$ a Banach space such that $\dim \mathcal{E} = \infty$ endowed with the norm $|.|_{\mathcal{E}}$, $\mathcal{A}_2 : \mathcal{D}_1^2 \times L^1(\mathcal{D}_2^2, \mathcal{E}) \to \mathcal{E}$ is a generic function and $\theta \in L^1(\mathcal{D}_2^2, \mathcal{E})$ with $\theta(0) = 0$. For any function ξ defined on $[-r, N]$ and any $t \in \mathcal{D}_1^2$, we denote by ξ_t the elements of $L^1([-r, 0], \mathbb{R})$ defined by

$$\xi_t(\curlywedge) = \xi(t + \curlywedge), \ \ \curlywedge \in [-r, 0].$$

Here $\xi_t(.)$ quantifies the history of the state from time $t - r$ up to the present time t. We denote $\mathcal{D}^2 = \mathcal{D}_1^2 \cup \mathcal{D}_2^2 = [-r, N]$.

2.2.1 Achieved results

Definition 2.2.1 *We say that $\xi \in L^1(\mathcal{D}^2)$ is a solution for (FDPLFDEVO) if and only if it (FDPLFDEVO(1)) and (FDPLFDEVO(2)) are fulfilled at the same time.*

In order to make the analysis issue easy in the BS $L^1(\mathcal{D}^2)$, we will provide an equivalent integral form of FDPLFDEVO(1).

Lemma 2.2.1 *For each $\xi \in L^1(\mathcal{D}^2)$. Then, equation (FDPLFDEVO(1)) is equivalent to*

$$\left(\mathcal{I}_{0^+}^{1-\sigma(.)}\xi\right)(t) = \int_0^t \frac{(t-\curlywedge)^{-\sigma(\curlywedge)}}{\Gamma(1-\sigma(\curlywedge))}\xi(\curlywedge)d\curlywedge = \int_0^t \mathcal{A}_2(\curlywedge, \xi_\curlywedge)d\curlywedge, t \in \mathcal{D}_1^2. \tag{2.11}$$

Proof 2.2.1 *Let $\xi \in L^p(\mathcal{D}^2, \mathbb{R})$, dy definition(1.3), (FDPLFDEVO(A)) is*

$$\left(\mathcal{D}_{0^+}^{\sigma(.)}\xi\right)(t) = \frac{d}{dt}\int_0^t \frac{(t-\curlywedge)^{-\sigma(\curlywedge)}}{\Gamma(1-\sigma(\curlywedge))}\xi(\curlywedge)d\curlywedge = \mathcal{A}_2(t, \xi_t). \tag{2.12}$$

Take the integral (2.12) from $[0, t]$, we get

$$\int_0^t \frac{(t-\curlywedge)^{-\sigma(\curlywedge)}}{\Gamma(1-\sigma(\curlywedge))}\xi(\curlywedge)d\curlywedge = c_0 + \int_0^t \mathcal{A}_2(\curlywedge, \xi_\curlywedge)d\curlywedge\,. \tag{2.13}$$

For $t = 0$ at (2.13) gives us $c_0 = 0$. The converse is a differentiation of (2.11), and it is insured by lemma 1.2.8 and we reach

$$\frac{d}{dt}\int_0^t \frac{(t-\curlywedge)^{-\sigma(\curlywedge)}}{\Gamma(1-\sigma(\curlywedge))}\xi(\curlywedge)d\curlywedge = \mathcal{A}_2(t, \xi_t), \tag{2.14}$$

from which we get (FDPLFDEVO(A)). The proof is concluded.

To proceed, we outline assumptions essential for the analysis.

(A1) $\sigma : \mathcal{D}_1^2 \times \to (0, \sigma_*]$ is a CF.

(A2) $\mathcal{A}_2 : \mathcal{D}_1^2 \times L^1(\mathcal{D}_2^2, \mathcal{E}) \to \mathcal{E}$ is a CF with respect to its first variable and such that:

$$\left|\mathcal{A}_2(t, \widetilde{f}_1) - \mathcal{A}_2(t, \widetilde{f}_2)\right|_{\mathcal{E}} \leq \tau \left|\left|\widetilde{f}_1 - \widetilde{f}_2\right|\right|_{L^1\left(\mathcal{D}_2^2, \mathcal{E}\right)}, \ \forall \ \widetilde{f}_1, \widetilde{f}_2 \in L^1\left(\mathcal{D}_2^2, \mathcal{E}\right),$$

$t \in \mathcal{D}_1^2$ and $\tau > 0$.

(A3) $\mathcal{A}_2 : \mathcal{D}_1^2 \times L^1(\mathcal{D}_2^2, \mathcal{E}) \to \mathcal{E}$ is a CF and for a sufficiently small $\delta > 0$ we assume further that

$$\left|\mathcal{A}_2(t+\delta, \widetilde{f}_1) - \mathcal{A}_2(t, \widetilde{f}_2)\right|_{\mathcal{E}} \leq \overline{\lambda}(\delta) + \tau \left|\left|\widetilde{f}_1 - \widetilde{f}_2\right|\right|_{L^1\left(\mathcal{D}_2^2, \mathcal{E}\right)}, \ \forall \ \widetilde{f}_1, \widetilde{f}_2 \in L^1\left(\mathcal{D}_2^2, \mathcal{E}\right), \quad (2.15)$$

$t \in \mathcal{D}_1^2$ and $\tau > 0$, $\overline{\lambda}$ a positive CF near the origine and $\overline{\lambda}(0) = 0$.

The 1^{rst} obtained result is concluded from lemma(1.4.1).

Theorem 2.2.1 *Under the assumptions (A1)-(A2) the (FDPLFDEVO) has a US in the BS $L^1(\mathcal{D}^2)$.*

Proof 2.2.2 *Let φ, ξ elements in $L^1(\mathcal{D}^2)$ and the operator*

$$\widetilde{\mathcal{W}}_\eta : L^1\left(\mathcal{D}^2\right) \to L^1\left(\mathcal{D}^2\right),$$

where

$$\left(\widetilde{\mathcal{W}}_\eta \xi\right)(t) = \begin{cases} \theta(t), & t \in \mathcal{D}_2^2, \\ \eta\xi(t) + \mathcal{I}_{0^+}^{1-\sigma(t)}\xi(t) - \int_0^t \mathcal{A}_2(\lambda, \xi_\lambda) d\lambda, & t \in \mathcal{D}_1^2. \end{cases} \quad (2.16)$$

Where $0 < \eta < 1$. We have from (A2) that ,

$$\begin{aligned} \left|\int_0^t \mathcal{A}_2(\lambda, \varphi_\lambda) - \mathcal{A}_2(\lambda, \xi_\lambda) d\lambda\right|_{\mathcal{E}} &\leq \tau \int_0^N ||\varphi_\lambda - \xi_\lambda||_{L^1\left(\mathcal{D}_2^2, \mathcal{E}\right)} d\lambda \\ &\leq N\tau ||\varphi - \xi||_{L^1(\mathcal{D}^2, \mathcal{E})}. \end{aligned} \quad (2.17)$$

Using Eq.(2.17), we get

$$\begin{aligned} \left|\left(\widetilde{\mathcal{W}}_\eta \varphi\right)(t) - \left(\widetilde{\mathcal{W}}_\eta \xi\right)(t)\right|_{\mathcal{E}} &\leq \eta|\varphi(t) - \xi(t)|_{\mathcal{E}} + |\mathcal{I}_{0^+}^{1-\sigma(t)}\varphi(t) - \mathcal{I}_{0^+}^{1-\sigma(t)}\xi(t)|_{\mathcal{E}} \\ &+ \left|\int_0^t \mathcal{A}_2(\lambda, \varphi(\lambda)) - \mathcal{A}_2(\lambda, \xi(\lambda)) d\lambda\right|_{\mathcal{E}} \\ &\leq \eta|\xi(t) - \varphi(t)|_{\mathcal{E}} + \left|\mathcal{I}_{0^+}^{1-\sigma(t)}\varphi(t) - \mathcal{I}_{0^+}^{1-\sigma(t)}\xi(t)\right|_{\mathcal{E}} \\ &+ N\tau ||\varphi - \xi||_{L^1(\mathcal{D}^2, \mathcal{E})}. \end{aligned} \quad (2.18)$$

We conclude

$$\begin{aligned} \left|\left|\widetilde{\mathcal{W}}_\eta \varphi - \widetilde{\mathcal{W}}_\eta \xi\right|\right|_{L^1(\mathcal{D}^2, \mathcal{E})} &\leq \eta ||\varphi - \xi||_{L^1(\mathcal{D}^2, \mathcal{E})} + \left|\left|\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi - \mathcal{I}_{0^+}^{1-\sigma(.)}\xi\right|\right|_{L^1(\mathcal{D}^2, \mathcal{E})} \\ &+ N^2\tau ||\varphi - \xi||_{L^1(\mathcal{D}^2, \mathcal{E})}. \end{aligned} \quad (2.19)$$

Using Eq.(1.9) from lemma 4, we get

$$\left|\left|\widetilde{\mathcal{W}}_\eta\varphi-\widetilde{\mathcal{W}}_\eta\xi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})}\leq\left(\eta+\frac{(N+r)\,F_\sigma\Sigma^*}{1-\sigma^*}+N^2\tau\right)||\varphi-\xi||_{L^1(\mathcal{D}^2,\mathcal{E})}. \tag{2.20}$$

Set

$$\varpi_3=\eta+\frac{(N+r)\,F_\sigma\Sigma^*}{1-\sigma^*}+N^2\tau. \tag{2.21}$$

Using an induction argument to prove that

$$\left|\left|\widetilde{\mathcal{W}}^k_\eta\varphi-\widetilde{\mathcal{W}}^k_\eta\xi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})}\leq\frac{\varpi_3^k}{k!}||\varphi-\xi||_{L^1(\mathcal{D}^2,\mathcal{E})}, \tag{2.22}$$

where $\widetilde{\mathcal{W}}^k_\eta$ *is in composition sens. It is a basic fact to prove that* $\frac{\varpi_3^k}{k!}$ *it tends to zero as* k *tends to infinity, and so a sufficiently large* k *we have.*

$$\frac{\varpi_3^k}{k!}<1.$$

From lemma 1.4.1 the existence of a unique $\xi\in L^1(\mathcal{D}^2)$ *such that*

$$\left(\widetilde{\mathcal{W}}_\eta\xi\right)(t)=\xi(t). \tag{2.23}$$

At this point we can fix $\eta=\frac{1}{2}$ *and* $t\in\mathcal{D}^2_1$ *we deduce*

$$\xi(0)=\theta(0)=0.$$

Letting $\eta\to 1$ *in the interval* $\mathcal{D}^2_1$*, we reach that*

$$\mathcal{I}^{1-\sigma(t)}_{0^+}\xi(t)=\int_0^t\mathcal{A}_2(\leftthreetimes,\xi_\leftthreetimes)d\leftthreetimes,t\in\mathcal{D}^2_1. \tag{2.24}$$

Using Lemma(2.2.1), then we can conclude that

$$\begin{cases}\mathcal{D}^{\sigma(t)}_{0^+}\xi(t)=\mathcal{A}_2(t,\xi_t), & t\in\mathcal{D}^2_1,\\ \xi(t)=\theta(t) & t\in\mathcal{D}^2_2.\end{cases} \tag{2.25}$$

This concludes our proof.

The 2^{end} obtained result is a consequence of Theorem (1.4.1).

Theorem 2.2.2 *Assume that (A1)-(A3) are satisfied and*

$$\widetilde{K}=\frac{(N+r)\,F_\sigma\Sigma^*}{1-\sigma^*}+N^2\tau<1. \tag{2.26}$$

Then the (FDPLFDEVO) has ALS in $L^1(\mathcal{D}^2)$.

Proof 2.2.3 *Let φ, ξ elements in $L^1(\mathcal{D}^2)$, $t \in \mathcal{D}^2$ and consider the operator*

$$\widetilde{\mathcal{W}}_{\eta_n} : L^1\left(\mathcal{D}^2\right) \to L^1\left(\mathcal{D}^2\right),$$

where

$$\left(\widetilde{\mathcal{W}}_{\eta_n}\xi\right)(t) = \begin{cases} \theta(t), & t \in \mathcal{D}_2^2, \\ \eta_n\xi(t) + \mathcal{I}_{0^+}^{1-\sigma(t)}\xi(t) - \int_0^t \mathcal{A}_2(\curlywedge, \xi_\curlywedge)d\curlywedge, & t \in \mathcal{D}_1^2. \end{cases} \tag{2.27}$$

Where $\eta_n = \dfrac{2a}{n+a}$ and $a > 0$ a real number made to be fixed after in the proof.For $t \in \mathcal{D}_2^2$

$$\left|\left(\widetilde{\mathcal{W}}_{\eta_n}\varphi\right)(t)\right|_{\mathcal{E}} \leq |\theta(t)|_{\mathcal{E}}. \tag{2.28}$$

From (A3) we have that

$$\begin{aligned} \left|\int_0^t \mathcal{A}_2(\curlywedge, \varphi_\curlywedge)d\curlywedge\right|_{\mathcal{E}} &\leq \int_0^N |\mathcal{A}_2(\curlywedge, \varphi_\curlywedge) - \mathcal{A}_2(\curlywedge, 0)|_{\mathcal{E}}\, d\curlywedge + N\,||\mathcal{A}_2(.,0)||_{\infty,\mathcal{E}} \\ &\leq N\tau||\varphi||_{L^1(\mathcal{D}^2)} + N\,||\mathcal{A}_2(.,0)||_{\infty}. \end{aligned} \tag{2.29}$$

For $t \in \mathcal{D}_2^2$ and by mean of Eq.(2.29), we get

$$\begin{aligned} \left|\left(\widetilde{\mathcal{W}}_{\eta_n}\varphi\right)(t)\right|_{\mathcal{E}} &\leq \eta_n|\varphi(t)|_{\mathcal{E}} + \left|\mathcal{I}_{0^+}^{1-\sigma(t)}\varphi(t)\right|_{\mathcal{E}} + \left|\int_0^t \mathcal{A}_2(\curlywedge, \varphi(\curlywedge))d\curlywedge\right|_{\mathcal{E}} \\ &\leq \eta_n|\varphi(t)|_{\mathcal{E}} + \left|\mathcal{I}_{0^+}^{1-\sigma(t)}\varphi(t)\right|_{\mathcal{E}} + N\tau||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})} + N\,||\mathcal{A}_2(.,0)||_{\infty,\mathcal{E}}. \end{aligned} \tag{2.30}$$

Combining Eq.(2.30) and Eq.(2.28), we get

$$\begin{aligned} \left|\left|\widetilde{\mathcal{W}}_{\eta_n}\varphi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} &\leq \eta_n||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})} + \left|\left|\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} + N^2\tau||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})} \\ &\quad + N^2\,||\mathcal{A}_2(.,0)||_{\infty,\mathcal{E}} + ||\theta||_{L^1\left(\mathcal{D}_2^2,\mathcal{E}\right)}. \end{aligned} \tag{2.31}$$

Using Eq.(1.9) from Lemma 4, we get

$$\left|\left|\widetilde{\mathcal{W}}_{\eta_n}\varphi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} \leq \left(\eta_n + \frac{(N+r)\,\digamma_x\Sigma^*}{1-\sigma^*} + N^2\tau\right)||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})} + N^2\,||\mathcal{A}_2(.,0)||_{\infty,\mathcal{E}} + ||\theta||_{L^1\left(\mathcal{D}_2^2,\mathcal{E}\right)}. \tag{2.32}$$

There is a sufficiently large integer N_1 for which

$$\eta_n < 1 - \widetilde{K} \;\; \forall n \geq N_1. \tag{2.33}$$

And the choice of N_1 is independent from the choice of $a > 0$ fixed in the binning of the proof. Set

$$R = \frac{N^2\,||\mathcal{A}_2(.,0)||_{\infty,\mathcal{E}} + ||\theta||_{L^1\left(\mathcal{D}_2^2,\mathcal{E}\right)}}{1 - \left(\eta_n + \widetilde{K}\right)}. \tag{2.34}$$

Consider

$$B_R = \left\{\varphi \in L^1\left(\mathcal{D}^2\right) \ \mid \ ||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})} \leq R\right\}. \tag{2.35}$$

It is well known that B_R is a CCBNE.

We will prove that $\widetilde{\mathcal{W}}_{\eta_n} : B_R \to B_R$ is CC in few steps.

step1: *The fact that $\widetilde{\mathcal{W}}_{\eta_n}(B_R) \subset B_R$ is bay construction.*

step2: *$\widetilde{\mathcal{W}}_{\eta_n}$ is CO, let $(\varphi_k)_{k\geq 0} \subset B_R$, and $\varphi \in B_R$ such that $\varphi_k \xrightarrow[k\to+\infty]{} \varphi$. We have to state and prove the following lemma*

Lemma 2.2.2 *We have the following convergence*

$$\left|\left|\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi_k - \mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} \xrightarrow[k\to+\infty]{} 0. \tag{2.36}$$

$$\int_0^N |\mathcal{A}_2(\curlywedge, \varphi_{\curlywedge,k}) - \mathcal{A}_2(\curlywedge, \varphi_{\curlywedge})|_{\mathcal{E}}\, d\curlywedge \xrightarrow[k\to+\infty]{} 0. \tag{2.37}$$

Proof 2.2.4 *For Eq.(2.36) using Eq.(1.9), we get*

$$\left|\left|\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi_k - \mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} \leq \frac{4\,(N+r)\,B\Sigma^*}{1-\sigma^*}\|\varphi_k - \varphi\|_{L^1(\mathcal{D}^2,\mathcal{E})} \xrightarrow[k\to+\infty]{} 0. \tag{2.38}$$

For Eq.(2.37) we have

$$\int_0^N |\mathcal{A}_2(\curlywedge, \varphi_{\curlywedge,k}) - \mathcal{A}_2(\curlywedge, \varphi_{\curlywedge})|_{\mathcal{E}}\, d\curlywedge \leq N\tau||\varphi_k - \varphi||_{L^1(\mathcal{D}^2,\mathcal{E})} \xrightarrow[k\to+\infty]{} 0, \tag{2.39}$$

and this is exactly (2.37).

As a consequence of the above lemma we have

$$\begin{aligned}\left|\left|\widetilde{\mathcal{W}}_{\eta_n}\varphi_k - \widetilde{\mathcal{W}}_{\eta_n}\varphi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} &\leq \eta_n||\varphi_k - \varphi||_{L^1(\mathcal{D}^2,\mathcal{E})} + \left|\left|\mathcal{I}_{0^+}^{1-\sigma(.)}\varphi_k - \mathcal{I}_{0^+}^{1-\sigma(.)}\varphi\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} \\ &+ \left|\left|\int_0^N |\mathcal{A}_2(\curlywedge, \varphi_{\curlywedge,k}) - \mathcal{A}_2(\curlywedge, \varphi_{\curlywedge})|_{\mathcal{E}}\, d\curlywedge\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} \xrightarrow[k\to+\infty]{} 0.\end{aligned} \tag{2.40}$$

Which means that $\widetilde{\mathcal{W}}_{\eta_n}$ is CO.

step3: *For two real numbers $0 < \rho < \delta$ we have to show that $\widetilde{\mathcal{W}}_{\eta_n}$ is a κ-Set, which means for $0 < \kappa < 1$ we have*

$$\Phi\left(\widetilde{\mathcal{W}}_{\eta_n}(B)\right) \leq \kappa\Phi(B), \tag{2.41}$$

where Φ is the (KMNC). We have for $t \in \mathcal{D}_2^2$

$$\left|\left(\widetilde{\mathcal{W}}_{\eta_n}\varphi\right)(t+\delta) - \left(\widetilde{\mathcal{W}}_{\eta_n}\varphi\right)(t)\right|_{\mathcal{E}} \leq |\theta(t+\delta) - \theta(t)|_{\mathcal{E}}\,. \tag{2.42}$$

From (A3) we have that

$$\begin{aligned}\left|\int_0^{t+\delta}\mathcal{A}_2(\curlywedge,\varphi_\curlywedge)d\curlywedge-\int_0^{t}\mathcal{A}_2(\curlywedge,\varphi_\curlywedge)d\curlywedge\right|_{\mathcal{E}} &\leq \left|\int_t^{t+\delta}\mathcal{A}_2(\curlywedge,\varphi_\curlywedge)d\curlywedge\right|_{\mathcal{E}}\\ &\leq \int_t^{t+\delta}\left|\mathcal{A}_2(\curlywedge,\varphi_\curlywedge)-\mathcal{A}_2(\curlywedge,0)\right|_{\mathcal{E}}d\curlywedge\\ &+\delta\left|\left|\mathcal{A}_2(.,0)\right|\right|_{\infty,\mathcal{E}}\\ &\leq \delta\tau||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})}+\delta\left|\left|\mathcal{A}_2(.,0)\right|\right|_{\infty,\mathcal{E}}.\end{aligned}\tag{2.43}$$

For $t\in\mathcal{D}_2^2$ and by mean of Eq.(2.43), we get

$$\begin{aligned}\left|\left(\widetilde{\mathcal{W}}_{\eta_n}\varphi\right)(t+\delta)-\left(\widetilde{\mathcal{W}}_{\eta_n}\varphi\right)(t)\right|_{\mathcal{E}} &\leq \eta_n|\varphi(t+\delta)-\varphi(t)|_{\mathcal{E}}+\left|\mathcal{I}_{0^+}^{1-\sigma(t)}\left(\varphi(t+\delta)-\varphi(t)\right)\right|_{\mathcal{E}}\\ &+\left|\int_0^{t+\delta}\mathcal{A}_2(\curlywedge,\varphi_\curlywedge)d\curlywedge-\int_0^{t}\mathcal{A}_2(\curlywedge,\varphi_\curlywedge)d\curlywedge\right|_{\mathcal{E}}\\ &\leq \eta_n|\varphi(t+\delta)-\varphi(t)|_{\mathcal{E}}+\left|\mathcal{I}_{0^+}^{1-\sigma(t)}\left(\varphi(t+\delta)-\varphi(t)\right)\right|_{\mathcal{E}}\\ &+\delta\tau||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})}+\delta\left|\left|\mathcal{A}_2(.,0)\right|\right|_{\infty,\mathcal{E}}.\end{aligned}\tag{2.44}$$

Combining Eq.(2.42) and Eq.(2.44), we conclude that

$$\begin{aligned}\left|\left|\widetilde{\mathcal{W}}_{\eta_n}\varphi(.+\delta)-\widetilde{\mathcal{W}}_{\eta_n}\varphi(.)\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} &\leq \eta_n||\varphi(.+\delta)-\varphi(.)||_{L^1(\mathcal{D}^2,\mathcal{E})}+\left|\left|\mathcal{I}_{0^+}^{1-\sigma(.)}\left(\varphi(.+\delta)-\varphi(.)\right)\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})}\\ &+N\delta\tau||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})}+N\delta\left|\left|\mathcal{A}_2(.,0)\right|\right|_{\infty,\mathcal{E}}.\end{aligned}\tag{2.45}$$

Using Eq.(1.9) from Lemma 1.2.7, we get

$$\begin{aligned}\left|\left|\widetilde{\mathcal{W}}_{\eta_n}\varphi(.+\delta)-\widetilde{\mathcal{W}}_{\eta_n}\varphi(.)\right|\right|_{L^1(\mathcal{D}^2,\mathcal{E})} &\leq \left(\eta_n+\frac{(N+r)\digamma_\sigma\Sigma^*}{1-\sigma^*}\right)||\varphi(.+\delta)-\varphi(.)||_{L^1(\mathcal{D}^2,\mathcal{E})}\\ &+N\delta\tau||\varphi||_{L^1(\mathcal{D}^2,\mathcal{E})}+N\delta\left|\left|\mathcal{A}_2(.,0)\right|\right|_{\infty,\mathcal{E}}.\end{aligned}\tag{2.46}$$

And this will give

$$\mu\left(\widetilde{\mathcal{W}}_{\eta_n}(B)\right)\leq\left(\eta_n+\frac{(N+r)\digamma_\sigma\Sigma^*}{1-\sigma^*}\right)\mu(B).\tag{2.47}$$

From Eq.(2.26) and as before there is a sufficiently large integer N_2 for which

$$\eta_n<1-\frac{(N+r)\digamma_\sigma\Sigma^*}{1-\sigma^*}\ \ \forall n\geq N_2.\tag{2.48}$$

Thus Eq.(2.41) is satisfied, take $N_0=\max\{N_1,N_2\}$ and hence by Theorem(1.4.1) $\widetilde{\mathcal{W}}_{\eta_n}$ has at least one FP in B_R.

$$\left(\widetilde{\mathcal{W}}_{\eta_n}\xi\right)(t)=\xi(t)\ \ \forall n\geq N_0.$$

At this point we can fix $a=N_0$, $n=2N_0$ and $t\in\mathcal{D}_1^2$ to deduce

$$\xi(0)=\theta(0)=0.$$

In the same way choose $n = N_0$ in the interval $\mathcal{D}_1^2$, we get

$$\mathcal{I}_{0^+}^{1-\sigma(t)}\xi(t) = \int_0^t \mathcal{A}_2(\lambda, \xi_\lambda)d\lambda, t \in \mathcal{D}_1^2. \tag{2.49}$$

Using Lemma(2.2.1), then we can conclude that

$$\begin{cases} \mathcal{D}_{0^+}^{\sigma(t)}\xi(t) = \mathcal{A}_2(t, \xi_t), & t \in \mathcal{D}_1^2, \\ \xi(t) = \theta(t) & t \in \mathcal{D}_2^2. \end{cases} \tag{2.50}$$

This concludes our proof.

Chapter 3

Initial and Boundary Value Problems with Implicit Variable Order Derivative

3.1 New solvability results for a variable-order fractional initial value problem

[1] We will study the existence of solutions for the initial value problem (IVP for short)

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t,y(t))}y(t) = \mathcal{A}_3(t,y(t)), \;\; t \in \mathcal{D}_3 := (0, F], \;\; 0 < F < \infty, & (A) \\ \\ y(0) = 0. & (B) \end{cases} \tag{IVPFDENVO}$$

where $\mathcal{D}_{0^+}^{\mu(t,y(t))}$ set forth the RLFDVO $\mu(t,y(t))$, $\mathcal{A}_3$ is a generic function and μ satisfies $0 < \mu_* \leq \mu(t,y(t)) \leq \mu^* < 1$.

3.1.1 Existence of solutions

Definition 3.1.1 *A function $y \in C_\varsigma(\mathcal{D}_3,\mathbb{R})$ or $y \in L^p(\mathcal{D}_3,\mathbb{R})$ is said to be a solution for (IVPFDENVO) if and only if it verifies (IVPFDENVO(A)) and (IVPFDENVO(B)), simultaneously.*

In order to present our new existence results in the BSs $C_\varsigma(\mathcal{D}_3,\mathbb{R})$ and $L^p(\mathcal{D}_3,\mathbb{R})$, we will analyze an equivalent integral form of the IVPFDENVO(A).

[1] **A. Hallouz**, M. S. Souid and J. Alzabut, New solvability and stability results for variable-order fractional initial value problem, *The Journal of Analysis*, **2024** (2024).

Lemma 3.1.1 *Let y be an element of $C_\varsigma(\mathcal{D}_3,\mathbb{R})$ or $L^p(\mathcal{D}_3,\mathbb{R})$. Then, equation (IVPF-DENVO(A)) is equivalent to*

$$\mathcal{I}_{0^+}^{1-\mu(t,y(t))}y(t) = \int_0^t \frac{(t-\curlywedge)^{-\mu(\curlywedge,y(\curlywedge))}}{\Gamma(1-\mu(\curlywedge,y(\curlywedge)))}y(\curlywedge)d\curlywedge = \int_0^t \mathcal{A}_3(\curlywedge,y(\curlywedge))d\curlywedge, t\in\mathcal{D}_3. \tag{3.1}$$

Proof 3.1.1 *Let $y\in C_\varsigma(\mathcal{D}_3,\mathbb{R})$ or $y\in L^p(\mathcal{D}_3,\mathbb{R})$. Then, equation (IVPFDENVO(A)) can be represented as*

$$\mathcal{D}_{0^+}^{\mu(t,y(t))}y(t) = \frac{d}{dt}\int_0^t \frac{(t-\curlywedge)^{-\mu(\curlywedge,y(\curlywedge))}}{\Gamma(1-\mu(\curlywedge,y(\curlywedge)))}y(\curlywedge)d\curlywedge = \mathcal{A}_3(t,y(t)). \tag{3.2}$$

Integrating both sides of (3.2) from $[0,t]$, we get

$$\int_0^t \frac{(t-\curlywedge)^{-\mu(\curlywedge,y(\curlywedge))}}{\Gamma(1-\mu(\curlywedge,y(\curlywedge)))}y(\curlywedge)d\curlywedge = c_0 + \int_0^t \mathcal{A}_3(\curlywedge,y(\curlywedge))d\curlywedge\,. \tag{3.3}$$

Evaluating (3.3) at $t=0$ gives us $c_0=0$. Conversely, differentiating both sides of (3.1) to reach

$$\frac{d}{dt}\int_0^t \frac{(t-\curlywedge)^{-\mu(\curlywedge,y(\curlywedge))}}{\Gamma(1-\mu(\curlywedge,y(\curlywedge)))}y(\curlywedge)d\curlywedge = \mathcal{A}_3(t,y(t)), \tag{3.4}$$

from which we get (IVPFDENVO(A)). The proof is concluded.

To proceed, we outline assumptions essential for the analysis.

(A1) $\mu:\mathcal{D}_3\times\mathbb{R}\to(0,\,\mu_*]$ is a CF.

(A2) $\mathcal{A}_3:\mathcal{D}_3\times\mathbb{R}\to\mathbb{R}$ is a CF with respect to its first variable and such that:

$$|\mathcal{A}_3(\curlywedge,\chi_1)-\mathcal{A}_3(\curlywedge,\chi_2)|\le k|\chi_1-\chi_2|,\ \forall\ \chi_1,\chi_2\in\mathbb{R},$$

$t\in\mathcal{D}_3$ and $k>0$.

3.1.2 Existence Result in $C_\varsigma(\mathcal{D}_3,\mathbb{R})$

The first obtained result is based on lemma (1.4.1).

Theorem 3.1.1 *Assume that (A1)-(A2) are satisfied. Then the (IVPFDENVO) has a US in $C_{1-\mu^*}(\mathcal{D}_3,\mathbb{R})$.*

Proof 3.1.2 *Let us consider $\varsigma=1-\mu^*$ and the set of elements Θ in the space $C_\varsigma(\mathcal{D}_3,\mathbb{R})$ such that $y(0)=0$. Define the following operator*

$$\Pi:\Theta\to\Theta,$$

where

$$(\Pi y)(t) = y(t) + \mathcal{I}_{0+}^{1-\mu(t,y(t))} y(t) - \int_0^t \mathcal{A}_3(\leftthreetimes, y(\leftthreetimes)) d\leftthreetimes . \tag{3.5}$$

First, for two $x, y : \mathcal{D}_3 \to \mathbb{R}$ *using (A2), we have*

$$\left| \int_0^t \left[\mathcal{A}_3(\leftthreetimes, x(\leftthreetimes)) - \mathcal{A}_3(\leftthreetimes, y(\leftthreetimes)) \right] d\leftthreetimes \right| \le k t^{-\varsigma} ||y - x||_\varsigma . \tag{3.6}$$

Then, from 3.6 we can get the following estimation

$$\begin{aligned} |(\Pi x)(t) - (\Pi y)(t)| &\le |x(t) - y(t)| + |\mathcal{I}_{0+}^{1-\mu(t,x(t))} x(t) - \mathcal{I}_{0+}^{1-\mu(t,y(t))} y(t)| \\ &+ \left| \int_0^t \left[\mathcal{A}_3(\leftthreetimes, x(\leftthreetimes)) - \mathcal{A}_3(\leftthreetimes, y(\leftthreetimes)) \right] d\leftthreetimes \right| \\ &\le t^{-\varsigma} ||y - x||_\varsigma + |\mathcal{I}_{0+}^{1-\mu(t,x(t))} x(t) - \mathcal{I}_{0+}^{1-\mu(t,y(t))} y(t)| + k t^{-\varsigma} ||y - x||_\varsigma . \end{aligned} \tag{3.7}$$

We multiply both sides of Eq.(3.7) with t^ς *and take the sup of both sides to get*

$$||\Pi x - \Pi y||_\varsigma \le ||y - x||_\varsigma + ||\mathcal{I}_{0+}^{1-\mu(.,x(.))} x - \mathcal{I}_{0+}^{1-\mu(.,y(.))} y||_\varsigma + k ||y - x||_\varsigma .$$

Using Eq.(1.15) we obtain

$$||\Pi x - \Pi y||_\varsigma \le ||y - x||_\varsigma + \frac{4 F B \Gamma(\mu^*)^2 \Sigma^*}{\Gamma(2\mu^*)} \|y - x\|_\varsigma + k ||y - x||_\varsigma .$$

If we set $\varpi_4 = \left(1 + \dfrac{4 F B \Gamma(\mu^*)^2 \Sigma^*}{\Gamma(2\mu^*)} + k \right)$, *then we have*

$$||\Pi x - \Pi y||_\varsigma \le \varpi_4 ||y - x||_\varsigma .$$

By induction it is trivial to prove that

$$||\Pi^n x - \Pi^n y||_\varsigma \le \frac{\varpi_4^n}{n!} ||y - x||_\varsigma ,$$

where $\Pi^n = \Pi \circ \Pi \circ \ldots \circ \Pi$ *n times. Since* $\frac{\varpi_4^n}{n!}$ *is the general term of the convergent exponential series* e^{ϖ_4}, *it tends to zero as n tends to infinity, and so for n sufficiently large we have.*

$$\frac{\varpi_4^n}{n!} < 1.$$

lemma 1.4.1 asserts that the operator Π *has a UFP in* Θ. *This implies that*

$$\mathcal{I}_{0+}^{1-\mu(t,y(t))} y(t) = \int_0^t \mathcal{A}_3(\leftthreetimes, y(\leftthreetimes)) d\leftthreetimes , \tag{3.8}$$

with $y(0) = 0$ *Finally, from lemma 3.1.1, we get*

$$\mathcal{D}_{0+}^{\mu(t,y(t))} y(t) = \mathcal{A}_3(t, y(t)) \ \ with \ \ y(0) = 0. \tag{3.9}$$

This concludes our proof.

3.1.3 Existence Result in $L^p(\mathcal{D}_3, \mathbb{R})$

Theorem 3.1.2 *Under the assumptions (A1)-(A2) the (IVPFDENVO) has a US in the BS $L^p(\mathcal{D}_3, \mathbb{R})$.*

Proof 3.1.3 *We consider the set Θ as an element in $L^p(\mathcal{D}_3, \mathbb{R})$ such that $y(0) = 0$, and the operator*

$$\Pi : \Theta \to \Theta,$$

where

$$(\Pi y)(t) = y(t) + \mathcal{I}_{0^+}^{1-\mu(t,y(t))} y(t) - \int_0^t \mathcal{A}_3(\leftthreetimes, y(\leftthreetimes)) d\leftthreetimes.$$

Then, we have from (A2) that for $x, y : \mathcal{D}_3 \to \mathbb{R}$,

$$\left| \int_0^t \mathcal{A}_3(\leftthreetimes, x(\leftthreetimes)) - \mathcal{A}_3(\leftthreetimes, y(\leftthreetimes)) d\leftthreetimes \right| \leq k F^{1/p} ||y - x||_p.$$

and

$$\begin{aligned} |(\Pi x)(t) - (\Pi y)(t)|^p &\leq 2^p \left(|y(t) - x(t)|^p + |\mathcal{I}_{0^+}^{1-\mu(t,x(t))} x(t) - \mathcal{I}_{0^+}^{1-\mu(t,y(t))} y(t))|^p \right. \\ &\quad \left. + \left| \int_0^t \mathcal{A}_3(\leftthreetimes, x(\leftthreetimes)) - \mathcal{A}_3(\leftthreetimes, y(\leftthreetimes)) d\leftthreetimes \right|^p \right) \\ &\leq 2^p \left(|y(t) - x(t)|^p + |\mathcal{I}_{0^+}^{1-\mu(t,x(t))} x(t) - \mathcal{I}_{0^+}^{1-\mu(t,y(t))} y(t)|^p \right. \\ &\quad \left. + k^p F ||y - x||_p^p \right). \end{aligned} \tag{3.10}$$

Integrating Eq.(3.10) on $[0, F]$, we get

$$||\Pi x - \Pi y||_p^p \leq 2^p \left(||y - x||_p^p + ||\mathcal{I}_{0^+}^{1-\mu(.,x(.))} x - \mathcal{I}_{0^+}^{1-\mu(.,y(.))} y||_p^p + k^p F^2 ||y - x||_p^p \right).$$

Using Eq.(1.19) from lemma 1.2.10, we get

$$\begin{aligned} ||\Pi x - \Pi y||_p^p &\leq 2^p (||y - x||_p^p + \left[\frac{4 F B \Sigma^*}{1 - \mu^*} \right]^p ||y - x||_p^p + k^p F^2 ||y - x||_p^p) \\ &\leq 2^p \left(1 + \left[\frac{4 F B \Sigma^*}{1 - \mu^*} \right]^p + k^p F^2 \right) ||y - x||_p^p. \end{aligned}$$

If we denote $\varpi_5 = 2 \left(1 + \left[\frac{4 F B \Sigma^*}{1 - \mu^*} \right]^p + k^p F^2 \right)^{\frac{1}{p}}$, then the rest of the proof is similar to the final part of the proof of theorem 3.1.1. This concludes our proof.

3.2 On The Finite Delayed Fractional Differential Equation Via R-Liouville Derivative of Non-linear Variable-Order

[2] We will study the existence of solutions for the finite delayed problem (FDP for short)

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t,u(t))}u(t) = \mathcal{A}_4(t,u_t), & t \in \mathcal{D}_1^4 := (0,\digamma], \ \ 0 < \digamma < \infty, \quad (A) \\ \\ u(t) = \psi(t), & t \in \mathcal{D}_2^4 := [-r,0]. \quad (B) \end{cases} \tag{FDPFDENVO}$$

Where $\mathcal{D}_{0^+}^{\mu(t,u(t))}$ set forth the RLFDVO $\mu(t,u(t))$, $r > 0$, μ satisfies

$$0 < \mu_* \leq \mu(t,u(t)) \leq \mu^* < 1,$$

$\mathcal{A}_4 : \mathcal{D}_1^4 \times L^1(\mathcal{D}_2^4) \to \mathbb{R}$ is a generic function and $\psi \in L^1(\mathcal{D}_2^4)$ with $\psi(0) = 0$. For any function u defined on $[-r,\digamma]$ and any $t \in \mathcal{D}_1^4$, we denote by u_t the elements of $L^1([-r,0])$ defined by

$$u_t(\lambda) = u(t+\lambda), \ \ \lambda \in [-r,0].$$

Here $u_t(.)$ quantifies the history of the state from time $t-r$ up to the present time t. We denote $\mathcal{D}^4 = \mathcal{D}_1^4 \cup \mathcal{D}_2^4 = [-r,\digamma]$.

3.2.1 Achieved Existence Results

Definition 3.2.1 *We say that $u \in L^1(\mathcal{D}^4)$ is a solution for (FDPFDENVO) if and only if u verifies (FDPFDENVO(A)) and (FDPFDENVO(B)), at the same time.*

Prior to asserting and showing the existence results in the BS $L^1(\mathcal{D}^4)$, We shall assert and demonstrate the following technical lemma.

Lemma 3.2.1 *For any $u \in L^1(\mathcal{D}^4)$, equation (FDPFDENVO(A)) take the following equivalent formula*

$$\mathcal{I}_{0^+}^{1-\mu(t,u(t))}u(t) = \int_0^t \frac{(t-\lambda)^{-\mu(\lambda,u(\lambda))}}{\Gamma(1-\mu(\lambda,u(\lambda)))}u(\lambda)d\lambda = \int_0^t \mathcal{A}_4(\lambda,u_\lambda)d\lambda, t \in \mathcal{D}_1^4. \tag{3.11}$$

[2] M. S. Souid, **A. Hallouz**, G. Hatira, On The Finite Delayed Fractional Differential Equation Via R-Liouville Derivative of Non-linear Variable-Order.

Proof 3.2.1 *Let $u \in L^p(\mathcal{D}^4, \mathbb{R})$, then from definition 1.2.6, we can write (FDPFDENVO(A)) simply as*

$$\mathcal{D}_{0^+}^{\mu(t,u(t))}u(t) = \frac{d}{dt}\int_0^t \frac{(t-\curlywedge)^{-\mu(\curlywedge,u(\curlywedge))}}{\Gamma(1-\mu(\curlywedge,u(\curlywedge)))}u(\curlywedge)d\curlywedge = \mathcal{A}_4(t,u_t). \tag{3.12}$$

Now we can integrate (3.12) from $[0,t]$, to get

$$\int_0^t \frac{(t-\curlywedge)^{-\mu(\curlywedge,u(\curlywedge))}}{\Gamma(1-\mu(\curlywedge,u(\curlywedge)))}u(\curlywedge)d\curlywedge = c_0 + \int_0^t \mathcal{A}_4(\curlywedge,u_\curlywedge)d\curlywedge\,. \tag{3.13}$$

Inserting $t=0$in (3.13) gives $c_0=0$, which is Eq(3.11) Conversely, let us consider (3.11), by the consideration in lemma 1.2.10, we can take its derivative which shows that

$$\frac{d}{dt}\int_0^t \frac{(t-\curlywedge)^{-\mu(\curlywedge,u(\curlywedge))}}{\Gamma(1-\mu(\curlywedge,u(\curlywedge)))}u(\curlywedge)d\curlywedge = \mathcal{A}_4(t,u_t), \tag{3.14}$$

and again definition 1.2.6 gives (FDPFDENVO(A)) and conclude the proof.

To proceed, we outline assumptions essential for the analysis.

(A1) $\mu : \mathcal{D}_1^4 \times \mathbb{R} \to [\mu^*, \mu_*]$ is a CF.

(A2) $\mathcal{A}_4 : \mathcal{D}_1^4 \times L^1(\mathcal{D}_2^4) \to \mathbb{R}$ is a CF with respect to its first variable and such that:

$$|\mathcal{A}_4(t,\widetilde{\varphi}_1) - \mathcal{A}_4(t,\widetilde{\varphi}_2)| \leq \tau\,||\widetilde{\varphi}_1 - \widetilde{\varphi}_2||_{L^1\left(\mathcal{D}_2^4\right)},\ \forall\ \widetilde{\varphi}_1,\widetilde{\varphi}_2 \in L^1\left(\mathcal{D}_2^4\right),$$

$t \in \mathcal{D}_1^4$ and $\tau > 0$.

(A3) $\mathcal{A}_4 : \mathcal{D}_1^4 \times L^1(\mathcal{D}_2^4) \to \mathbb{R}$ is a CF and such that:

$$|\mathcal{A}_4(t,\widetilde{\varphi})| \leq |a(t)| + \tau\,||\widetilde{\varphi}||_{L^1\left(\mathcal{D}_2^4\right)},\ \forall\ \widetilde{\varphi} \in L^1\left(\mathcal{D}_2^4\right), \tag{3.15}$$

$t \in \mathcal{D}_1^4$ and $\tau > 0$, $a \in L^1(\mathcal{D}_1^4)$.

In this first existence and uniqueness result we are going to use lemma 1.4.1.

Theorem 3.2.1 *If we consider (A1)-(A2), then (FDPFDENVO) has a US in the BS* $L^1(\mathcal{D}^4)$.

Proof 3.2.2 *Let v, u elements in $L^1(\mathcal{D}^4)$, and the operator*

$$\widetilde{\mathcal{X}_\varepsilon} : L^1\left(\mathcal{D}^4\right) \to L^1\left(\mathcal{D}^4\right),$$

where

$$\left(\widetilde{\mathcal{X}_\varepsilon}u\right)(t) = \begin{cases} \psi(t), & t \in \mathcal{D}_2^4, \\ \varepsilon u(t) + \mathcal{I}_{0^+}^{1-\mu(t,u(t))}u(t) - \displaystyle\int_0^t \mathcal{A}_4(\curlywedge,u_\curlywedge)d\curlywedge, & t \in \mathcal{D}_1^4. \end{cases} \tag{3.16}$$

Where $0 < \varepsilon < 1$. We have from (A2) that ,

$$\begin{aligned} \left|\int_0^t \mathcal{A}_4(\lambda, v_\lambda) - \mathcal{A}_4(\lambda, u_\lambda) d\lambda\right| &\leq \tau \int_0^F ||v_\lambda - u_\lambda||_{L^1(\mathcal{D}_2^4)} d\lambda \\ &\leq F\tau ||v-u||_{L^1(\mathcal{D}^4)}. \end{aligned} \tag{3.17}$$

Using Eq.(3.17), we get

$$\begin{aligned} \left|\left(\widetilde{\mathcal{X}}_\varepsilon v\right)(t) - \left(\widetilde{\mathcal{X}}_\varepsilon u\right)(t)\right| &\leq \varepsilon|v(t)-u(t)| + |\mathcal{I}_{0^+}^{1-\mu(t,v(t))} v(t) - \mathcal{I}_{0^+}^{1-\mu(t,u(t))} u(t))| \\ &+ \left|\int_0^t \mathcal{A}_4(\lambda, v(\lambda)) - \mathcal{A}_4(\lambda, u(\lambda)) d\lambda\right| \\ &\leq \varepsilon|u(t)-v(t)| + \left|\mathcal{I}_{0^+}^{1-\mu(t,v(t))} v(t) - \mathcal{I}_{0^+}^{1-\mu(t,u(t))} u(t))\right| \\ &+ F\tau||v-u||_{L^1(\mathcal{D}^4)}. \end{aligned} \tag{3.18}$$

Integrating Eq.(3.18) on $\mathcal{D}^4$, we get

$$\begin{aligned} \left|\left|\widetilde{\mathcal{X}}_\varepsilon v - \widetilde{\mathcal{X}}_\varepsilon u\right|\right|_{L^1(\mathcal{D}^4)} &\leq \varepsilon||v-u||_{L^1(\mathcal{D}^4)} + \left|\left|\mathcal{I}_{0^+}^{1-\mu(.,v(.))} v - \mathcal{I}_{0^+}^{1-\mu(.,u(.))} u\right|\right|_{L^1(\mathcal{D}^4)} \\ &+ F^2\tau||v-u||_{L^1(\mathcal{D}^4)}. \end{aligned} \tag{3.19}$$

Using Eq.(1.19) from lemma 1.2.10, we get

$$\left|\left|\widetilde{\mathcal{X}}_\varepsilon v - \widetilde{\mathcal{X}}_\varepsilon u\right|\right|_{L^1(\mathcal{D}^4)} \leq \left(\varepsilon + \frac{4\left(F + r\right) B\Sigma^*}{1-\mu^*} + F^2\tau\right) ||v-u||_{L^1(\mathcal{D}^4)}. \tag{3.20}$$

Set $\varpi_6 = \varepsilon + \dfrac{4\left(F + r\right) B\Sigma^*}{1-\mu^*} + F^2\tau$. *By induction it is trivial to prove that*

$$\left|\left|\widetilde{\mathcal{X}}_\varepsilon^n v - \widetilde{\mathcal{X}}_\varepsilon^n u\right|\right|_{L^1(\mathcal{D}^4)} \leq \frac{\varpi_6^n}{n!}||v-u||_{L^1(\mathcal{D}^4)},$$

where $\widetilde{\mathcal{X}}_\varepsilon^n$ is the composition of the same function n times. We can remark that $\dfrac{\varpi_6^n}{n!}$ tends to zero as n tends to infinity, hence for n sufficiently large we have.

$$\frac{\varpi_6^n}{n!} < 1.$$

Now applying lemma 1.4.1, there exist a unique $u \in L^1\left(\mathcal{D}^4\right)$ such that

$$\left(\widetilde{\mathcal{X}}_\varepsilon u\right)(t) = u(t). \tag{3.21}$$

If we consider $\varepsilon = \dfrac{1}{2}$ and $t \in \mathcal{D}_1^4$ we deduce

$$u(0) = \psi(0) = 0.$$

Take the limit as $\varepsilon \to 1$ in the interval $\mathcal{D}_1^4$, we reach that

$$\mathcal{I}_{0^+}^{1-\mu(t,u(t))} u(t) = \int_0^t \mathcal{A}_4(\lambda, u_\lambda) d\lambda, t \in \mathcal{D}_1^4. \tag{3.22}$$

At this point Lemma 3.2.1 going to be crucial and give

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t,u(t))}u(t) = \mathcal{A}_4(t,u_t), & t \in \mathcal{D}_1^4, \\ u(t) = \psi(t) & t \in \mathcal{D}_2^4. \end{cases} \tag{3.23}$$

This concludes our proof.

In this second existence result we are going to use Theorem 1.4.2.

Theorem 3.2.2 *If we consider (A1)-(A3) and the condition*

$$\widetilde{K} = \frac{(\digamma + r)\digamma_x \Sigma^*}{1-\mu^*} + \digamma^2\tau < 1. \tag{3.24}$$

Then the (FDPFDENVO) has ALS in $L^1(\mathcal{D}^4)$.

Proof 3.2.3 *Let* v, u *elements in* $L^1(\mathcal{D}^4)$, $t \in \mathcal{D}^4$ *and consider the operator*

$$\widetilde{\mathcal{X}}_{\varepsilon_n} : L^1\left(\mathcal{D}^4\right) \to L^1\left(\mathcal{D}^4\right),$$

where

$$\left(\widetilde{\mathcal{X}}_{\varepsilon_n} u\right)(t) = \begin{cases} \psi(t), & t \in \mathcal{D}_2^4, \\ \varepsilon_n u(t) + \mathcal{I}_{0^+}^{1-\mu(t,u(t))}u(t) - \int_0^t \mathcal{A}_4(\lambda, u_\lambda)d\lambda, & t \in \mathcal{D}_1^4. \end{cases} \tag{3.25}$$

Where $\varepsilon_n = \dfrac{2b}{n+b}$ *and* $b > 0$ *a real number made to be fixed after in the proof.For* $t \in \mathcal{D}_2^4$

$$\left|\left(\widetilde{\mathcal{X}}_{\varepsilon_n} v\right)(t)\right| \leq |\psi(t)|. \tag{3.26}$$

From (A3) we have that

$$\begin{aligned} \left|\int_0^t \mathcal{A}_4(\lambda, v_\lambda)d\lambda\right| &\leq ||a||_{L^1\left(\mathcal{D}_1^4\right)} + \tau\int_0^{\digamma} ||v_\lambda||_{L^1\left(\mathcal{D}_2^4\right)}d\lambda \\ &\leq ||a||_{L^1\left(\mathcal{D}_1^4\right)} + \digamma\tau||v||_{L^1(\mathcal{D}^4)}. \end{aligned} \tag{3.27}$$

For $t \in \mathcal{D}_2^4$ *and by mean of Eq.(3.27), we get*

$$\begin{aligned} \left|\left(\widetilde{\mathcal{X}}_{\varepsilon_n} v\right)(t)\right| &\leq \varepsilon_n|v(t)| + \left|\mathcal{I}_{0^+}^{1-\mu(t,v(t))}v(t)\right| + \left|\int_0^t \mathcal{A}_4(\lambda, v(\lambda))d\lambda\right| \\ &\leq \varepsilon_n|v(t)| + \left|\mathcal{I}_{0^+}^{1-\mu(t,v(t))}v(t)\right| + ||a||_{L^1\left(\mathcal{D}_1^4\right)} + \digamma\tau||v||_{L^1(\mathcal{D}^4)}. \end{aligned} \tag{3.28}$$

Combining Eq.(3.28) and Eq.(3.26), we get

$$\begin{aligned} \left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n} v\right|\right|_{L^1(\mathcal{D}^4)} &\leq \varepsilon_n||v||_{L^1(\mathcal{D}^4)} + \left|\left|\mathcal{I}_{0^+}^{1-\mu(.,v(.))}v\right|\right|_{L^1(\mathcal{D}^4)} \\ &+\digamma||a||_{L^1\left(\mathcal{D}_1^4\right)} + \digamma^2\tau||v||_{L^1(\mathcal{D}^4)} + ||\psi||_{L^1\left(\mathcal{D}_2^4\right)}. \end{aligned} \tag{3.29}$$

Using Eq.(1.19) from Lemma 1.2.10, we get

$$\left\|\widetilde{\mathcal{X}}_{\varepsilon_n} v\right\|_{L^1(\mathcal{D}^4)} \leq \left(\varepsilon_n + \frac{(F+r)\, F_x \Sigma^*}{1-\mu^*} + F^2\tau\right) ||v||_{L^1(\mathcal{D}^4)} + F\, ||a||_{L^1\left(\mathcal{D}_1^4\right)} + ||\psi||_{L^1\left(\mathcal{D}_2^4\right)}. \tag{3.30}$$

There is a sufficiently large integer N_0 for which

$$\varepsilon_n < 1 - \widetilde{K} \;\; \forall n \geq N_0. \tag{3.31}$$

And the choice of N_0 is independent from the choice of $b > 0$ fixed in the beginning of the proof. Set

$$R = \frac{||\psi||_{L^1\left(\mathcal{D}_2^4\right)} + F\, ||a||_{L^1\left(\mathcal{D}_1^4\right)}}{1 - \left(\varepsilon_n + \widetilde{K}\right)}. \tag{3.32}$$

Consider

$$B_R = \left\{ v \in L^1\left(\mathcal{D}^4\right) \;\mid\; ||v||_{L^1(\mathcal{D}^4)} \leq R \right\}. \tag{3.33}$$

It is well known that B_R is a CCBNE.
We will prove that $\widetilde{\mathcal{X}}_{\varepsilon_n} : B_R \to B_R$ is CC in few steps.
***step1:** The fact that $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R) \subset B_R$ is bay construction.*
***step2:** $\widetilde{\mathcal{X}}_{\varepsilon_n}$ is CO, let $(v_k)_{k\geq 0} \subset B_R$, and $v \in B_R$ such that $v_k \xrightarrow[k\to+\infty]{} v$. We have to state and prove the following lemma*

***Lemma 3.2.2** We have the following convergence*

$$\left\|\mathcal{I}_{0^+}^{1-\mu(.,v_k(.))} v_n - \mathcal{I}_{0^+}^{1-\mu(.,v(.))} v\right\|_{L^1(\mathcal{D}^4)} \xrightarrow[k\to+\infty]{} 0. \tag{3.34}$$

$$\int_0^F |\mathcal{A}_4(\leftthreetimes, v_{\leftthreetimes,k}) - \mathcal{A}_4(\leftthreetimes, v_{\leftthreetimes})|\, d\leftthreetimes \xrightarrow[k\to+\infty]{} 0. \tag{3.35}$$

***Proof 3.2.4** For Eq.(3.34) using Eq.(1.19), we get*

$$\left\|\mathcal{I}_{0^+}^{1-\mu(.,v_k(.))} v_n - \mathcal{I}_{0^+}^{1-\mu(.,v(.))} v\right\|_{L^1(\mathcal{D}^4)} \leq \frac{4\,(F+r)\, B\Sigma^*}{1-\mu^*} \|v_k - v\|_{L^1(\mathcal{D}^4)} \xrightarrow[k\to+\infty]{} 0. \tag{3.36}$$

For Eq.(3.35), we have $|\mathcal{A}_4(\leftthreetimes, v_{k,\leftthreetimes}) - \mathcal{A}_4(\leftthreetimes, v_{\leftthreetimes})| \xrightarrow[k\to+\infty]{} 0$, and

$$|\mathcal{A}_4(\leftthreetimes, v_{\leftthreetimes,k}) - \mathcal{A}_4(\leftthreetimes, v_{\leftthreetimes})| \leq 2|a(t)| + c\tau ||v||_{L^1(\mathcal{D}^4)},$$

as a consequence of (A3) and this insures that the DCT is applied and this is exactly (3.35).

We have

$$\left\|\widetilde{\mathcal{X}}_{\varepsilon_n} v_k - \widetilde{\mathcal{X}}_{\varepsilon_n} v\right\|_{L^1(\mathcal{D}^4)} \begin{aligned}[t] &\leq \varepsilon_n ||v_k - v||_{L^1(\mathcal{D}^4)} \\ &+ \left\|\int_0^F |\mathcal{A}_4(\curlywedge, v_{\curlywedge,k}) - \mathcal{A}_4(\curlywedge, v_{\curlywedge})|\, d\curlywedge\right\|_{L^1(\mathcal{D}^4)} \\ &+ \left\|\mathcal{I}_{0^+}^{1-\mu(.,v_k(.))} v_k - \mathcal{I}_{0^+}^{1-\mu(.,v(.))} v\right\|_{L^1(\mathcal{D}^4)} \xrightarrow[k\to+\infty]{} 0. \end{aligned} \tag{3.37}$$

Which means that $\widetilde{\mathcal{X}}_{\varepsilon_n}$ is continuous.

step3: *$\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R)$ is uniformly bounded by construction, it remains to prove that*

$$\left\|\frac{1}{h}\int_t^{t+h} \left(\widetilde{\mathcal{X}}_{\varepsilon_n} v\right)(\curlywedge) d\curlywedge - \widetilde{\mathcal{X}}_{\varepsilon_n} v\right\|_{L^1(\mathcal{D}^4)} \xrightarrow[h\to 0]{} 0 \text{ Uniformly.} \tag{3.38}$$

We have

$$\left\|\frac{1}{h}\int_t^{t+h} \left(\widetilde{\mathcal{X}}_{\varepsilon_n} v\right)(\curlywedge) d\curlywedge - \widetilde{\mathcal{X}}_{\varepsilon_n} v\right\|_{L^1(\mathcal{D}^4)} \begin{aligned}[t] &\leq \varepsilon_n \int_0^F \left|\frac{1}{h}\int_t^{t+h} v(\curlywedge) - v(t) d\curlywedge\right| dt \\ &+ \int_{-r}^0 \left|\frac{1}{h}\int_t^{t+h} \psi(\curlywedge) - \psi(t) d\curlywedge\right| dt \\ &+ \int_0^F \left|\frac{1}{h}\int_t^{t+h} \mathcal{I}_{0^+}^{1-\mu(\curlywedge,v(\curlywedge))} v(\curlywedge) - \mathcal{I}_{0^+}^{1-\mu(t,v(t))} v(t) d\curlywedge\right| dt \\ &+ \int_0^F \left|\frac{1}{h}\int_t^{t+h} \int_0^{\curlywedge} \mathcal{A}_4(\rho, v_\rho) d\rho - \int_0^t \mathcal{A}_4(\sigma, v_\sigma) d\sigma d\curlywedge\right| dt. \end{aligned} \tag{3.39}$$

Since $v, \mathcal{I}_{0^+}^{1-\mu(\curlywedge,v(\curlywedge))} v, \int_0^{\curlywedge} \mathcal{A}_4(\rho, v_\rho) d\rho \in L^1\left(\mathcal{D}_1^4\right)$, and $\psi \in L^1\left(\mathcal{D}_1^4\right)$ and as a consequence of Theorem1.3.1 and another application of DCT we conclude that Eq.(3.38) is satisfied.
By Kolmogorov's theorem $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R)$ is RC, and hence by Theorem(1.4.2) $\widetilde{\mathcal{X}}_{\varepsilon_n}$ has at least one FP in B_R.

$$\left(\widetilde{\mathcal{X}}_{\varepsilon_n} u\right)(t) = u(t) \;\; \forall n \geq N_0.$$

At this point we can fix $b = N_0$, $n = 2N_0$ and $t \in \mathcal{D}_1^4$ to deduce

$$u(0) = \psi(0) = 0.$$

In the same way choose $n = N_0$ in the interval $\mathcal{D}_1^4$, we get

$$\mathcal{I}_{0^+}^{1-\mu(t,u(t))} u(t) = \int_0^t \mathcal{A}_4(\curlywedge, u_{\curlywedge}) d\curlywedge, t \in \mathcal{D}_1^4. \tag{3.40}$$

Using Lemma(3.2.1), then we can conclude that

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t,u(t))} u(t) = \mathcal{A}_4(t, u_t), & t \in \mathcal{D}_1^4, \\ u(t) = \psi(t) & t \in \mathcal{D}_2^4. \end{cases} \tag{3.41}$$

This concludes our proof.

3.2.2 An Approval Example

Example 1: Let us consider the following (FDPFDENVO)

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t,u(t))}u(t) = \dfrac{e^{ct}\|u_t\|_{L^1(\mathcal{D}^4)}}{3(e^{ct}+e^{-ct})(1+\|u_t\|_{L^1(\mathcal{D}^4)})}, & t \in \mathcal{D}_1^4 := (0,2], \quad (C) \\ \\ u(t) = \psi(t) & t \in \mathcal{D}_2^4 := [-r,0]. \quad (D) \end{cases} \tag{FDPFDENVO2}$$

with

$$\mu(t,u) = \frac{1}{4}t + \frac{3}{17(2+3u^4)}. \tag{3.42}$$

and

$$\mathcal{A}_4(t,u) = \frac{e^{ct}u}{3(e^{ct}+e^{-ct})(1+u)} \quad (t,u) \in \mathcal{D}_1^4 \times (0,+\infty), \quad c>0. \tag{3.43}$$

From equation (3.42) we can see that μ is a CF on $\mathcal{D}_1^4 \times \mathbb{R}$ and $0 < \mu(t,y) < 1$. And from Eq.(3.43) we can see that

$$\begin{aligned} |\mathcal{A}_4(t,u) - \mathcal{A}_4(t,v)| &\leq \frac{e^{ct}}{3(e^{ct}+e^{-ct})}\left| \frac{\|u_t\|_{L^1(\mathcal{D}^4)}}{\left(1+\|u_t\|_{L^1\left(\mathcal{D}_1^4\right)}\right)^2} - \frac{\|v_t\|_{L^1\left(\mathcal{D}_1^4\right)}}{\left(1+\|v_t\|_{L^1(\mathcal{D}^4)}\right)^2} \right| \\ &\leq \frac{e^{ct}}{3(e^{ct}+e^{-ct})} \frac{|\|u\|_{L^1(\mathcal{D}^4)} - \|v\|_{L^1(\mathcal{D}^4)}|}{\left(1+\|u_t\|_{L^1(\mathcal{D}^4)}\right)^2\left(1+\|v_t\|_{L^1(\mathcal{D}^4)}\right)^2} \\ &\leq \frac{e^{ct}}{3(e^{ct}+e^{-ct})}\|u-v\|_{L^1(\mathcal{D}^4)} \\ &\leq \tfrac{1}{3}\|u-v\|_{L^1(\mathcal{D}^4)}, \end{aligned} \tag{3.44}$$

It is easy to check that for the given choice of nonlinear functions μ and $\mathcal{A}_4$ assumptions (A1)-(A2) are satisfied. Therefore by theorem(3.2.1), the problem (FDPFDENVO2) has a US.

3.3 Border Value Problem For R-Liouville Differential Equations Of Nonlinear Variable Order

[3]

[3]M. S. Souid, **A. Hallouz**, Border Value Problem For R-Liouville Differential Equations Of Nonlinear Variable Order.

We will study the existence of solutions for the boundary value problem (BVP for short)

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t,\omega(t))}\omega(t) = \mathcal{A}_5(t,\omega(t)), \;\; t \in \mathcal{D}_5 := (0,D), \;\; 0 < D < \infty, & (A) \\ \\ \omega(0) = \omega(D) = 0, & (B) \end{cases} \tag{BVPFDENVO1}$$

where $\mathcal{D}_{0^+}^{\mu(t,\omega(t))}$ set forth RLFDVO, $\mu(t,\omega(t))$, $\mathcal{A}_5$ is a generic function, $1 < \mu_* \le \mu(t,\omega(t)) \le \mu^* < 2$.

3.3.1 Existence of solutions

Definition 3.3.1 *A function $y \in C_{2-\mu^*}(\overline{\mathcal{D}_5})$ or $y \in L^1(\overline{\mathcal{D}_5})$ is said to be a solution for (BVPFDENVO1) if and only if it verifies (BVPFDENVO1(A)) and (BVPFDENVO1(B)), simultaneously.*

In order to present our new existence results in the BS $C_\varrho(\overline{\mathcal{D}_5})$ and $L^1(\overline{\mathcal{D}_5})$, we will analyse an equivalent integral form of the BVPFDENVO1(A).

Lemma 3.3.1 *Let y be an element of $C_{2-\mu^*}(\overline{\mathcal{D}_5})$ or $L^1(\overline{\mathcal{D}_5})$. Then, equation (BVPFDENVO1(A)) is equivalent to*

$$\begin{aligned} \mathcal{I}_{a_1^+}^{2-\mu(t,\omega(t))}\omega(t) = \; & \frac{t}{D}\left(\mathcal{I}_{a_1^+}^{2-\mu(t,\omega(t))}\omega(D) - \int_0^D (D-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda\right) \\ & + \int_0^t (t-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda)\Big)\, d\lambda, t \in \overline{\mathcal{D}_5}. \end{aligned} \tag{3.45}$$

Proof 3.3.1 *Let $y \in C_{2-\mu^*}(\overline{\mathcal{D}_5})$ or $y \in L^1(\overline{\mathcal{D}_5})$. Then, equation (BVPFDENVO1(A)) can be represented as*

$$\mathcal{D}_{0^+}^{\mu(t,\omega(t))}\omega(t) = \left(\frac{d}{dt}\right)^2 \mathcal{I}_{a_1^+}^{2-\mu(t,\omega(t))}\omega(t) = \mathcal{A}_5(t,\omega(t)). \tag{3.46}$$

Integrating both sides of (BVPFDENVO1(A)) on $[0,t]$ we get

$$\int_0^t \frac{(t-\lambda)^{1-\mu(\lambda,\omega(\lambda))}}{\Gamma(2-\mu(\lambda,\omega(\lambda)))}\omega(\lambda)d\lambda = c_0 + c_1 t + \int_0^t (t-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda\,. \tag{3.47}$$

Evaluating (3.47) at 0 and D gives us

$$\begin{cases} c_0 &= 0, \\ c_1 &= \dfrac{1}{D}\left(\displaystyle\int_0^D \frac{(D-\lambda)^{1-\mu(\lambda,\omega(\lambda))}}{\Gamma(2-\mu(\lambda,\omega(\lambda)))}\omega(\lambda)d\lambda - \int_0^D (D-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda\right). \end{cases} \tag{3.48}$$

Inserting these in Eq.(3.47), then Eq.(3.45) yields. Conversely, differentiating twice both sides of (3.45) gives

$$\left(\frac{d}{dt}\right)^2 \mathcal{I}_{a_1^+}^{2-\mu(t,\omega(t))}\omega(t) = \mathcal{A}_5(t,\omega(t)), \tag{3.49}$$

from which we get (BVPFDENVO1(A)) and this concludes the proof.

To proceed, we outline assumptions essential for the analysis.

(A1) $\mu : \overline{\mathcal{D}_5} \times \mathbb{R} \to [\mu_*, \mu^*]$ is a CF.

(A2) $\mathcal{A}_5 : \overline{\mathcal{D}_5} \times \mathbb{R} \to \mathbb{R}$ is a CF with respect to its first variable and such that:

$$\left|\mathcal{A}_5(\leftthreetimes, \widehat{E}_1) - \mathcal{A}_5(\leftthreetimes, \widehat{E}_2)\right| \leq k|\widehat{E}_1 - \widehat{E}_2|, \ \forall \ \widehat{E}_1, \widehat{E}_2 \in \mathbb{R},$$

$\leftthreetimes \in \overline{\mathcal{D}_5}$ and $k > 0$.

(A3) $\mathcal{A}_5 : \overline{\mathcal{D}_5} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:
$|\mathcal{A}_5(t, \widehat{E})| \leq a(t) + b(t)|\widehat{E}|$, $\forall x \in \mathbb{R}$, $t \in \overline{\mathcal{D}_5}$. Where $a \in C_\varrho(\overline{\mathcal{D}_5})$, b is a CF on $\overline{\mathcal{D}_5}$ and both are positive.

3.3.2 Existence Result in $L^1(\overline{\mathcal{D}_5})$

Theorem 3.3.1 *Under the assumptions (A1)-(A2) the (BVPFDENVO1) has a US in the BS* $L^1(\overline{\mathcal{D}_5}, \mathbb{R})$.

Proof 3.3.2 *Consider the set* $\tilde{\Omega}$ *as the elements in* $L^1(\overline{\mathcal{D}_5})$ *such that* $\omega(0) = \omega(D) = 0$*, and the operator*

$$\widetilde{\mathcal{A}_5} : \tilde{\Omega} \to \tilde{\Omega},$$

where

$$\begin{aligned} \left(\widetilde{\mathcal{A}_5}\omega\right)(t) = \ & \omega(t) + \mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(t) - \int_0^t (t-\leftthreetimes)\mathcal{A}_5(\leftthreetimes, \omega(\leftthreetimes))d\leftthreetimes \\ & -\frac{t}{D}\left(\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(D) - \int_0^D (D-\leftthreetimes)\mathcal{A}_5(\leftthreetimes, \omega(\leftthreetimes))d\leftthreetimes\right). \end{aligned} \tag{3.50}$$

By mean of Hölder's inequality, we have from (A2) that for $x, y : \overline{\mathcal{D}_5} \to \mathbb{R}$,

$$\begin{aligned} \left|\int_0^t (t-\leftthreetimes)\left(\mathcal{A}_5(\leftthreetimes, \overline{z}(\leftthreetimes)) - \mathcal{A}_5(\leftthreetimes, \omega(\leftthreetimes))\right) d\leftthreetimes\right| & \leq k||\overline{z} - \omega||_1 \int_0^D (D-\leftthreetimes)d\leftthreetimes \\ & \leq k\frac{\mathcal{D}^2}{2}||\overline{z} - \omega||_1. \end{aligned} \tag{3.51}$$

In the same way we can get

$$\begin{aligned}\left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}(t))}\overline{z}(D) - \mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(D)\right| &\leq \frac{4B\nu^*}{D^{1-\mu^*}}\int_0^D (D-\lambda)^{1-\mu^*}|\overline{z}(\lambda)-\omega(\lambda)|d\lambda \\ &\leq \frac{4B\nu^*}{(2-\mu^*)D}||\overline{z}-\omega||_1.\end{aligned} \tag{3.52}$$

Then using Eq.(3.51) and Eq.(3.52) will give the following estimation

$$\begin{aligned}\left|\left(\widetilde{\mathcal{A}_5}\overline{z}\right)(t) - \left(\widetilde{\mathcal{A}_5}\omega\right)(t)\right| &\leq |\overline{z}(t)-\omega(t)| + \left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}(t))}\overline{z}(t) - \mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(t))\right| \\ &+ \left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}(t))}\overline{z}(D) - \mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(D)\right| \\ &+2\left|\int_0^D (D-\lambda)\mathcal{A}_5(\lambda,\overline{z}(\lambda)) - \mathcal{A}_5(\lambda,\omega(\lambda))d\lambda\right| \\ &\leq ||\overline{z}-\omega||_1 + \left\|\mathcal{I}_{0^+}^{1-\mu(.,\overline{z}(.))}\overline{z} - \mathcal{I}_{0^+}^{1-\mu(.,\omega(.))}\omega\right\|_1 \\ &+\frac{4B\nu^*}{(2-\mu^*)D}||\overline{z}-\omega||_1 + kD^2||\overline{z}-\omega||_1.\end{aligned} \tag{3.53}$$

Using Eq.(1.19) from lemma 1.2.10, we get

$$\left\|\widetilde{\mathcal{A}_5}\overline{z} - \widetilde{\mathcal{A}_5}\omega\right\|_1 \leq \left(1 + \widetilde{Q_4} + \frac{4B\nu^*}{(2-\mu^*)D} + kD^2\right)||\overline{z}-\omega||_1. \tag{3.54}$$

If we denote $\varpi_7 = 1 + \widetilde{Q_4} + \dfrac{4B\nu^*}{(2-\mu^*)D} + kD^2$ *then we have*

$$\left\|\widetilde{\mathcal{A}_5}\overline{z} - \widetilde{\mathcal{A}_5}\omega\right\|_1 \leq \varpi_7||\overline{z}-\omega||_1.$$

By induction it is trivial to prove that

$$\left\|\widetilde{\mathcal{A}_5}^n\overline{z} - \widetilde{\mathcal{A}_5}^n\omega\right\|_1 \leq \frac{\varpi_7^n}{n!}||\overline{z}-\omega||_1,$$

where $\widetilde{\mathcal{A}_5}^n = \widetilde{\mathcal{A}_5}\circ\widetilde{\mathcal{A}_2}\circ ...\circ\widetilde{\mathcal{A}_2}$ *n times. Since* $\dfrac{\varpi_7^n}{n!}$ *is the general term of the convergent exponential series* e^{ϖ_7}*, it tends to zero as n tends to infinity, and so for n sufficiently large we have.*

$$\frac{\varpi_7^n}{n!} < 1.$$

lemma 1.4.1 asserts that the operator $\widetilde{\mathcal{A}_2}$ *has a FP point in* $\widetilde{\Omega}$*. This implies that*

$$\begin{aligned}\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(t) = &\frac{t}{D}(\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(D) - \int_0^D (D-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda) \\ &+\int_0^t (t-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda, t\in\overline{\mathcal{D}_5},\end{aligned} \tag{3.55}$$

with $\omega(0)=\omega(D)=0$*. Finally, from lemma 3.3.1, we get*

$$\mathcal{D}_{0^+}^{\mu(t,\omega(t))}\omega(t) = \mathcal{A}_5(t,\omega(t)) \ \textit{with}\ \ \omega(0)=\omega(D)=0. \tag{3.56}$$

This concludes our proof.

3.3.3 Existence Result in $C_{\varrho}(\overline{\mathcal{D}_5})$

The first obtained result in this subsection is based on Theorem (1.4.2).

Theorem 3.3.2 *Assume that (A1)-(A3) are satisfied and*

$$\widetilde{K} = \widetilde{Q_1} + \frac{2D^{\mu^*}\Gamma(\mu^*-1)}{\Gamma(\mu^*+1)}\|b\|_\infty + \frac{\Gamma(2-\mu^*)\Gamma(\mu^*-1)F_\omega \nu^*}{D^{1-\mu^*}} < 1. \tag{3.57}$$

Then the (BVPFDENVO1) has at least a solution in $C_{2-\mu_*}(\overline{\mathcal{D}_5})$.

Proof 3.3.3 *Let $d > 0$ made to be fixed later in the proof, $\varrho = 2-\mu^*$, $\upsilon_n = \dfrac{2d}{n+d}$ and and the operator*

$$\widetilde{\mathcal{A}}_{5n} : C_{2-\mu^*}(\overline{\mathcal{D}_5}) \to C_{2-\mu^*}(\overline{\mathcal{D}_5}), \tag{3.58}$$

where

$$\begin{aligned}(\widetilde{\mathcal{A}}_{5n}\omega)(t) = \quad & \upsilon_n\omega(t) + \mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(t) - \int_0^t (t-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda \\ & -\frac{t}{D}(\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(D) - \int_0^D (D-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda).\end{aligned} \tag{3.59}$$

For all $t \in \overline{\mathcal{D}_5}$, by mean of (A3) and Eq.(1.14) the operator $\widetilde{\mathcal{A}}_{5n}$ is well defined, and:

$$\begin{aligned}\left|(\widetilde{\mathcal{A}}_{5n}\omega)(t)\right| \leq \quad & \upsilon_n|\omega(t)| + \left|\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(t)\right| + 2\left|\int_0^D (D-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda\right| \\ & + \left|\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\omega(D)\right| \\ & \leq \upsilon_n t^{-\varrho}\|\omega\|_\varrho + t^{-\varrho}\|\mathcal{I}_{0^+}^{2-\mu(.,\omega(.))}\omega\|_\varrho + 2\int_0^D (D-\lambda)(a(\lambda)+b(\lambda)|\omega(\lambda)|)d\lambda \\ & + \frac{F_\omega \nu^*}{\mathcal{D}^{1-\mu^*}}\|\omega\|_\varrho \int_0^D (D-\lambda)^{1-\mu^*}\lambda^{-\varrho}\, d\lambda \\ & \leq \upsilon_n t^{-\varrho}\|\omega\|_\varrho + t^{-\varrho}\widetilde{Q_1}\|\omega\|_\varrho + t^{-\varrho}\frac{2D^{\mu^*}\Gamma(\mu^*-1)}{\Gamma(\mu^*+1)}\|a\|_\varrho \\ & + t^{-\varrho}\frac{2D^{\mu^*}\Gamma(\mu^*-1)}{\Gamma(\mu^*+1)}\|b\|_\infty\|\omega\|_\varrho \\ & + t^{-\varrho}\frac{\Gamma(2-\mu^*)\Gamma(\mu_*-1)F_\omega \nu^*}{D^{1-\mu^*}}\|\omega\|_\varrho.\end{aligned} \tag{3.60}$$

We multiply both sides of Eq.(3.60) with t^ϱ and take the sup of both sides to get

$$\begin{aligned}\left\|\widetilde{\mathcal{A}}_{5n}\omega\right\|_\varrho \leq \quad & \left(\upsilon_n + \widetilde{K}\right)\|\omega\|_\varrho \\ & + \frac{2D^{\mu^*}\Gamma(\mu^*-1)}{\Gamma(\mu^*+1)}\|a\|_\varrho.\end{aligned} \tag{3.61}$$

There is a sufficiently large integer N_0 for wich

$$\upsilon_n < 1 - \widetilde{K} \;\; \forall n \geq N_0. \tag{3.62}$$

And the choice of N_0 is independent from the choice of $d > 0$ fixed in the beginning of the proof Set

$$R = \frac{\frac{2D^{\mu^*}\Gamma(\mu^*-1)}{\Gamma(\mu^*+1)}\|a\|_\varrho}{1-\left(\upsilon_n+\widetilde{K}\right)}. \tag{3.63}$$

Consider

$$B_R = \left\{\overline{z} \in C_{2-\mu^*}(\overline{\mathcal{D}_5}) \;\mid\; ||\overline{z}||_\varrho \leq R\right\}. \tag{3.64}$$

It is well known that B_R is a CCBNE.

We will prove that $\widetilde{\mathcal{A}}_{5n} : B_R \to B_R$ is CC in few steps.

step1: *The fact that $\widetilde{\mathcal{A}}_{5n}(B_R) \subset B_R$ is bay construction.*

step2: *$\widetilde{\mathcal{A}}_{5n}$ is CO, let $(\overline{z}_k)_{n\geq 0} \subset B_R$, and $\overline{z} \in B_R$ such that $\overline{z}_k \xrightarrow[k\to+\infty]{} \overline{z}$,*

Lemma 3.3.2 *We have the following convergence*

$$\left|\left|\mathcal{I}_{0^+}^{2-\mu(.,\overline{z}_k(.))}\overline{z}_k - \mathcal{I}_{0^+}^{2-\mu(.,\overline{z}(.))}\overline{z}\right|\right|_\varrho \xrightarrow[k\to+\infty]{} 0. \tag{3.65}$$

$$\left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}_k(t))}\overline{z}_k(D) - \mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\overline{z}(D)\right| \xrightarrow[k\to+\infty]{} 0. \tag{3.66}$$

$$\int_0^D (D-\curlywedge)\left|\mathcal{A}_5(\curlywedge,\overline{z}_k(\curlywedge)) - \mathcal{A}_5(\curlywedge,\overline{z}(\curlywedge))\right| d\curlywedge \xrightarrow[k\to+\infty]{} 0. \tag{3.67}$$

Proof 3.3.4 *For Eq.(3.65) and using Eq.(1.15), we get*

$$\left|\left|\mathcal{I}_{0^+}^{2-\mu(.,\overline{z}_k(.))}\overline{z}_k - \mathcal{I}_{0^+}^{2-\mu(.,\overline{z}(.))}\overline{z}\right|\right|_\varrho \leq \widetilde{Q_2}\|\overline{z}_k - \overline{z}\|_\varrho \xrightarrow[k\to+\infty]{} 0. \tag{3.68}$$

For Eq.(3.66) and using Eq.(1.15), we get

$$\begin{aligned}\left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}_k(t))}\overline{z}_k(D) - \mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\overline{z}(D)\right| &\leq \frac{4B\nu^*}{D^{1-\mu^*}}\int_0^t (t-\curlywedge)^{1-\mu^*}|\overline{z}_k(\curlywedge)-\overline{z}(\curlywedge)|d\curlywedge \\ &\leq \frac{4B\Gamma(\mu^*-1)\Gamma(2-\mu^*)\nu^*}{D^{1-\mu^*}}\|\overline{z}_k-\overline{z}\|_\varrho \xrightarrow[k\to+\infty]{} 0.\end{aligned} \tag{3.69}$$

For Eq.(3.67) we have $(D-\curlywedge)\left|\mathcal{A}_5(\curlywedge,\overline{z}_k(\curlywedge)) - \mathcal{A}_5(\curlywedge,\overline{z}(\curlywedge))\right| \xrightarrow[k\to+\infty]{} 0$ and

$$(D-\curlywedge)\left|\mathcal{A}_5(\curlywedge,\overline{z}_k(\curlywedge)) - \mathcal{A}_5(\curlywedge,\overline{z}(\curlywedge))\right| \leq D(2a(t)+cb(t)||\overline{z}||_\varrho).$$

as a consequence of (A3) and these insure that the DCT is applied and this is exactly (3.67)

We have

$$\begin{aligned}\left|(\widetilde{\mathcal{A}}_{5n}\overline{z}_k)(t)-(\widetilde{\mathcal{A}}_{5n}\overline{z})(t)\right| = & \Bigg|\upsilon_n(\overline{z}_k(t)-\overline{z}(t))+\frac{t}{D}(\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}_k(t))}\overline{z}_k(D)-\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\overline{z}(D)\\ & +\int_0^D(D-\lambda)(\mathcal{A}_5(\lambda,\overline{z}_k(\lambda))-\mathcal{A}_5(\lambda,\overline{z}(\lambda)))d\lambda)\\ & +\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}_k(t))}\overline{z}_k(t)-\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}(t))}\overline{z}(t)\\ & +\int_0^t(t-\lambda)(\mathcal{A}_5(\lambda,\overline{z}_k(\lambda))-\mathcal{A}_5(\lambda,\overline{z}(\lambda)))d\lambda\Bigg|\\ & \leq t^{-\varrho}\upsilon_n||\overline{z}_k-\overline{z}||_\varrho\\ & +2\int_0^D(D-\lambda)\left|\mathcal{A}_5(\lambda,\overline{z}_k(\lambda))-\mathcal{A}_5(\lambda,\overline{z}(\lambda))\right|d\lambda\\ & +t^{-\varrho}\left|\left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}_k(t))}\overline{z}_k-\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}(t))}\overline{z}\right|\right|_\varrho\\ & +t^{-\varrho}D^\varrho\left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}_k(t))}\overline{z}_k(D)-\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\overline{z}(D)\right|.\end{aligned} \tag{3.70}$$

We multiply both sides of Eq.(3.70) by t^ϱ, take the sup of both sides and then use Lemma(3.3.2) we get

$$\begin{aligned}\left|\left|\widetilde{\mathcal{A}}_{5n}\overline{z}_k-\widetilde{\mathcal{A}}_{5n}\overline{z}\right|\right|_\varrho & \leq \upsilon_n\,||\overline{z}_k-\overline{z}||_\varrho+2\int_0^D(D-\lambda)\left|\mathcal{A}_5(\lambda,\overline{z}_k(\lambda))-\mathcal{A}_5(\lambda,\overline{z}(\lambda))\right|d\lambda\\ & +\left|\left|\mathcal{I}_{0^+}^{2-\mu(.,\overline{z}_k(.))}\overline{z}_k-\mathcal{I}_{0^+}^{2-\mu(.,\overline{z}(.))}\overline{z}\right|\right|_\varrho\\ & +D^\varrho\left|\mathcal{I}_{0^+}^{2-\mu(t,\overline{z}_k(t))}\overline{z}_k(D)-\mathcal{I}_{0^+}^{2-\mu(t,\omega(t))}\overline{z}(D)\right|\xrightarrow[k\to+\infty]{}0.\end{aligned} \tag{3.71}$$

Which means that $\widetilde{\mathcal{A}}_{5n}$ is CO.

step3: *$\widetilde{\mathcal{A}}_{2n}(B_R)$ is UB by construction, it remains to prove that $\widetilde{\mathcal{A}}_{5n}(B_R)$ is EC.*
Let $\overline{t},\underline{t}\in\overline{\mathcal{D}_5}$, without loss of generality we can assume $\overline{t}<\underline{t}$, and $x\in B_r$

$$\begin{aligned}\left|\overline{t}^\varrho(\widetilde{\mathcal{A}}_{5n}\overline{z})(\overline{t})-\underline{t}^\varrho(\widetilde{\mathcal{A}}_{5n}\overline{z})(\underline{t})\right| = & \Bigg|\upsilon_n(\overline{t}^\varrho\overline{z}(\overline{t})-\underline{t}^\varrho\overline{z}(\underline{t}))+\frac{\overline{t}^{\varrho+1}-\underline{t}^{\varrho+1}}{D}\Bigg(\int_0^D\frac{(D-\lambda)^{1-\mu(\lambda,\overline{z}(\lambda))}}{\Gamma(2-\mu(\lambda,\overline{z}(\lambda)))}\overline{z}(\lambda)d\lambda\\ & -\int_0^D(D-\lambda)\mathcal{A}_5(\lambda,\overline{z}(\lambda))d\lambda\Bigg)\\ & +\overline{t}^\varrho\int_0^{\overline{t}}\frac{(\overline{t}-\lambda)^{1-\mu(\lambda,\overline{z}(\lambda))}}{\Gamma(2-\mu(\lambda,\overline{z}(\lambda)))}\overline{z}(\lambda)d\lambda-\underline{t}^\varrho\int_0^{\underline{t}}\frac{(\underline{t}-\lambda)^{1-\mu(\lambda,\overline{z}(\lambda))}}{\Gamma(2-\mu(\lambda,\overline{z}(\lambda)))}\overline{z}(\lambda)d\lambda\\ & +\overline{t}^\varrho\int_0^{\overline{t}}(\overline{t}-\lambda)\mathcal{A}_5(\lambda,\overline{z}(\lambda))d\lambda-\underline{t}^\varrho\int_0^{\underline{t}}(\underline{t}-\lambda)\mathcal{A}_5(\lambda,\overline{z}(\lambda))d\lambda\Bigg|\\ & \leq\upsilon_n\left|\overline{t}^\varrho\overline{z}(\overline{t})-\underline{t}^\varrho\overline{z}(\underline{t})\right|+\left(\frac{\underline{t}^{\varrho+1}-\overline{t}^{\varrho+1}}{D}\right)C^*\\ & +F_{\overline{z}}||\overline{z}||_\varrho\nu^*\int_0^{\overline{t}}\left(\left(\frac{\overline{t}-\lambda}{D}\right)^{1-\mu(\lambda,\overline{z}(\lambda))}-\left(\frac{\underline{t}-\lambda}{D}\right)^{1-\mu(\lambda,\overline{z}(\lambda))}\right)d\lambda\end{aligned}$$

$$
\begin{aligned}
&+F_{\overline{z}}||\overline{z}||_{\varrho}\nu^* D\left(\int_{\overline{t}}^{\underline{t}}(\underline{t}-\lambda)^{1-\mu^*}\lambda^{-\varrho}\,d\lambda\right)\\
&+\underline{t}\int_{\overline{t}}^{\underline{t}}|\mathcal{A}_5(\lambda,\overline{z}(\lambda))|d\lambda+\int_{\overline{t}}^{\underline{t}}\lambda|\mathcal{A}_5(\lambda,\overline{z}(\lambda))|d\lambda\\
&\leq \upsilon_n\left|\overline{t}^{\varrho}\overline{z}(\overline{t})-\underline{t}^{\varrho}\overline{z}(\underline{t})\right|+\left(\frac{\underline{t}^{\varrho+1}-\overline{t}^{\varrho+1}}{D}\right)C^*\\
&+\frac{F_{\overline{z}}||\overline{z}||_{\varrho}\nu^*}{\mathcal{D}^{1-\mu^*}}\int_0^{\overline{t}}\left((\overline{t}-\lambda)^{1-\mu^*}-(\underline{t}-\lambda)^{1-\mu^*}\right)d\lambda\\
&+\frac{F_{\overline{z}}||\overline{z}||_{\varrho}\nu^* D}{(2-\mu^*)\overline{t}^{\varrho}}(\underline{t}-\overline{t})^{2-\mu^*}\\
&+2D\left(||a||_{\varrho}+||b||_{\infty}||\overline{z}||_{\varrho}\right)\left(\underline{t}^{\varrho+1}-\overline{t}^{\varrho+1}\right)\\
&\leq \upsilon_n|\overline{t}^{\varrho}\overline{z}(\overline{t})-\underline{t}^{\varrho}\overline{z}(\underline{t})|+\left(\frac{\underline{t}^{\varrho+1}-\overline{t}^{\varrho+1}}{D}\right)C^*\\
&+\frac{F_{\overline{z}}||\overline{z}||_{\varrho}\nu^*}{(2-\mu^*)D^{1-\mu_*}}\left(\overline{t}^{2-\mu^*}-\underline{t}^{2-\mu^*}+(\underline{t}-\overline{t})^{2-\mu^*}\right)\\
&+\frac{F_{\overline{z}}||\overline{z}||_{\varrho}\nu^* D}{(2-\mu^*)\overline{t}^{\varrho}}(\underline{t}-\overline{t})^{2-\mu^*}\\
&+2D\left(||a||_{\varrho}+||b||_{\infty}||\overline{z}||_{\varrho}\right)\left(\underline{t}^{\varrho+1}-\overline{t}^{\varrho+1}\right)\\
&\xrightarrow[\overline{t}\to\underline{t}]{} 0.
\end{aligned}
\tag{3.72}
$$

Where $C^*=\dfrac{F_{\overline{z}}\Gamma(\mu^*-1)\Gamma(2-\mu^*)\nu^*||\overline{z}||_{\varrho}}{D^{1-\mu^*}}+(||a||_{\varrho}+||b||_{\infty}||\overline{z}||_{\varrho})\dfrac{2\Gamma(\mu^*-1)D^{\mu^*}}{\Gamma(1+\mu^*)}$

By AAT $\widetilde{\mathcal{A}}_{5n}(B_R)$ *is RC, and hence by theorem(1.4.2)* $\widetilde{\mathcal{A}}_{5n}$ *has at least one FP in* B_R.

$$(\widetilde{\mathcal{A}}_{5n}\omega)(t)=\omega(t)\ \ \forall n\geq N_0.$$

At this point we can fixe $d=N_0$ *and take* $n=3N_0$ *we deduce*

$$
\begin{cases}
\omega(0) & =0,\\
\omega(D) & =0.
\end{cases}
\tag{3.73}
$$

Take $n=N_0$ *and this implies that*

$$
\begin{aligned}
\mathcal{I}_{a_1^+}^{2-\mu(t,\omega(t))}\omega(t)= \quad & \frac{t}{D}\left(\mathcal{I}_{a_1^+}^{2-\mu(t,\omega(t))}\omega(D)-\int_0^D(D-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda\right)\\
&+\int_0^t(t-\lambda)\mathcal{A}_5(\lambda,\omega(\lambda))d\lambda,\, t\in\overline{\mathcal{D}_5}.
\end{aligned}
\tag{3.74}
$$

Using Lemma(3.3.1)

$$\mathcal{D}_{0^+}^{\mu(t,\omega(t))}\omega(t)=\mathcal{A}_5(t,\omega(t))\ \ with\ \ \omega(0)=\omega(D)=0. \tag{3.75}$$

This concludes our proof

3.4 An Approval Example

Example 1: Let us consider the following fractional Border value problem

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t,\omega(t))}\omega(t) = \mathcal{A}_5(t,\omega(t)), \ \ t \in \widehat{\Omega} := (0,1) & (1) \\ \\ \omega(0) = \omega(1) = 0, & (2) \end{cases} \tag{BVPFDENVO2}$$

with

$$\mu(t,y) = \frac{1}{4}t + \frac{3}{17(2+3y^4)} + 1. \tag{3.76}$$

and

$$\mathcal{A}_5(t,y) = \frac{5e^{-t} \mid y \mid}{(4+7e^{2t})(1+ \mid y \mid)^2}. \tag{3.77}$$

From equation (3.76) we can see that μ is a CF on $\overline{\mathcal{D}_5} \times \mathbb{R}$ and $1 < \mu(t,y) < 2$. And from Eq.(3.77) we can see that

$$\begin{aligned} |\mathcal{A}_5(t,x) - \mathcal{A}_5(t,y)| &\leq \frac{5e^{-t}}{(4+7e^{2t})}\left|\frac{|x|}{(1+\mid x \mid)^2} - \frac{|y|}{(1+\mid y \mid)^2}\right| \\ &\leq \frac{5e^{-t}|x-y|}{(4+7e^{2t})(1+\mid x \mid)^2(1+\mid y \mid)^2} \\ &\leq \frac{5e^{-t}}{(4+7e^{2t})}|x-y| \\ &\leq \frac{5}{13}|x-y|. \end{aligned} \tag{3.78}$$

It is easy to check that for the given choice of nonlinear functions μ and $\mathcal{A}_5$ assumptions (A1)-(A2)are satisfied. Therefore by theorem(3.3.1), the problem (BVPFDENVO2) has a US.

Chapter 4

On The Finite Delayed Fractional Differential Equation Via The Weighted R-Liouville Derivative of Variable-Order

We will study the existence of solutions for the finite delayed problem (FDP for short)

$$\begin{cases} {}_{0^+}\mathcal{D}_w^{\mu(t)}u(t) = \mathcal{A}_6(t, u_t), & t \in \mathcal{D}_1^6 := (0, \digamma], \ \ 0 < \digamma < \infty, \quad (A) \\ \\ u(t) = \zeta(t), & t \in \mathcal{D}_2^6 := [-r, 0]. \quad (B) \end{cases} \tag{FDPWFDENVO}$$

Where ${}_{0^+}\mathcal{D}_w^{\mu(t)}$ set forth the WRLFDVO$\mu(t)$, $r > 0$ and μ satisfies $0 < \mu_* \le \mu(t) \le \mu^* < 1$, $\mathcal{A}_6 : \mathcal{D}_1^6 \times L_w^1(\mathcal{D}_2^6) \to \mathbb{R}$ is a generic function and $\zeta \in L_w^1(\mathcal{D}_2^6)$ with $\zeta(0) = 0$. For any function u defined on $[-r, \digamma]$ and any $t \in \mathcal{D}_1^6$, we denote by u_t the elements of $L_w^1([-r, 0])$ defined by

$$u_t(\leftthreetimes) = u(t + \leftthreetimes), \ \ \leftthreetimes \in [-r, 0].$$

Here $u_t(.)$ quantifies the history of the state from time $t - r$ up to the present time t. We denote $\mathcal{D}_6 = \mathcal{D}_1^6 \cup \mathcal{D}_2^6 = [-r, \digamma]$.

4.1 Achieved Existence Results

Definition 4.1.1 *We say that $u \in L_w^1(\mathcal{D}_6)$ is a solution for (FDPWFDENVO) if and only if u verifies (FDPWFDENVO(A)) and (FDPWFDENVO(B)), at the same time.*

Prior to asserting and showing the existence results in the BS $L^1_w(\mathcal{D}_6)$, We shall assert and demonstrate the following technical lemma.

Lemma 4.1.1 *For any $u \in L^1_w(\mathcal{D}_6)$, equation (FDPWFDENVO(A)) take the following Equivalent formula*

$$
{}_{0^+}\mathcal{I}_w^{1-\mu(t)}u(t) = \frac{1}{w(t)\Gamma(1-\mu(t))}\int_0^t (g(t)-g(\curlywedge))^{-\mu(\curlywedge)}w(\curlywedge)u(\curlywedge)g'(\curlywedge)d\curlywedge = \int_0^t \mathcal{A}_6(\curlywedge, u_\curlywedge)d\curlywedge, t \in \mathcal{D}_1^6. \tag{4.1}
$$

Proof 4.1.1 *Let $u \in L^1_w(\mathcal{D}_6)$, then from definition(1.3.2) we can write (FDPWFDENVO(A)) simply as*

$$
\mathcal{D}_{0^+}^{\mu(t)}u(t) = \frac{d}{dt}\left(\frac{1}{w(t)\Gamma(1-\mu(t))}\int_0^t (g(t)-g(\curlywedge))^{-\mu(\curlywedge)}w(\curlywedge)u(\curlywedge)g'(\curlywedge)d\curlywedge\right) = \mathcal{A}_6(t, u_t) \quad t \in \mathcal{D}_1^6. \tag{4.2}
$$

Now we can integrate (4.2) from $[0, t]$, to get

$$
\frac{1}{w(t)\Gamma(1-\mu(t))}\int_0^t (g(t)-g(\curlywedge))^{-\mu(\curlywedge)}w(\curlywedge)u(\curlywedge)g'(\curlywedge)d\curlywedge = c_0 + \int_0^t \mathcal{A}_6(\curlywedge, u_\curlywedge)d\curlywedge\,. \tag{4.3}
$$

Inserting $t = 0$in (4.3) gives $c_0 = 0$, which is Eq(4.1). Conversely, let us consider (4.1), by the consideration in lemma(1.3.2), we can take its derivative which shows that

$$
\frac{d}{dt}\left(\frac{1}{w(t)\Gamma(1-\mu(t))}\int_0^t (g(t)-g(\curlywedge))^{-\mu(\curlywedge)}w(\curlywedge)u(\curlywedge)g'(\curlywedge)d\curlywedge\right) = \mathcal{A}_6(t, u_t), \tag{4.4}
$$

and again definition(1.3.2) gives (FDPWFDENVO(A)) and conclude the proof.

To proceed, we outline assumptions essential for the analysis.

(A1) $\mu : \mathcal{D}_1^6 \to (0, \mu_*]$ is a CF, w is a CF and $w(x) > 0$.

(A2) $\mathcal{A}_6 : \mathcal{D}_1^6 \times L^1_w(\mathcal{D}_2^6) \to \mathbb{R}$ is a CF with respect to its first variable and such that:

$$
|\mathcal{A}_6(t, \widetilde{\varphi}_1) - \mathcal{A}_6(t, \widetilde{\varphi}_2)| \le \tau \,||\widetilde{\varphi}_1 - \widetilde{\varphi}_2||_{1,w}\,,\ \forall\ \widetilde{\varphi}_1, \widetilde{\varphi}_2 \in L^1_w\left(\mathcal{D}_2^6\right),
$$

$t \in \mathcal{D}_1^6$ and $\tau > 0$.

(A3) $\mathcal{A}_6 : \mathcal{D}_1^6 \times L^1_w(\mathcal{D}_2^6) \to \mathbb{R}$ is a CF and such that:

$$
|\mathcal{A}_6(t, \widetilde{\varphi})| \le |a(t)| + \tau \,||\widetilde{\varphi}||_{1,w}\,,\ \forall\ \widetilde{\varphi} \in L^1_w\left(\mathcal{D}_2^6\right), \tag{4.5}
$$

$t \in \mathcal{D}_1^6$ and $\tau > 0$, $a \in L^1(\mathcal{D}_1^6)$.

Remark 4.1.1 *Let the hypothesis on g and w holds as in (A1), for $-r \leq \leftthreetimes \leq 0$, then we get*

$$\begin{aligned} ||v_{\leftthreetimes}||_{1,w,A_1} &= \int_0^{F} |w(x)v(x+\leftthreetimes)|g'(x)dx \\ &= \int_{\leftthreetimes}^{F+\leftthreetimes} |w(t-\leftthreetimes)v(t)|g'(t-\leftthreetimes)dt \quad t = x+\leftthreetimes \\ &\leq C_*||v||_{1,w,A}, \end{aligned} \tag{4.6}$$

where $C_* = \max\left(\frac{g'(t-\leftthreetimes)}{g'(t)}\frac{w(t-\leftthreetimes)}{w(t)}\right)$.

Example 4.1.1 *As an example of $L^1_w(\mathcal{D}^6_2)$ we can take $w(x) = e^{-Kx}$ and $g(x) = e^{Kx}$ where K is a real number with $K > 0$.*

In this first existence and uniqueness result we are going to use lemma (1.4.1).

Theorem 4.1.1 *If we consider (A1)-(A2), then (FDPWFDENVO) has a US in the BS $L^1_w(\mathcal{D}_6)$.*

Proof 4.1.2 *Let v, u elements in $L^1_w(\mathcal{D}_6)$, $t \in \mathcal{D}_6$ and consider the operator*

$$\widetilde{\mathcal{X}}_{\varepsilon_n} : L^1_w(\mathcal{D}_6) \to L^1_w(\mathcal{D}_6),$$

where

$$\left(\widetilde{\mathcal{X}}_{\varepsilon_n}u\right)(t) = \begin{cases} \zeta(t), & t \in \mathcal{D}^6_2, \\ \varepsilon_n u(t) + \left({}_{0^+}\mathcal{I}^{1-\mu(\cdot)}_w u\right)(t) - \int_0^t \mathcal{A}_6(\leftthreetimes, u_{\leftthreetimes})d\leftthreetimes, & t \in \mathcal{D}^6_1. \end{cases} \tag{4.7}$$

Where $\varepsilon_n = \dfrac{2b}{n+b}$ and $b > 0$ a real number made to be fixed after in the proof. We have from (A2) and Remark 4.1.1 that

$$\begin{aligned} \left|\int_0^t \mathcal{A}_6(\leftthreetimes, v_{\leftthreetimes}) - \mathcal{A}_6(\leftthreetimes, u_{\leftthreetimes})d\leftthreetimes\right| &\leq \tau \int_0^F ||v_{\leftthreetimes} - u_{\leftthreetimes}||_{1,w,\underline{\Delta_1}}d\leftthreetimes \\ &\leq F C_*\tau||v-u||_{1,w,\mathcal{D}_6}. \end{aligned} \tag{4.8}$$

Using Eq.(4.8) we get

$$\begin{aligned} \left|\left(\widetilde{\mathcal{X}}_{\varepsilon_n}v\right)(t) - \left(\widetilde{\mathcal{X}}_{\varepsilon_n}u\right)(t)\right| &\leq \varepsilon_n|v(t)-u(t)| + \left|\left({}_{0^+}\mathcal{I}^{1-\mu(\cdot)}_w v\right)(t) - \left({}_{0^+}\mathcal{I}^{1-\mu(\cdot)}_w u\right)(t)\right| \\ &+ \left|\int_0^t \mathcal{A}_6(\leftthreetimes, v(\leftthreetimes)) - \mathcal{A}_6(\leftthreetimes, u(\leftthreetimes))d\leftthreetimes\right| \\ &\leq \varepsilon_n|u(t)-v(t)| + \left|\left({}_{0^+}\mathcal{I}^{1-\mu(\cdot)}_w v\right)(t) - \left({}_{0^+}\mathcal{I}^{1-\mu(\cdot)}_w u\right)(t)\right| \\ &+ F C_*\tau||v-u||_{1,w,\mathcal{D}_6}. \end{aligned} \tag{4.9}$$

Integrating Eq.(4.9) on $\mathcal{D}_6$, we get

$$\left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n}v-\widetilde{\mathcal{X}}_{\varepsilon_n}u\right|\right|_{1,w,\mathcal{D}_6}\leq\varepsilon_n||v-u||_{1,w,\mathcal{D}_6}+\left|\left|{}_{0^+}\mathcal{I}_w^{1-\mu(.)}v-{}_{0^+}\mathcal{I}_w^{1-\mu(.)}u\right|\right|_{1,w,\mathcal{D}_6}+F^2C_*\tau||v-u||_{1,w,\mathcal{D}_6}. \tag{4.10}$$

Using Eq.(1.26) from lemma 1.3.2, we get

$$\left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n}v-\widetilde{\mathcal{X}}_{\varepsilon_n}u\right|\right|_{1,w,\mathcal{D}_6}\leq\left(\varepsilon_n+\frac{(g(F)-g(-r))\,F_\mu\theta^*}{\mu^*}+F^2\tau C_*\right)||v-u||_{1,w,\mathcal{D}_6}. \tag{4.11}$$

Set $\varpi_8=\varepsilon_n+\dfrac{(g(F)-g(-r))\,F_\mu\theta^*}{\mu^*}+F^2\tau C_*$*. By induction it is trivial to prove that*

$$\left|\left|\widetilde{\mathcal{X}}^n_{\varepsilon_n}v-\widetilde{\mathcal{X}}^n_{\varepsilon_n}u\right|\right|_{1,w,\mathcal{D}_6}\leq\frac{\varpi_8^n}{n!}||v-u||_{1,w,\mathcal{D}_6},$$

where $\widetilde{\mathcal{X}}^n_{\varepsilon_n}$ *is the composition of the same function n times.*
We can remark that $\dfrac{\varpi_8^n}{n!}$ *tends to zero as n tends to infinity, hence for n sufficiently large we have.*

$$\frac{\varpi_8^n}{n!}<1.$$

Now applying lemma 1.4.1, there exist a unique $u\in L^1_w(\mathcal{D}_6)$ *such that*

$$\left(\widetilde{\mathcal{X}}_{\varepsilon_n}u\right)(t)=u(t). \tag{4.12}$$

Choose $b=1$*,* $n=1$ *and* $t\in\mathcal{D}^6_1$ *to deduce*

$$u(0)=\zeta(0)=0.$$

Now take $n=0$ *and t in the interval* $\mathcal{D}^6_1$*, we get*

$${}_{0^+}\mathcal{I}_w^{1-\mu(t)}u(t)=\int_0^t\mathcal{A}_6(\curlywedge,u_\curlywedge)d\curlywedge,t\in\mathcal{D}^6_1. \tag{4.13}$$

Using Lemma(4.1.1), then we can conclude that

$$\begin{cases}{}_{0^+}\mathcal{D}_w^{\mu(t)}u(t)=\mathcal{A}_6(t,u_t), & t\in\mathcal{D}^6_1,\\ u(t)=\zeta(t) & t\in\mathcal{D}^6_2.\end{cases} \tag{4.14}$$

This concludes our proof.

In this second existence result we are going to use Theorem (1.4.2).

Theorem 4.1.2 *If we consider (A1)-(A3) and the condition*

$$\widetilde{K}=\frac{(g(F)-g(-r))\,F_\mu\theta^*}{\mu^*}+C_*\tau<1. \tag{4.15}$$

Then the (FDPWFDENVO) has ALS in $L^1_w(\mathcal{D}_6)$*.*

Proof 4.1.3 *Let v, u elements in $L^1_w(\mathcal{D}_6)$, $t \in \mathcal{D}_6$ and consider the operator*

$$\widetilde{\mathcal{X}}_{\varepsilon_n} : L^1_w(\mathcal{D}_6) \to L^1_w(\mathcal{D}_6),$$

where $\widetilde{\mathcal{X}}_{\varepsilon_n}$ is defined by Eq.(4.7).For $t \in \mathcal{D}_2^6$

$$\left|\left(\widetilde{\mathcal{X}}_{\varepsilon_n} v\right)(t)\right| \leq |\zeta(t)|. \tag{4.16}$$

From (A3) we have that

$$\begin{aligned} \left|\int_0^t \mathcal{A}_6(\curlywedge, v_\curlywedge) d\curlywedge\right| &\leq ||a||_1 + \tau \int_0^{\digamma} ||v_\curlywedge||_{1,w} d\curlywedge \\ &\leq ||a||_1 + \digamma C_* \tau ||v||_{1,w}. \end{aligned} \tag{4.17}$$

For $t \in \mathcal{D}_2^6$ and by mean of Eq.(4.17), we get

$$\begin{aligned} \left|\left(\widetilde{\mathcal{X}}_{\varepsilon_n} v\right)(t)\right| &\leq \varepsilon_n |v(t)| + \left|\left({}_{0^+}\mathcal{I}_w^{1-\mu(.)} v\right)(t)\right| + \left|\int_0^t \mathcal{A}_6(\curlywedge, v(\curlywedge)) d\curlywedge\right| \\ &\leq \varepsilon_n |v(t)| + \left|\left({}_{0^+}\mathcal{I}_w^{1-\mu(.)} v\right)(t)\right| + ||a||_1 + \digamma C_* \tau ||v||_{1,w,\mathcal{D}_6}. \end{aligned} \tag{4.18}$$

Combining Eq.(4.16) and Eq.(4.18), we get

$$\left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n} v\right|\right|_{1,w} \leq \varepsilon_n ||v||_{1,w,\mathcal{D}_6} + \left|\left|{}_{0^+}\mathcal{I}_w^{1-\mu(.)} v\right|\right|_{1,w,\mathcal{D}_6} + ||a||_1 + C_* \tau ||v||_{1,w,\mathcal{D}_6} + ||\zeta||_{1,w,\underline{\Delta_2}}. \tag{4.19}$$

Using Eq.(1.26) from Lemma 1.3.2, we get

$$\left|\left|\widetilde{\mathcal{X}}_{\varepsilon_n} v\right|\right|_{1,w} \leq \left(\varepsilon_n + \frac{(g(\digamma) - g(-r)) \digamma_\mu \theta^*}{\mu^*} + C_* \tau\right) ||v||_{1,w,\mathcal{D}_6} + ||a||_1 + ||\zeta||_{1,w,\underline{\Delta_2}}. \tag{4.20}$$

There is a sufficiently large integer N_0 for wich

$$\varepsilon_n < 1 - \widetilde{K} \ \ \forall n \geq N_0. \tag{4.21}$$

And the choice of N_0 is independent from the choice of $d > 0$ fixed in the beginning of the proof Set

$$R = \frac{||\zeta||_{1,w,\underline{\Delta_2}} + ||a||_1}{1 - \left(\varepsilon_n + \widetilde{K}\right)}. \tag{4.22}$$

Consider

$$B_R = \left\{v \in L^1(\mathcal{D}_6) \ \mid \ ||v||_{1,w} \leq R\right\}. \tag{4.23}$$

It is well known that B_R is a CCBNE.
We will prove that $\widetilde{\mathcal{X}}_{\varepsilon_n} : B_R \to B_R$ is CC in few steps.
step1: *The fact that $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R) \subset B_R$ is by construction.*
step2: *$\widetilde{\mathcal{X}}_{\varepsilon_n}$ is CO, let $(v_k)_{k\geq 0} \subset B_R$, and $v \in B_R$ such that $v_k \xrightarrow[k\to+\infty]{} v$. We have to state and prove the following lemma*

Lemma 4.1.2 *We have the following convergence*

$$\left\|{}_{0^+}\mathcal{I}_w^{1-\mu(.)}v_k - {}_{0^+}\mathcal{I}_w^{1-\mu(.)}v\right\|_{1,w,\mathcal{D}_6} \xrightarrow[k\to+\infty]{} 0. \tag{4.24}$$

$$\int_0^{F} |\mathcal{A}_6(\lambda, v_{\lambda,k}) - \mathcal{A}_6(\lambda, v_\lambda)|\, d\lambda \xrightarrow[k\to+\infty]{} 0 \quad In \quad L_w^1\left(\mathcal{D}_6^1\right). \tag{4.25}$$

Proof 4.1.4 *For Eq.(4.24) usig Eq.(1.26) we get*

$$\left\|{}_{0^+}\mathcal{I}_w^{1-\mu(.)}v_k - {}_{0^+}\mathcal{I}_w^{1-\mu(.)}v\right\|_{1,w\mathcal{D}_6} \leq \frac{(g(F) - g(-r))\, F\, {}_\mu\theta^*}{\mu^*}\|v_k - v\|_{1,w,\mathcal{D}_6} \xrightarrow[k\to+\infty]{} 0. \tag{4.26}$$

For Eq.(4.25) we have $|w(\lambda)(\mathcal{A}_6(\lambda, v_{k,\lambda}) - \mathcal{A}_6(\lambda, v_\lambda))|\, g'(\lambda) \xrightarrow[k\to+\infty]{} 0,$ *and*

$$|\mathcal{A}_6(\lambda, v_{\lambda,k}) - \mathcal{A}_6(\lambda, v_\lambda)| \leq 2|a(t)| + c\tau||v||_{1,w,\mathcal{D}_6}.$$

as a consequence of (A3) and this insures that the DCT is applied and this is exactly (4.25).

We have

$$\begin{aligned}\left\|\widetilde{\mathcal{X}}_{\varepsilon_n}v_k - \widetilde{\mathcal{X}}_{\varepsilon_n}v\right\|_{1,w} &\leq \varepsilon_n||v_k - v||_{1,w} + \left\|\int_0^{F} |\mathcal{A}_6(\lambda, v_{\lambda,n}) - \mathcal{A}_6(\lambda, v_\lambda)|\, d\lambda\right\|_{1,w} \\ &+ \left\|\mathcal{I}_{0^+}^{1-\mu(.)}v_k - \mathcal{I}_{0^+}^{1-\mu(.)}v\right\|_{1,w} \xrightarrow[k\to+\infty]{} 0.\end{aligned} \tag{4.27}$$

Which means that $\widetilde{\mathcal{X}}_{\varepsilon_n}$ *is CO.*

step3: $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R)$ *is UB by construction, it remains to prove that*

$$\left\|\frac{1}{h}\int_t^{t+h}\left(\widetilde{\mathcal{X}}_{\varepsilon_n}v\right)(\lambda)d\lambda - \widetilde{\mathcal{X}}_{\varepsilon_n}v\right\|_{1,w} \xrightarrow[h\to0]{} 0, \; Uniformly. \tag{4.28}$$

We have

$$\begin{aligned}\left\|\frac{1}{h}\int_t^{t+h}\left(\widetilde{\mathcal{X}}_{\varepsilon_n}v\right)(\lambda)d\lambda - \widetilde{\mathcal{X}}_{\varepsilon_n}v\right\|_{1,w} &\leq \varepsilon_n\int_0^{F}\left|w(t)\left(\frac{1}{h}\int_t^{t+h}v(\lambda) - v(t)d\lambda\right)\right|g'(t)dt \\ &+ \int_{-r}^{0}\left|w(t)\left(\frac{1}{h}\int_t^{t+h}\zeta(\lambda) - \zeta(t)d\lambda\right)\right|g'(t)dt \\ &+ \int_0^{F}\left|w(t)\left(\frac{1}{h}\int_t^{t+h}\left({}_{0^+}\mathcal{I}_w^{1-\mu(\lambda)}v\right)(\lambda) - \left({}_{0^+}\mathcal{I}_w^{1-\mu(\lambda)}v\right)(t)d\lambda\right)\right| \\ &\times g'(t)dt \\ &+ \int_0^{F}\left|w(t)\left(\frac{1}{h}\int_t^{t+h}\int_0^{\lambda}\mathcal{A}_6(\rho, v_\rho)d\rho - \int_0^{t}\mathcal{A}_6(\sigma, v_\sigma)d\sigma d\lambda\right)\right| \\ &\times g'(t)dt.\end{aligned} \tag{4.29}$$

Since $v,_{0^+}\mathcal{I}_w^{1-\mu(.)}v, \int_0^{\leftthreetimes} \mathcal{A}_6(\rho, v_\rho)d\rho \in L_w^1\left(\mathcal{D}_1^6\right)$, *and* $\zeta \in L_w^1(\mathcal{D}_1^6)$ *and as a consequence of Theorem1.3.1 and another application of DCT we conclude that Eq.(4.28) is satisfied.*
By Kolmogorov's theorem $\widetilde{\mathcal{X}}_{\varepsilon_n}(B_R)$ *is RC, and hence by Theorem(1.4.2)* $\widetilde{\mathcal{X}}_{\varepsilon_n}$ *has at least one FP in* B_R.

$$\left(\widetilde{\mathcal{X}}_{\varepsilon_n}u\right)(t) = u(t) \;\; \forall n \geq N_0.$$

At this point we can fix $b = N_0$, $n = 2N_0$ *and* $t \in \mathcal{D}_1^6$ *to deduce*

$$u(0) = \zeta(0) = 0.$$

In the same way choose $n = N_0$ *in the interval* $\mathcal{D}_1^6$, *we get*

$$_{0^+}\mathcal{I}_w^{1-\mu(t)}u(t) = \int_0^t \mathcal{A}_6(\leftthreetimes, u_\leftthreetimes)d\leftthreetimes, t \in \mathcal{D}_1^6. \tag{4.30}$$

Using Lemma(4.1.1), then we can conclude that

$$\begin{cases} {}_{0^+}\mathcal{D}_w^{\mu(t)}u(t) = \mathcal{A}_6(t, u_t), & t \in \mathcal{D}_1^6, \\ u(t) = \zeta(t) & t \in \mathcal{D}_2^6. \end{cases} \tag{4.31}$$

This concludes our proof.

4.2 An Approval Example

Example 1: Let us consider the following (FDPFDENVO)

$$\begin{cases} \mathcal{D}_{0^+}^{\mu(t)}u(t) = \dfrac{e^{at}\|u_t\|_{1,w}}{3(e^{at}+e^{-at})(1+\|u_t\|_{1,w})}, & t \in \mathcal{D}_1^6 := (0,2], \quad (C) \\ \\ u(t) = \zeta(t) & t \in \mathcal{D}_2^6 := [-r, 0]. \quad (D) \end{cases} \tag{FDPWFDENVO2}$$

with

$$\mu(t) = \frac{1}{4}t, \tag{4.32}$$

and

$$\mathcal{A}_6(t,u) = \frac{7e^{at}u}{3(e^{at}+e^{-at})(1+u)} \quad (t,u) \in \mathcal{D}_1^6 \times (0, +\infty). \tag{4.33}$$

From equation (4.32) we can see that μ is a CF on $\mathcal{D}_1^6 \times \mathbb{R}$ and $0 < \mu(t) < 1$. And from Eq.(4.33), we can see that

$$\begin{aligned} |\mathcal{A}_6(t,u) - \mathcal{A}_6(t,v)| &\leq \frac{7e^{at}}{3(e^{at}+e^{-at})}\left|\tfrac{\|u_t\|_{1,w}}{(1+\|u_t\|_{1,w})^2} - \tfrac{\|v_t\|_{1,w}}{(1+\|v_t\|_{1,w})^2}\right| \\ &\leq \frac{7e^{at}}{3(e^{at}+e^{-at})}\frac{|\|u\|_{1,w} - \|v\|_{1,w}|}{(1+\|u_t\|_{1,w})^2(1+\|v_t\|_{1,w})^2} \\ &\leq \frac{7e^{at}}{3(e^{at}+e^{-at})}\|u-v\|_{1,w} \\ &\leq \tfrac{7}{3}\|u-v\|_{1,w}. \end{aligned} \tag{4.34}$$

It is easy to check that for the given choice of nonlinear functions μ and $\mathcal{A}_6$ assumptions (A1)-(A2) are satisfied. Therefore by theorem(4.1.1), the problem (FDPWFDENVO2) has a US.

Conclusion

This thesis presents an abstract version of the R-Liouville variable-order boundary value problems. The variable order of the provided systems is denoted by the function ($\mu(t) : J \to (0,1]$, $\mu(t) : J \to (1,2]$ or $\mu(t, x(t)) : J \to (0,1]$). The primary challenge that we addressed was that the semi-group property does not apply to variable-order integrals, which we discovered after reviewing certain crucial tools (definitions and notations) of the multiplied variable-order operator.

Next, we gave an analogous perturbed integral equation for each system in each chapter. The results in this study are established with the help of the Darbo's fixed point theorem combined with Kuratowski measure of noncompactness , Schauder and Banach fixed point theorems.

All of the results we have obtained from our examination into this intriguing particular research topic are distinct and outstanding.

Moreover, there is great potential for applying all of the findings in this thesis to a wide range of trans-disciplinary science applications. We might be able to do additional research on this open research topic with the help of our initial research study findings.

Stated differently, the suggested IVPs or BVPs may eventually be expanded to more complex real mathematical fractional models.

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