PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
Ministry of Higher Education and Scientific Research
IBN KHALDOUN UNIVERSITY OF TIARET
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
Department of Mathematics

MASTER MEMORY

In order to obtain the degree of master

Specialty:
« MATHEMATICS »

Option:
« Functional Analysis and Differential Equations»

Presented by:
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Under the title:

A study on some classes of fractional differential equations with retardation and anticipation

Defended publicly on 30/06/2024
in Tiaret in front the jury composed of:

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Academic year : 2023/2024

Acknowledgements

*In In the name of Allah, Most Gracious, Most Merciful. We express our gratitude to our supervisor Mr; Boumaaza Mokhtar, for his multiple advice and for all the hours he devoted to directing this research from the beginning to the end of this work. We also thank the jury member Mr.Khelifa Hizia and Mr. Ziane Mohamed for dedicating their time to evaluate our modest work. We extend our sincere thanks to all our families and in particular to our parents who were always there when we needed them, our teachers from primary school to university, our friends, our loved ones, and finally, we thank all those who helped us. directly or indirectly contributed to the development of this work. ***THANKS.*

Dedication

In the first place, I thank Allah and praise Him for His guidance in this humble work. This work is dedicated to everyone who supported me throughout this academic journey. To my parents, for your endless encouragement and believed in me, and that has meant everything. To my professors and mentors, for your wisdom and guidance. To my friends and peers. To my partner, and to myself, for sticking with it through the challenges and hard work

Abstract

Functional differential equations occur in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. This work memoir the existence of solutions and random solutions for some implicit fractional differential equations, involving both retarded and advanced arguments, with generalized Caputo fractional derivative. Our results will be obtained by means of fixed points theorems and by the technique of measures of noncompactness.

Rsume

Les quations diffrentielles fonctionnelles apparaissent dans divers domaines dapplica- tions biologiques, physiques et dingnierie, et ces quations ont reu beaucoup dattention ces

dernires annes. Cette thse examine lexistence de solutions et de solutions alatoires pour certaines quations diffrentielles fractionnaires implicites, impliquant la fois des arguments retards et avancs, avec une drive fractionnaire gnralise de Caputo. Nos rsultats seront obtenus au moyen de thormes de points fixes et par la technique des mesures de non compacit.

Contents

0.1 Introduction

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator $D^{\alpha}_{a^+}$ where $a, \alpha \in \mathbb{R}$. Several approaches to fractional derivatives exist : Riemann-Liouville (RL), Hadamard, Erdélyi-Kober, Grunwald-Letnikov (GL), Weyl and Caputo etc. The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to the books [8, 28, 29, 34] and the references therein. In this thesis, we always use the generalized Caputo derivative.

Fractional differential equations and inclusions appear in several areas such as engineering, mathematics, bio-engineering, physics, viscoelasticity, electrochemistry, control, etc. For current advances of fractional calculus, we refer the reader to the monographs [1, 2, 29] and the references therein. In particular, time fractional differential equations are used when attempting to describe transport processes with long memory. Recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo and generalized Caputo derivative. See for example [6, 7, 16] and references therein.

The differential equation with delay is a special type of functional differential equations. Delay differential equations arise in many biological and physical applications and it often forces us to consider variable or state-dependent delays. The functional differential equations with state-dependent delay have many important applications in mathematical models of real phenomena and the study of this type of equations has received much attention in recent years. We refer the reader to the monographs [11, 12, 19, 20].

The authors studied the existence and uniqueness of solutions for boundary value problems of Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments;see [5, 14, 17] and the references therein.

The measure of noncompactness which is one of the fundamental tools in the theory of nonlinear analysis was initiated by the pioneering articles of Kuratowski [32], Darbo [21] and was developed by Bana's and Goebel [9] and many researchers in the literature. The applications of the measure of noncompactness can be seen in the wide range of applied mathematics: theory of differential equations (see [4, 17, 18] and references therein).

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year ; see for instance[3, 10, 13, 17] and the references therein.

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems. Random differential equations, used in many on cases, to describe phenomena in biology, physics, engineering, and systems sciences contain certain parameters or coefficients which have specific interpretations, but whose values are unknown. We refer the reader to the monographs [37, 36], the papers [40, 38, 39] and references therein.

In the following we give an outline of our thesis organization consisting of four chapters. The first chapter gives some notations, definitions, lemmas and fixed point theorems which are used throughout this memoir.

In Chapter 2, we establish the existence of solutions for a class of problems for nonlinear implicit generalized Caputo fractional differential equations(NIFDE) involving both retarded and advanced arguments. Here results are discussed, the first is based on the Banach contraction principle and Schauder's , Schaefer's fixed point theorems.

In Chapter 3, we establish the existence of solutions for a class of problems for nonlinear implicit generalized Caputo fractional differential equations(NIFDE) involving both retarded and advanced arguments in Banach space. Here results are discussed, is based on the method associated with the technique of measures of non compactness and the fixed point theorems of Darbo and Mönch.

In Chapter 4, we establish the existence of Random solutions For Mixed Fractional Differential Equations with Retarded and Advanced Arguments. Here results are discussed, is based on the Banach contraction principle, Schauder's fixed point theorems.

Chapter 1

Preliminaries

1.1 Notations and Definitions

Let $(E, \|.\|)$ be the Banach space. We denote by $C([-r,\beta],E)$ the Banach space of all continuous functions from $[-r,\beta]$ into E equipped with the norm

$$\|x\|_{[-r,\beta]} = \sup\{\|x(t)\| : -r \le t \le \beta\}$$

and $C([a,T],E)$ is the Banach space endowed with the norm

$$\|x\|_{[a,T]} = \sup\{\|x(t)\| : a \le t \le T\}.$$

Also, let $E_1 = C([a-r,a],E)$, $E_2 = C([T,T+\beta],E)$
and let the space

$$AC^1(I) := \{w : I \longrightarrow E : w' \in AC(I)\},$$

where

$$w'(t) = t\frac{d}{dt}w(t),\ t \in I,$$

$AC(I,E)$ is the space of absolutely continuous functions on I,
$\mathcal{C} = \{x : [a-r,T+\beta] \longmapsto E : x\mid_{[a-r,a]} \in C([a-r,a]), x\mid_{[a,T]} \in AC^1([a,T])$

$$\text{and } x\mid_{[T,T+\beta]} \in C([T,T+\beta])\}$$

be the spaces endowed, respectively, with the norms

$$\|x\|_{[a-r,a]} = \sup\{\|x(t)\| : a-r \le t \le a\},$$

and

$$\|x\|_{[T,T+\beta]} = \sup\{\|x(t)\| : T \le t \le T+\beta\},$$

$$\|x\|_{\Omega} = \sup\{\|x(t)\| : a-r \le t \le T+\beta\}.$$

Let $L^1(I)$, be the Banach space of measurable functions $v : I \longrightarrow E$ which are Bochner integrable, equipped with the norm

$$\|v\|_{L^1} = \int_a^T \|v(t)\| dt.$$

Consider the space $X_c^p(a,b)$, $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex-valued Bochner measurable functions f on $[a,b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by :

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, c \in \mathbb{R}).$$

In particular, where $c = \frac{1}{p}$ the space $X_c^p(a,b)$ coincides $L^p(a,b)$ space, i.e., $X_{\frac{1}{p}}^p(a,b) = L^p(a,b)$.

Denote by $L^\infty(I, \mathbb{R})$, the Banach space of essentially bounded measurable functions $u : I \longrightarrow \mathbb{R}$ equipped with the norm

$$\|f\|_{L^\infty} = \inf\{c \geq 0; \quad |f(x)| \leq c \text{ a.e. on } \quad I\}.$$

1.2 Fractional Calculus

Definition 1.1 *([28, 30, 31]): **(The Caputo-type generalized fractional integral)** Let $\alpha \in \mathbb{R}, c \in \mathbb{R}$ and $g \in X_c^p(a,b)$, the Erdélyi-Kober fractional integral of order α is defined by :*

$$({}^\rho I_{a^+}^\alpha g)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} g(s) ds, \qquad t > a, \rho > 0 \tag{1.1}$$

where Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \ \alpha > 0.$$

Definition 1.2 *([27]) The generalized fractional derivative, corresponding to the fractional integral (1.1), is defined, for $0 \leq a < t$, by:*

$$\begin{aligned} {}^\rho D_{a^+}^\alpha g(t) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\alpha}} g(s) ds \\ &= \delta_\rho^n ({}^\rho I_{a^+}^{n-\alpha} g)(t), \end{aligned} \tag{1.2}$$

where $\delta_\rho^n = \left(t^{1-\rho} \frac{d}{dt} \right)^n$.

Definition 1.3 *([27, 33]) The Caputo-type generalized fractional derivative ${}_c^\rho D_{a^+}^\alpha$ is defined via the above generalized fractional derivative (1.2) as follows*

$$({}_c^\rho D_{a^+}^\alpha g)(t) = \left({}^\rho D_{a^+}^\alpha \left[g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (s-a)^k \right] \right). \tag{1.3}$$

Lemma 1.4 *([27]) Let $\alpha, \rho \in \mathrm{I\!R}^+$, then*

$$({}^{\rho}I^{\alpha}_{a^+}\,{}^{C}D^{\alpha,\rho}_{a^+}g)(t) = g(t) - \sum_{k=0}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho}\right)^k, \tag{1.4}$$

and

$$({}^{\rho}I^{\alpha}_{a^+}\,{}^{RL}D^{\alpha,\rho}_{a^+}g)(t) = g(t) - \sum_{k=1}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho}\right)^{k-\alpha}, \tag{1.5}$$

for some $c_k \in \mathrm{I\!R}$, $n = [\alpha] + 1$.

Lemma 1.5 *([27]) If $x > n$, then we have*

$$\left[{}^{\rho}I^{\alpha}_{a^+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}\right](x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha+\beta-1}. \tag{1.6}$$

1.3 Random Operator

Let $B_{\mathbb{R}}$ be the σ-algebra of Borel subsets of $\mathbb{R}$. A mapping $v : \Omega \to \mathbb{R}$ is said to be measurable if for any $D \in B_{\mathbb{R}^m}$, one has

$$v^{-1}(D) = \{w \in \Omega : v(w) \in D\} \subset \mathcal{A}.$$

To define integrals of sample paths of a random process, it is necessary to define a jointly measurable map.

Definition 1.6 *A mapping $T : \Omega \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if for any $D \in B_{\mathbb{R}^m}$, one has*

$$T^{-1}(D) = \{(w, v) \in \Omega \times E : T(w, v) \in D\} \subset \mathcal{A} \times B_{\mathbb{R}},$$

where $\mathcal{A} \times B_{\mathbb{R}}$ is the direct product of the σ-algebras $\mathcal{A}$ and $B_{\mathbb{R}}$, those defined in Ω and $\mathbb{R}$, respectively.

Definition 1.7 *A function $T : \Omega \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.*

A mapping $T : \Omega \times \mathbb{R} \to \mathbb{R}$ is called a random operator if $T(w, u)$ is measurable in w for all $u \in \mathbb{R}$, and it expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on $\mathbb{R}$. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [26].

Definition 1.8 *[23] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C, if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called a random (stochastic) fixed point of T if for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$, and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.*

Definition 1.9 *A function $f : I \times C([-r,\beta], \mathbb{R}) \times \Omega \to \mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:*

(i) The map $(t,w) \to f(t,u,w)$ is jointly measurable for all $u \in C([-r,\beta], \mathbb{R})$ and

(ii) The map $u \to f(t,u,w)$ is continuous for all $t \in I$ and $w \in \Omega$.

1.4 Measure of Noncompactness and Auxiliary Results

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness .

Definition 1.10 *([9]) Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty)$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}.$$

The Kuratowski measure of noncompactness satisfies the following properties.

Lemma 1.11 *([22]) Let A and B bounded sets.*

(1) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(2) $\alpha(cov(B)) = \alpha(B)$,($cov(B)$ denote the convex hull of B)

(3) $\alpha(B) = \alpha(\overline{B})$, ($\overline{B}$ denote the closure of B.)

(4) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(5) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, where $A + B = \{x + y : x \in A,\quad y \in B\}$.

(6) $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in \mathbb{R}$, where $\lambda B = \{\lambda x : x \in B\}$.

(7) $\alpha(A \cup B) = max\{\mu(A), \alpha(B)\}$.

(8) $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in E$.

Lemma 1.12 *([25]) Let $V \subset C(I,E)$ is a bounded and equicontinuous set, then*

(i) the function $t \longmapsto \alpha(V(t))$ is continuous on I, and

$$\alpha_C(V) = \max_{t\in I} \alpha(V(t)),$$

(ii)

$$\alpha\left(\int_a^T x(s)ds : x \in V\right) = \int_a^T \alpha(V(s))ds,$$

where

$$V(t) = \{x(t) : x \in V\},\ t \in I.$$

and α_C is the Kuratowski measure of noncompactness defined on the bounded sets of $C(I)$.

Theorem 1.13 *([17])**(AscoliArzela)** . Let $A \subset C(I,E)$,A is relatively compact (i.e.A is compact) if:*
1. A is uniformly bounded i.e., there exists $M > 0$ such that

$$\|f(t)\| < M \quad for \quad every \quad f \in A \quad and \quad t \in I.$$

.
2.A is equicontinuous i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each t,$\bar{t} \in I$,

$$|t - \bar{t}| \leq \delta \Longrightarrow \|f(t) - f(\bar{t})\| \leq \epsilon, \quad for \quad every \quad f \in A.$$

.
3. The set $\{f(t) : f \in A; t \in I\}$ is relatively compact in E.

1.5 Some Fixed Point Theorems

Theorem 1.14 *([24])**(Schauder's)**. Let X be a Banach space, $D \subset X$ a nonempty convex bounded closed set and let $N : D \longmapsto D$ be a completely continuous operator. Then N has at least one fixed point.*

Theorem 1.15 *([26]). Let X be a nonempty, closed convex bounded subset of the separable Banach space E and let $N : \Omega \times X \longmapsto X$ be a compact and continuous random operator. Then the random equation $N(w, u(w)) = u(w)$ has a random solution.*

Theorem 1.16 *([24])**(Schaefer's)**. Let X be a Banach space, and $N : X \longmapsto X$ be a completely continuous operator. If the set*

$$\xi = \{x \in X : x = \lambda Nx,\ for\ some\ \lambda \in (0,1)\} \quad is\ bounded,$$

then N has a fixed point.

Lemma 1.17 *(Darbo , [21]). Let D be a bounded, closed and convex subset of Banach space X. If the operator $N : D \to D$ is a strict set contraction, i.e there is a constant $0 \leq \lambda < 1$ such that $\alpha(N(S)) \leq \lambda\alpha(S)$ for any set $S \subset D$ then N has a fixed point in D.*

Theorem 1.18 *(Mönch , [35]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup 0 \Longrightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

.

Chapter 2

Nonlinear Neutral IFDE with Retarded and Advanced Arguments

2.1 Introduction

In this chapter, we establish, the existence and uniqueness of solutions for implicit generalized Caputo fractional differential equations with retarded and advanced arguments.

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}(x(t)-k(t,x^t)) = f(t,x^t,{}^{\rho}_{c}D^{\alpha}_{a^+}x(t)), \text{ for } \ t\in I:=[a,T],\ 1<\alpha\leq 2, \tag{2.1}$$

$$ x(t)=\phi(t),\ t\in[a-r,a], r>0 \tag{2.2}$$

$$ x(t)=\psi(t),\ t\in[T,T+\beta],\ \beta>0, \tag{2.3}$$

where ${}^{\rho}_{c}D^{\alpha}_{a^+}$ is the Caputo type modification of the generalized fractional derivative, $f : I\times C([-r,\beta],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is a given function, $\phi\in C([a-r,a],\mathbb{R})$ with $\phi(a)=0$ and $\psi\in C([T,T+\beta],\mathbb{R})$ with $\psi(T)=0$. We denote by x^t the element of $C([-r,\beta])$ defined by:

$$x^t(s)=x(t+s) : s\in[-r,\beta]$$

here $x^t(\cdot)$ represents the history of the state from time $t-r$ up to time $t+\beta$.

2.2 Existence Results for the NIFDE with Retarded and Advanced Arguments

Lemma 2.1 *Let* $1<\alpha\leq 2$, $\phi\in C([a-r,a],\mathbb{R})$ *with* $\phi(a)=0$, $\psi\in C([T,T+\beta],\mathbb{R})$ *with* $\psi(T)=0$ *and* $h : I\to\mathbb{R}$ *be a continuous function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}x(t)=h(t), \text{ for a.e. } \ t\in I:=[a,T],\ 1<\alpha\leq 2, \tag{2.4}$$

$$ x(t)=\phi(t),\ t\in[a-r,a],\ r>0 \tag{2.5}$$

$$x(t) = \psi(t),\ t \in [T, T+\beta],\ \beta > 0, \tag{2.6}$$

has a unique solution, which is given by

$$x(t) = \begin{cases} \phi(t), \ \text{if} \quad t \in [a-r, a], \\ k(t, x^t) - \displaystyle\int_a^T G(t,s)h(s)ds, \ \text{if} \quad t \in I \\ \psi(t), \ \text{if} \quad t \in [T, T+\beta], \end{cases} \tag{2.7}$$

where

$$G(t,s) = \frac{\rho^{1-\alpha}}{\alpha(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - c^\rho)^{\alpha-1} c^{\rho-1}}{(T^\rho - a^\rho)} - c^{\rho-1}(t^\rho - c^\rho)^{\alpha-1}, & a \le c \le t \le T, \\ \dfrac{(t^\rho - a^\rho)(T^\rho - c^\rho)^{\alpha-1} c^{\rho-1}}{(T^\rho - a^\rho)}, & a \le t \le c \le T. \end{cases} \tag{2.8}$$

Here $G(t,s)$ is called the Green function of the boundary value problem (2.4)-(2.6).
Proof. *From* (1.4)*, we have*

$$x(t) = c_0 + c_1 \left(\frac{t^\rho - a^\rho}{\rho} \right) a +^\rho I^\alpha_{a^+} h(s), \quad c_0, c_1 \in \mathbb{R}, \tag{2.9}$$

therefore

$$x(a) = c_0 = 0,$$

$$x(T) = c_1 \left(\frac{T^\rho - a^\rho}{\rho} \right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T (T^\rho - c^\rho)^{\alpha-1} s^{\rho-1} h(s) ds,$$

and

$$c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^{rho})\alpha(\alpha)} \int_a^T (T^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s) ds.$$

Substitute the value of c_0 and c_1 into equation (2.9), we get equation (2.7).

$$x(t) = \begin{cases} \phi(t), \ \ if \quad t \in [a-r, a], \\ \\ k(t, x^t) - \displaystyle\int_a^T G(t,s)h(s)ds, \ \ if \quad t \in I \\ \\ \psi(t), \ \ if \quad t \in [T, T+\beta], \end{cases}$$

where G is defined by equation(2.8), the proof is complete.

Lemma 2.2 *Let $f : I \times C[-r, \beta] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. A function $x \in C$ is solution of problem* $(2.1) - (2.3)$ *if and only if x satisfies the following integral equation*

$$x(t) = \begin{cases} \phi(t), \ \ if \quad t \in [a-r, a], \\ \\ k(t, x^t) - \displaystyle\int_a^T G(t,s)h(s)ds, \ \ if \quad t \in I \\ \\ \psi(t), \ \ if \quad t \in [T, T+\beta], \end{cases}$$

where $h \in C(J)$ satisfies the functional equation

$$h(t) = f(t, x_t, h(t)).$$

The following hypotheses will be used in the sequel:

(H_1) The function $f : I \times C[-r, \beta] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $k : I \times C[-r, \beta] \longrightarrow \mathbb{R}$ are continuous.

(H_2) There exist $S > 0$, $P > 0$, $0 < \overline{S} < 1$ such that

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq S\|u - \bar{u}\|_{[-r,\beta]} + \overline{S}|v - \bar{v}|$$

and

$$|k(t, m) - k(t, \bar{m})| \leq P\|m - \bar{m}\|_{[-r,\beta]}$$

for any $u, \bar{u}, m, \bar{m} \in C([-r, \beta])$ and $v, \bar{v} \in \mathbb{R}$.

(H_3) There exists $q, b \in L^\infty([a, T], \mathbb{R}_+)$ such that

$$|f(t, u, v)| \leq q(t) \text{ for a.e. } t \in I, \quad \text{and each} \quad u \in C([-r, \beta]) \text{ and } v \in \mathbb{R},$$

and

$$|k(t, u)| \leq b(t) \text{ for a.e. } t \in I, \quad \text{and each} \quad u \in C([-r, \beta])$$

Set

$$q^* = ess \sup_{t \in I} q(t)$$

and

$$b^* = ess \sup_{t\in I} q(t)$$

$$\widetilde{G} = \sup \left\{ \int_a^T |G(t,s)|ds, t \in I \right\}.$$

(H_4) For each bounded set D_N in $\mathcal{C}$, the set $\{t \longrightarrow k(t,x^t) : x \in D_N\}$ is equicontinuous in $C(I, \mathrm{IR})$.

Now, we state and prove our existence result for (2.1)-(2.3) based on the Banach contraction principle.

Theorem 2.3 *Assume* (H_1) *and* (H_2) *hold. If*

$$(P + \frac{S\widetilde{G}}{(1-\overline{S})}) < 1, \tag{2.10}$$

then the problem (2.1)-(2.3) has a unique solution.

Proof: Let the operator $L : \mathcal{C} \longrightarrow \mathcal{C}$ defined by

$$(Lx)(t) = \begin{cases} \phi(t), \text{ if } \quad t \in [a-r, a], \\ \\ k(t,x^t) - \displaystyle\int_a^T G(t,s)h_x(s)ds, \text{ if } \quad t \in I \\ \\ \psi(t), \text{ if } \quad t \in [T, T+\beta]. \end{cases} \tag{2.11}$$

By Lemma 2.2 it is clear that the fixed points of L are solutions (2.1)-(2.3) .
Let $x_1, x_2 \in \mathcal{C}$. If $t \in [a-r, a]$ or $t \in [T, T+\beta]$ then

$$|(Lx_1)(t) - (Lx_2)(t)| = 0.$$

For $t \in I$, we have

$$|(Lx_1)(t) - (Lx_2)(t)| \leq |k(t,x_1^t) - k(t,x_2^t)| + \int_a^T |G(t,s)||h_{x_1}(s) - h_{x_2}(s)|ds, \tag{2.12}$$

and by (H_2) we have

$$\begin{aligned} |h_{x_1}(t) - h_{x_2}(t)| &= |f(t, x_1^t, {}^\rho_c D^\alpha_{a^+} x_1(t)) - f(t, x_2^t, {}^\rho_c D^\alpha_{a^+} x_2(t))| \\ &\leq S\|x_1 - x_2\|_{[-r,\beta]} + \overline{S}|h_{x_1}(t) - h_{x_2}(t)|. \end{aligned}$$

Then

$$|h_{x_1}(t) - h_{x_2}(t)| \leq \frac{S}{(1-\overline{S})}\|x_1 - x_2\|_{[-r,\beta]}. \tag{2.13}$$

By replacing (2.15) in (2.12) we obtain,

$$\begin{aligned} |(Lx_1)(t)-(Lx_2)(t)| &\leq P\|x_1-x_2\|_{[-r,\beta]}+\frac{S}{(1-\overline{S})}\int_a^T |G(t,s)|\|x_1-x_2\|_{[-r,\beta]}ds \\ &\leq (P+\frac{S\widetilde{G}}{(1-\overline{S})})\|x_1-x_2\|_{[-r,\beta]}. \end{aligned}$$

Therfore ,For each $t \in I$, we have

$$|(Lx_1)(t)-(Lx_2)(t)| \leq \left(P+\frac{S\widetilde{G}}{1-\overline{S}}\right)\|x_1-x_2\|_{[-r,\beta]}. \tag{2.14}$$

Thus

$$\|Lx_1)-Lx_2\|_c \leq \left(P+\frac{S\widetilde{G}}{1-\overline{S}}\right)\|x_1-x_2\|_C. \tag{2.15}$$

We now prove an existence result for (2.1)-(2.3) by using the Schauder's fixed point theorem.

Theorem 2.4 *Assume that the hypotheses* (H_1) *and* (H_3) *hold. Then problem (2.1)-(2.3) has at least one solution.*

Step 1. L is continuous. Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathcal{C}$. If $t \in [a-r,a]$ or $t \in [T, T+\beta]$ then

$$|(Lx_n)(t)-(Lx)(t)| = 0.$$

For $t \in I$, we have

$$|(Lx_n)(t)-(Lx)(t)| \leq |k(t,x_n^t)-k(t,x^t)|+\int_a^T |G(t,s)||h_n(s)-h(s)|ds, \tag{2.16}$$

where

$$h_n(t) = f(t,x_n^t,h_n(t)),$$

and

$$h(t) = f(t,x^t,h(t)).$$

Since $x_n \longrightarrow x$, and by (H_1) we get $h_n(t) \longrightarrow h(t)$ and $k(t,x_n^t) \longrightarrow k(t,x^t)$ as $n \longrightarrow \infty$ for each $t \in I$.
By (H_3) we have for each $t \in I$,

$$|h_n(t)| \leq q^*. \tag{2.17}$$

Then,

$$\begin{aligned} |G(t,s)||h_n(t)-h(t)| &\leq |G(t,s)|\,[|h_n(t)|+|(t)|] \\ &\leq 2q^*|G(t,s)|. \end{aligned}$$

For each $t \in I$ the functions $s \longmapsto 2q^*|G(t,s)|$ are integrable on $[a,t]$, then by Lebesgue dominated convergence theorem, equation (2.16) implies

$$|(Lx_n)(t) - (Lx)(t)| \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

and hence

$$\|L(x_n) - L(x)\|_{\mathcal{C}} \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Consequently, N is continuous.

Let the constant R be such that:

$$R \geq \max\left\{q^*\widetilde{G}, \|\phi\|_{[a-r,a]}, \|\psi\|_{[T,T+\beta]}\right\}, \tag{2.18}$$

and define

$$D_R = \{x \in \mathcal{C} : \|x\|_{\mathcal{C}} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of β.

Step 2. $L(D_R) \subset D_R$.

Let $x \in D_R$ we show that $Lx \in D_R$.
If $t \in [a-r, a]$, then

$$|L(x)(t)| \leq \|\phi\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$|L(x)(t)| \leq \|\psi\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$|(Lx)(t)| \leq |k(t, x^t)| + \int_a^T |G(t,s)||h(s)|ds.$$

By (H_3), we have

$$\begin{aligned} |(Lx)(t)| &\leq b^* + q^* \int_a^T |G(t,s)|ds \\ &\leq b^* + q^*\widetilde{G} \\ &\leq R, \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $|Lx(t)| \leq R$, which implies that $\|Lx\|_c \leq R$. Consequently,

$$L(D_R) \subset D_R.$$

Step 3: $L(D_R)$ is bounded and equicontinuous.
By Step 2 we have $L(D_R)$ is bounded.
Let $t_1, t_2 \in I = [a, T], t_1 < t_2, \ and \ x \in D_R$ then

$$\begin{aligned} |(Lx)(t_2) - (Lx)(t_1)| &\leq |k(t_2, x^t) - k(t_1, x^t)| + \int_a^T |G(t_2, s) - G(t_1, s)|h(s)|ds \\ &\leq |k(t_2, x^t) - k(t_1, x^t)| + q^* \int_a^T |G(t_2, s) - G(t_1, s)|ds. \end{aligned}$$

By (H_4) and as $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous. From Schauder's theorem, we conclude that N has a fixed point with is a solution of the problem (2.1)-(2.3).

We prove an existence result for the (2.1)-(2.3) problem by using the Schaefer's fixed point theorem.

Theorem 2.5 *Assume that* (H_1) *and*
(H_4) *There exist* $d, q, m, b \in C(J, \mathbb{R})$ *with* $m^* = \sup_{t \in I} m(t) < 1$ *such that*

$$|f(t,u,v)| \leq d(t) + q(t)\|u\|_{[-r,\beta]} + m(t)|v| \quad where \quad t \in I, \quad u \in C([-r,\beta],\mathbb{R}) \quad and \quad v \in \mathbb{R}.$$

And

$$|k(t,u)| \leq b(t) \quad where \quad t \in I, \quad u \in C([-r,\beta],\mathbb{R}) \quad and \quad v \in \mathbb{R}.$$

If

$$\frac{q^* \widetilde{G}}{(1-m^*)} < 1, \tag{2.19}$$

then problem (2.1)-(2.3) has at least one solution.

Proof. Consider the operator L defined in (2.11). We shall show that L satisfies the assumption of Schaefer's fixed point theorem. As shown in Theorem 4.4, we see that the operator L is continuous, and completely continuous.
Now it remains to show that the set

$$\xi = \{x \in C : x = \lambda L x, \quad \text{for some } \lambda \in (0,1)\} \quad \text{is bounded.}$$

Let $x \in \xi$, then $x = \lambda L y$ for some $0 < \lambda < 1$. Thus for each $t \in I$ we have

$$x(t) = \lambda \left(k(t, x^t) - \int_a^T G(t,s) h_x(s) ds \right), \tag{2.20}$$

where

$$h_x(t) = f(t, x^t, h_x(t)).$$

By (H_4), we have for each $t \in I$

$$\begin{aligned} |h_x(t)| &\leq d(t) + q(t)\|x\|_{[-r,\beta]} + m(t)|h_x(t)| \\ &\leq d^* + q^*\|x\|_{[-r,\beta]} + m^*|h_x(t)| \end{aligned}$$

Thus

$$|h_x(t)| \leq \frac{1}{1-m^*} \left(d^* + q^*\|x\|_{[-r,\beta]} \right).$$

This implies, by (2.20) that for each $t \in I$ we have

$$\begin{aligned}|x(t)| &\leq b^* + \int_a^T |G(t,s)| \frac{1}{1-m^*}\left(d^* + q^*\|x\|_{[-r,\beta]}\right) ds \\ &\leq b^* + \frac{\left(d^* + q^*\|x\|_{[-r,\beta]}\right)\widetilde{G}}{(1-m^*)}.\end{aligned}$$

Then

$$\|x\|_{[-r,\beta]} \leq b^* + \frac{d^*\widetilde{G}}{(1-m^*)} + \frac{q^*\widetilde{G}\|x\|_{[-r,\beta]}}{(1-m^*)}.$$

Thus

$$\left[1 - \frac{q^*\widetilde{G}}{(1-m^*)}\right]\|x\|_{[-r,\beta]} \leq b^* + \frac{d^*\widetilde{G}}{(1-m^*)}.$$

Finally, by (2.19) we have

$$\|x\|_{[-r,\beta]} \leq \frac{b^* + \frac{d^*\widetilde{G}}{(1-m^*)}}{\left[1 - \frac{q^*\widetilde{G}}{(1-m^*)}\right]} = b_0.$$

If $t \in [a-r, a]$, then

$$|x(t)| \leq \|\phi\|_{[a-r,a]} \leq b_1,$$

and if $t \in [T, T+\beta]$, then

$$|x(t)| \leq \|\psi\|_{[T,T+\beta]} \leq b_2.$$

From which it follows that for each $t \in [a-r, T+\beta]$, we have $|x(t)| \leq \max\{b_2, b_1, b_0\}$, which implies that $\|x\|_{\mathcal{C}} \leq \max\{b_2, b_1, b_0\}$, this implies that ξ is bounded As a consequence of Schaefer's fixed point theorem, L admits a fixed point which is a solution of the problem (2.1)-(2.3).

2.2.1 Examples

Example 1: Consider the boundary value problem of implicit generalized Caputo fractional differential equation:

$$\begin{cases} x(t) = e^{t-2} - 1, \quad t \in [1,2], \\ {}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}x(t) = \dfrac{1}{10e^{t+2}\left(1 + |x^t| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{2^+}x(t)\right|\right)} + \dfrac{\sin(t)}{\ln(t^2+1)}, \quad t \in I = [2,4] \\ x(t) = t - 4, \quad t \in [4,6]. \end{cases} \tag{2.21}$$

Set

$$f(t,u,v)=\frac{1}{10e^{t+2}(1+|u|+|v|)}+\frac{\sin(t)}{\ln(t^2+1)} \quad k(t,u)=0, \quad t\in[2,4], \quad u\in C([-r,\beta])$$

k(t,u)=0 and $v\in \mathbb{R}, \alpha=\frac{3}{2}, \rho=\frac{1}{2}, r=1, \beta=2$. For each $u,\bar{u}\in C([-r,\beta])$, $v,\bar{v}\in \mathbb{R}$ and $t\in[2,4]$, we have

$$\begin{aligned}
|f(t,u,v)-f(t,\bar{u},\bar{v})| &\leq \left|\frac{1}{10e^{t+2}(1+|u|+|v|)}-\frac{1}{10e^{t+2}(1+|\bar{u}|+|\bar{v}|)}\right| \\
&\leq \frac{1}{10e^{t+2}}\left(|u-\bar{u}|+|v-\bar{v}|\right) \\
&\leq \frac{1}{10e^{t+2}}\left(\|u-\bar{u}\|_{[-r,\beta]}+|v-\bar{v}|\right).
\end{aligned}$$

Therefore, (H_2) is verified with $S=\overline{S}=\frac{1}{10e^4}$.
For each $t\in I$ we have

$$\int_a^T |G(t,s)|ds \leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{T^\rho-a^\rho}\right)\int_a^T \left|\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t \left|\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds.$$

Then

$$\int_a^T |G(t,s)|ds \leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^\alpha.$$

Therefore

$$\widetilde{G}\leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^\alpha.$$

The condition

$$\begin{aligned}
P+\frac{S\widetilde{G}}{(1-\overline{S})} &\leq 2\frac{\frac{1}{10e^4}}{(1-\frac{1}{10e^4})\Gamma(\frac{5}{2})}\left(\frac{2-2^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}} \approx 0.0035008 \\
&< 1,
\end{aligned}$$

is satisfied with $T=4, a=2$ and $\alpha=\frac{3}{2}$. Hence all conditions of Theorem 2.3 are satisfied, it follows that the problem (2.21) admit a unique solution defined on I.

Example 2: Consider the boundary value problem of implicit generalized Caputo fractional differential equation:

$$\begin{cases} x(t) = e^t - 1, \quad t \in [-1, 0], \\ \\ {}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+}x(t) = \dfrac{\sin(2t)\left(2 + |x^t| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+}x(t)\right|\right)}{20e^{t+4}\left(1 + |x^t| + \left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+}x(t)\right|\right)}, \qquad t \in I = [0, e] \\ \\ x(t) = \ln(t) - 1, \quad t \in [e, 4], \end{cases} \tag{2.22}$$

with

$$f(t, u, v) = \frac{\sin(2t)\,(2 + |u| + |v|)}{10e^{t+2}(1 + |u| + |v|)}, \quad k(t, u) = 0 \;\; t \in I = [0, e], u \in C([-r, \beta]) \;\text{ and } v \in \mathrm{IR}$$

$$\alpha = \frac{3}{2}, \rho = \frac{1}{2}, r = 1, \beta = 4 - e.$$

Condition (H_4) is satisfied for each $u, \in C([-r, \beta])$, $v \in \mathrm{IR}$ and $t \in [0, e]$:

$$\begin{aligned} |f(t, u, v)| &\leq \frac{2 + |u| + |v|}{20e^{t+4}} \\ &\leq \frac{1}{20e^{t+4}}\left(2 + |v| + \|u\|_{[-r,\beta]}\right). \end{aligned}$$

Therefore, (H_4) is verified with

$$d(t) = \frac{1}{10e^{t+4}}, \quad q(t) = m(t) = \frac{1}{20e^{t+4}} \quad \textit{and} \quad m^* = \frac{1}{20e^4} < 1.$$

Condition:

$$\begin{aligned} \frac{q^*\widetilde{G}}{(1 - m^*)} &\leq 2\frac{\frac{1}{20e^4}}{(1 - \frac{1}{20e^4})\Gamma(\frac{5}{2})}\left(\frac{e^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}} \approx 0.0082575 \\ &< 1, \end{aligned}$$

is satisfied with $T = e, a = 0$ and $\alpha = \frac{3}{2}$. Hence all conditions of Theorem 2.5 are satisfied, it follows that the problem (2.22) has at least one solution on I.

Chapter 3

Nonlinear Neutral IFDE with Retarded and Advanced Arguments

3.1 Introduction

In this chapter, we establish, the existence of solutions for implicit generalized Caputo fractional differential equations in Banach space with retarded and advanced arguments.

3.2 Existence Results for the NIFDE with Retarded and Advanced Arguments in Banach Spaces

$$ {}_{c}^{\rho}D_{a^+}^{\alpha}\left(x(t)-k(t,x^t)\right)=f(t,x^t,{}_{c}^{\rho}D_{a^+}^{\alpha}x(t)),\ t\in I:=[a,T],\ 1<\alpha\leq 2, \tag{3.1}$$

$$ y(t)=\phi(t),\ t\in[a-r,a],\ r>0 \tag{3.2}$$

$$ y(t)=\psi(t),\ t\in[T,T+\beta],\ \beta>0, \tag{3.3}$$

where ${}_{c}^{\rho}D_{a^+}^{\alpha}$ is the generalized Caputo fractional derivative, $(E,\|\cdot\|)$ is a real Banach space and $f: I\times C([-r,\beta],E)\times E\to E$ is a given function, $\phi\in C([a-r,a],E)$ with $\phi(a)=0$ and $\psi\in C([T,T+\beta,E)$ with $\psi(T)=0$.
We denote by x^t the element of $C([-r,\beta])$ defined by:

$$x^t(s)=x(t+s): s\in[-r,\beta]$$

here $x^t(\cdot)$ represents the history of the state from time $t-r$ up to time $t+\beta$.

To prove the existence of solutions to (3.1)–(3.3), we need the following auxiliary Lemma.

Lemma 3.1 *Let $f : I \times C[-r,\beta] \times E \longrightarrow E$ be a continuous function. A function $x \in C$ is solution of problem $(3.1)-(3.3)$ if and only if x satisfies the following integral equation*

$$x(t) = \begin{cases} \phi(t), \ \ if \quad t \in [a-r,a], \\ \\ k(t,x^t) - \displaystyle\int_a^T G(t,s)h(s)ds, \ \ if \quad t \in I \\ \\ \psi(t), \ \ if \quad t \in [T,T+\beta], \end{cases}$$

where $h \in C(I)$ satisfies the functional equation

$$h(t) = f(t,x^t,h(t)),$$

and

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \leq s \leq t \leq T, \\ \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)}, & a \leq t \leq s \leq T. \end{cases}$$

The following hypotheses will be used in the sequel:

(H_1) The function $f : I \times C[-r,\beta] \times E \longrightarrow E$ and $k : I \times C[-r,\beta] \longrightarrow E$ are continuous.

(H_2) There exist $d, q, m \in C(I, \mathrm{I\!R})$ with $m^* = \sup_{t\in I} m(t) < 1$ such that

$$\|f(t,u,v)\| \leq d(t) + q(t)\|u\|_{[-r,\beta]} + m(t)\|v\|, \ u \in C([-r,\beta],E), \ v \in E, \ t \in I,$$

and

$$\|k(t,u)\| \leq b(t) \text{ for a.e. } t \in I, \quad \text{and each} \quad u \in C([-r,\beta],E)$$

(H_3) for each bounded set $B \subset \mathcal{C}$, and for each $t \in I$, we have

$$\alpha(f(t,B_1,B_2)) \leq q(t) \sup_{t\in[-r,\beta]} \alpha(B_1) + m(t) \sup_{t\in[-r,\beta]} \alpha(B_2),$$

and

$$\alpha(k(t,B_1)) \leq b(t) \sup_{t\in[-r,\beta]} \alpha(B_1),$$

for any bounded sets, $B_1 \subset C([-r,\beta]), B_2 \subset E$.
Set

$$q^* = \sup_{t\in I} q(t), \ \ m^* = \sup_{t\in I} m(t), b^* = \sup_{t\in I} b(t), \ \ \widetilde{G} = \sup\left\{ \int_a^T |G(t,s)|ds, \ t \in I \right\}.$$

(H_4) For each bounded set D_N in $\mathcal{C}$, the set $\{t \longrightarrow k(t, x^t) : x \in D_N\}$ is equicontinuous in $C(I, E)$

We prove an existence result for the (3.1)-(3.3) problem, by using the Darbo fixed point theorem.

Theorem 3.2 *Assume that the hypotheses $(H_1) - (H_3)$ hold. If*

$$b^* + \frac{q^* \widetilde{G}}{1 - m^*} < 1, \tag{3.4}$$

then problem (3.1)-(3.3) has at least one solution.

Proof. Let the operator $N : \mathcal{C} \longrightarrow \mathcal{C}$ defined by

$$(Nx)(t) = \begin{cases} \phi(t), \text{ if } \quad t \in [a - r, a], \\ \\ g_x(s) - \displaystyle\int_a^T G(t, s) h_x(s) ds, \text{ if } \quad t \in I \\ \\ \psi(t), \text{ if } \quad t \in [T, T + \beta]. \end{cases} \tag{3.5}$$

By Lemma 3.1 it is clear that the fixed points of N are solutions (3.1)-(3.3).
Step 1: N is continuous. Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathcal{C}$. If $t \in [a-r, a]$ or $t \in [T, T + \beta]$ then

$$\|(Nx_n)(t) - (Nx)(t)\| = 0.$$

For $t \in I$, we have

$$\|(Nx_n)(t) - (Nx)(t)\| \leq \|g_n(s) - g(s)\| - \int_a^T |G(t, s)| \|h_n(s) - h(s)\| ds, \tag{3.6}$$

where

$$h_n(t) = f(t, x_n^t, h_n(t))$$

and

$$g_n(t) = k(t, x_n^t).$$

Since $x_n \longrightarrow x$, bx (H_1) we get $h_n(t) \longrightarrow h(t)$ as $n \longrightarrow \infty$ for each $t \in I$.
And let $\eta > 0$, such that, for each $t \in I$, we have $\|h_n(t)\| \leq \eta$ and $\|h(t)\| \leq \eta$.
Therefore

$$\begin{aligned} |G(t, s)| \|h_n(t) - h(t)\| &\leq |G(t, s)| \left[\|h_n(t)\| + \|h(t)\|\right] \\ &\leq 2\eta |G(t, s)|. \end{aligned}$$

For each $t \in I$ the function $s \longmapsto 2\eta|G(t,s)|$ is integrable on $[a,t]$, then by Lebesgue dominated convergence theorem, equation (3.6) implies

$$\|(Nx_n)(t) - (Nx)(t)\| \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

and hence

$$\|N(x_n) - N(x)\|_{\mathcal{C}} \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Thus N is continuous.

Let the constant R be such that:

$$R \geq \max\left\{b^* + A\widetilde{G}, \|\phi\|_{[a-r,a]}, \|\psi\|_{[T,T+\beta]}\right\}, \tag{3.7}$$

and define

$$D_R = \{x \in \mathcal{C} : \|x\|_{\mathcal{C}} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of C.

Step 2: N maps D_R into itself.
Let $x \in D_R$ we show that $Nx \in D_R$.
If $t \in [a-r,a]$, then

$$\|N(x)(t)\| \leq \|\phi\|_{[a-r,a]} \leq R,$$

and if $t \in [T, T+\beta]$, then

$$\|N(x)(t)\| \leq \|\psi\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$\|(Nx)(t)\| \leq \|k(t,x^t)\| + \int_a^T |G(t,s)|\|h(s)\|ds.$$

By (H_2) we have for each $t \in I$

$$\begin{aligned} \|h(t)\| &\leq d(t) + q(t)\|x\|_{[-r,\beta]} + m(t)\|h(t)\| \\ &\leq d^* + q^*\|x\|_{[-r,\beta]} + m^*\|h(t)\| \\ &\leq d^* + q^*R + m^*\|h(t)\|, \end{aligned}$$

and

$$\begin{aligned} \|k(t,u)\| &\leq b(t) \\ &\leq b^* \end{aligned}$$

where

$$b^* = \sup_{t\in I} b(t), \quad d^* = \sup_{t\in I} d(t), \quad q^* = \sup_{t\in I} q(t) \ \ and \ \ m^* = \sup_{t\in I} m(t).$$

Then

$$\|h(t)\| \leq \frac{d^* + q^* R}{1 - m^*} = A. \tag{3.8}$$

By (3.8), for $t \in I$, we have

$$\begin{aligned} \|(Nx)(t)\| &\leq b^* + A \int_a^T |G(t,s)| ds \\ &\leq b^* + A\widetilde{G} \\ &\leq R, \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $\|Nx(t)\| \leq R$, which implies that $\|Nx\|_{[-r;\beta]} \leq R$. This proves that N transforms the set D_R into itself.
Step 3: $N(D_R)$ is bounded and equicontinuous.
Since $N(D_R) = \{N(x) : x \in D_R\} \subset D_R$ and D_R is bounded, then $N(D_R)$ is bounded.
Now, let $t_1, t_2 \in I = [a, T], t_1 < t_2,\ and\ x \in D_R$ then

$$\begin{aligned} \|(Nx)(t_2) - (Nx)(t_1)\| &\leq \|k(t_2, x^{t_2}) - k(t_1, x^{t_1})\| + \int_a^T |G(t_2, s) - G(t_1, s)| \|h(s)\| ds \\ &\leq \|k(t_2, x^{t_2}) - k(t_1, x^{t_1})\| + A \int_a^T |G(t_2, s) - G(t_1, s)| ds. \end{aligned}$$

By (H_4) as $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero.
Step 4: The operator $N : D_R \longmapsto D_R$ is a strict set contraction.
Let $V \subset D_R$ if t $[a-r, a]$, then

$$\begin{aligned} \alpha(N(V)(t)) &= \alpha(N(x)(t), x \in V) \\ &= \alpha(\phi(t)) \\ &= 0, \end{aligned}$$

also if t $[T, T+\beta]$, then

$$\begin{aligned} \alpha(N(V)(t)) &= \alpha(N(x)(t), x \in V) \\ &= \alpha(\psi(t)) \\ &= 0. \end{aligned}$$

And if $t \in I$, we have

$$\begin{aligned} \alpha(N(V)(t)) &= \alpha(N(x)(t), x \in V) \\ &\leq \left\{ \alpha(g(t)) + \int_a^T |G(t,s)| \alpha(h(s)) ds, x \in V \right\}. \end{aligned}$$

By (H_3) we have

$$\begin{aligned}\alpha(h(s), x \in V) &= \alpha(\{f(s, x(s), h(s)), x \in V\}) \\ &\leq q(t)\alpha(\{x(s), x \in V\}) + m(t)\alpha(\{h(s), x \in V\}) \\ &\leq q^*\alpha(\{x(s), x \in V\}) + m^*\alpha(\{h(s), x \in V\}),\end{aligned}$$

and

$$\begin{aligned}\alpha(g(t), x \in V) &= \alpha(\{k(t, x(t)), x \in V\}) \\ &\leq b(t)\alpha(\{x(s), x \in V\}) \\ &\leq b^*\alpha(\{x(s), x \in V\}).\end{aligned}$$

Then

$$\alpha(\{h(s), x \in V\}) \leq \frac{q^*}{1 - m^*}\alpha(\{x(s), x \in V\}). \tag{3.9}$$

Thus

$$\begin{aligned}\alpha(N(V)(t)) &\leq b^*\alpha(\{x(s), x \in V\}) + \frac{q^*}{1 - m^*}\int_a^T |G(t, s)|\alpha(\{x(s), x \in V\})ds \\ &\leq \left(b^* + \frac{q^*\widetilde{G}}{1 - m^*}\right)\alpha_c(V).\end{aligned}$$

Therefore

$$\alpha_c(NV) \leq \left(b^* + \frac{q^*\widetilde{G}}{1 - m^*}\right)\alpha_c(V).$$

So by (3.4) the operator N is a set contraction. And thus, by Theorem 1.17, N has a fixed point, which is solution to problem (3.1) – (3.3) .

We prove an existence result for the (3.1)-(3.3) problem, by using the Mönch's fixed point theorem.

Theorem 3.3 *Assume that* $(H_1) - (H_3)$ *hold. If*

$$b^* + \frac{q^*\widetilde{G}}{1 - m^*} < 1, \tag{3.10}$$

then problem (3.1) – (3.3) *has at least one solution.*

Proof: Consider the operator N defined in (3.5). According to Theorem 3.2, the operator $N(D_R)$ is bounded into itself, and equicontinuous.
Now let V be a subset of D_R such that $V \subset conv(N(V) \cup \{0\})$. Since V is bounded and

equicontinuous, the function $t \longmapsto v(t) = \alpha(V(t))$ is continuous on $[a-r, T+\beta]$. By $(H_1)-(H_3)$, Lemma 1.12, and the properties of measure α, for each $t \in I$, we have

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha\left(\{(Nx)(t), x \in V\}\right) \\ &\leq b^*\alpha_c(V) + \int_a^T |G(t,s)| \frac{q^*}{1-m^*}\alpha(V(s))ds \\ &\leq \left(b^* + \frac{q^*\widetilde{G}}{1-m^*}\right)\alpha_c(V). \end{aligned}$$

Thus

$$\alpha_c(V) \leq \left(b^* + \frac{q^*\widetilde{G}}{1-m^*}\right)\alpha_c(V).$$

From (3.10), we get $\alpha_c(V) = 0$, that is $\alpha(V(t)) = 0$ for each $t \in I$.
For $t \in [a-r, a]$, we have

$$\begin{aligned} v(t) &= \alpha(\phi(t)) \\ &= 0. \end{aligned}$$

Also for $t \in [T, T+\beta]$ we have

$$\begin{aligned} v(t) &= \alpha(\psi(t)) \\ &= 0, \end{aligned}$$

then $V(t)$ is relatively compact in E. In view of Ascoli-Arzela theorem, V is relatively compact in D_R. Applying Theorem 1.18, we conclude that N has a fixed point which is a solution of the problem (3.1) – (3.3).

An Example :

Let

$$E = l^1 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots), \sum_{k=1}^{\infty} |x_n| < \infty \right\},$$

be the Banach space with the norm

$$\|x\|_E = \sum_{k=1}^{\infty} |x_n|.$$

Consider the boundary value problem of implicit generalized Caputo fractional differential equation

$$\begin{cases} x(t) = \ln(t) - 1, \quad t \in [e, 4], \\ \\ {}^{3}_{c}D^{\frac{3}{2}}_{2^+} x_n(t) = f(t, x^t_n, {}^{3}_{c}D^{\frac{3}{2}}_{2^+} x_n(t)), \quad t \in I = [2, e] \\ \\ x(t) = \frac{1}{2}t - 1, \quad t \in [-1, 2], \end{cases} \tag{3.11}$$

here $T = e, \quad a = 2, \quad \alpha = \dfrac{3}{2}, \quad \rho = 3.$
Set

$$x = (x_1, x_2, \ldots, x_n, \ldots), \quad f = (f_1, f_2, \ldots, f_n, \ldots)$$

$$f(t, x^t, {}^3_c D^{\frac{3}{2}}_{2^+} x(t)) = \frac{cos(t) + \|x_t\|_{C([-3,4-e])} + \|{}^3_c D^{\frac{3}{2}}_{2^+} x(t)\|}{2e^{t-2}\left(1 + \|x_t\|_{C([-3,4-e])} + \|{}^3_c D^{\frac{3}{2}}_{2^+} x\|_E\right)}, \quad k(t, x^t)$$

For each $x \in E$ and $t \in [2, e]$, we have

$$\|f(t, x(t), {}^3_c D^{\frac{3}{2}}_{2^+} x(t))\| \leq \frac{1}{2e^{t-2}} \left(cos(t) + \|x_t\|_{C([-3,4-e])} + \|{}^3_c D^{\frac{3}{2}}_{2^+} x(t)\|\right),$$

hence. (H_2) is satisfied with $m^* = q^* = \frac{1}{2}$.
For each $t \in I$ we have

$$\int_a^T |G(t, s)| ds \leq \frac{1}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right) \int_a^T \left| \left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha - 1} s^{\rho - 1} \right| ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^t \left| \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha - 1} s^{\rho - 1} \right| ds,$$

then

$$\int_a^T |G(t, s)| ds \leq \frac{2}{\Gamma(\nu + 1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^\nu.$$

Therefore

$$\widetilde{G} \leq \frac{2}{\Gamma(\nu + 1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^\nu.$$

Condition (3.4) holds, indeed,

$$\begin{aligned} \frac{q^* \widetilde{G}}{1 - m^*} &\leq \frac{2}{\Gamma(\frac{3}{2} + 1)} \left(\frac{e^{\frac{3}{2}} - 2^{\frac{3}{2}}}{3}\right)^{\frac{3}{2}} \approx 0.61549 \\ &< 1. \end{aligned}$$

Hence all conditions of Theorem 3.2 are satisfied. It follows that the problem (3.11) has at least one solution.

Chapter 4

Random Solutions For Mixed Fractional Differential Equations with Retarded and Advanced Arguments

4.1 Introduction

In this chapter, we study the existence of random solutions for a class of problem involving both generalized Caputo and generalized Riemann-Liouville fractional derivatives differential equations with retarded and advanced arguments:

$$ {}^{RL}D^{\alpha,\rho}_{a^+}({}^{C}D^{\delta,\rho}_{a^+}(x(t,w)-\sum_{i=1}^{m}({}^{\rho}I^{\nu_i}_{a}g_i(t,x^t(w),w)))=f(t,x^t(w),w),\ \text{for}\ \ t\in I:=[a,T], \tag{4.1}$$

$$x(t,w)=\phi(t,w),\ t\in[a-r,a],\ r>0 \tag{4.2}$$

$$x(t,w)=\psi(t,w),\ t\in[T,T+\beta],\ \beta>0, \tag{4.3}$$

where $0<\alpha,\delta\leq 1$, ${}^{RL}D^{\alpha,\rho}_{a^+}$, ${}^{C}D^{\delta,\rho}_{a^+}$ is the generalized RL and generalized Caputo and (as it is, respectively) fractional derivative, ${}^{\rho}I^{\nu_i}_{a^+}$ represents the generalized RL integral with $\nu_i>0$ $f:I\times C([-r,\beta],\mathbb{R})\times\Omega\to\mathbb{R}$ and $g_i:I\times C([-r,\beta],\mathbb{R})\times\Omega\to\mathbb{R}$, with $g_i(T,x^T(w))=g_i(a,x^a(w))=0$, $i=1,2,\ldots,m$, is a given function, $\phi\in C([a-r,a],\mathbb{R})$ with $\phi(a)=0$ and $\psi\in C([T,T+\beta],\mathbb{R})$ with $\psi(T)=0$. We denote by x^t the element of $C([-r,\beta])$.
we denote by x^t the element of $C([-r,\beta])$ defined by:

$$x^t(s)=x(t+s),\quad s\in[-r,\beta].$$

4.2 Existence of Solutions

Lemma 4.1 *Let $1 < \alpha \leq 2$, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a)=0$, $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T)=0$ and $h : I \to \mathbb{R}$ be a continuous function. Then the linear problem*

$$^{RL}D^{\alpha,\rho}_{a^+}(^{C}D^{\delta,\rho}_{a^+}x(t)-\sum_{i=1}^{m}(^{\rho}I_a^{\nu_i}g_i(t,x^t)))=h(t),\ \text{for a.e. } t\in I:=[a,T],\ 1<\alpha\leq 2, \quad (4.4)$$

$$x(t)=\phi(t),\ t\in[a-r,a],\ r>0 \quad (4.5)$$

$$x(t)=\psi(t),\ t\in[T,T+\beta],\ \beta>0, \quad (4.6)$$

has a unique solution, which is given by

$$x(t)=\begin{cases}\phi(t),\ \text{if} \quad t\in[a-r,a],\\ \\ \sum_{i=1}^{m}(^{\rho}I_a^{\nu_i}g_i(t,x^t,w))-\displaystyle\int_a^T G(t,s)h(s)ds,\ \text{if} \quad t\in I\\ \\ \psi(t),\ \text{if} \quad t\in[T,T+\beta],\end{cases} \quad (4.7)$$

where

$$G(t,s)=\frac{\rho^{\alpha+\delta-1}s^{\rho-1}}{\Gamma(\alpha+\delta)}\begin{cases}\left(\dfrac{(t^\rho-a^\rho)(T^\rho-s^\rho)}{(T^\rho-a^\rho)}\right)^{\alpha+\delta-1}-(t^\rho-s^\rho)^{\alpha+\delta-1}, & a\leq s\leq t\leq T,\\ \\ \left(\dfrac{(t^\rho-a^\rho)(T^\rho-s^\rho)}{(T^\rho-a^\rho)}\right)^{\alpha+\delta-1}, & a\leq t\leq s\leq T.\end{cases} \quad (4.8)$$

Here $G(t,s)$ is called the Green function of the boundary value problem (4.4)-(4.6).
Proof. *To obtain the integral equation modeled by the BVP(4.7), we apply the generalization Riemann-Liouville fractional integral of order to both sides of (1.5), and we get*

$$^{c}D^{\delta}_{a^+}(x(t)-\sum_{i=1}^{m}(^{\rho}I_a^{\nu_i}g_i(t,x^t))={}^{\rho}I^{\alpha}_{a^+}h(s)+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}. \quad (4.9)$$

Next, applying the generalization Caputo fractional integral of order β to both sides of (1.4)*, we have*

$$x(t)-\sum_{i=1}^{m}(^{\rho}I_a^{\nu_i}g_i(t,x^t)={}^{\rho}I^{\alpha+\delta}_{a^+}h(s)+c_1^{\rho}I^{\delta}_{a^+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}+c_2, \quad (4.10)$$

where c_1 and $c_2\in\mathbb{R}$. By using lemma (1.6) , we get

$$x(t) = \sum_{i=1}^{m} ({}^{\rho}I_a^{\nu_i} g_i(t, x^t(w)) +^{\rho} I_{a^+}^{\alpha+\delta} h(s) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\delta)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\delta+\alpha-1} + c_2, \qquad (4.11)$$

therefore

$$x(a) = c_2 = 0,$$

$$x(T) = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\delta)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha+\delta-1} + \frac{1}{\Gamma(\alpha+\delta)} \int_a^T \left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha+\delta-1} s^{\rho-1} h(s) ds,$$

and

$$c_1 = -\frac{\rho^{\delta+\alpha-1}}{(T^\rho - a^\rho)^{\delta+\alpha-1} \Gamma(\alpha)} \int_a^T (T^\rho - s^\rho)^{\delta+\alpha-1} s^{\rho-1} h(s) ds.$$

Substitute the value of c_1 *and* c_2 *into equation (4.11), we get equation (4.7).*

$$x(t) = \begin{cases} \phi(t), \ \ if \quad t \in [a-r, a], \\ \\ \sum_{i=1}^{m} ({}^{\rho}I_a^{\nu_i} g_i(t, x^t(w)) - \displaystyle\int_a^T G(t,s) h(s) ds, \ \ if \quad t \in I \\ \\ \psi(t), \ \ if \quad t \in [T, T+\beta], \end{cases}$$

where G is defined by equation (4.8), the proof is complete.

Lemma 4.2 *Let* $f : I \times C[-r, \beta] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function.*

A function $x \in \mathcal{C}$ *is a random solution of problem* $(4.1) - (4.3)$ *if and only if* x *satisfies the following integral equation*

$$x(t,w) = \begin{cases} \phi(t,w), \ \ if \quad t \in [a-r, a], \\ \\ \sum_{i=1}^{m} ({}^{\rho}I_a^{\nu_i} g_i(t, x^t(w)) - \displaystyle\int_a^T G(t,s) h(s,w) ds, \ \ if \quad t \in I \\ \\ \psi(t,w), \ \ if \quad t \in [T, T+\beta], \end{cases}$$

where $h \in C(I)$ *satisfies the functional equation*

$$h(t) = f(t, x^t, w).$$

The following hypotheses will be used in the sequel:

(H_1) The function $f, g_i : I\times C[-r,\beta]\times\Omega \longrightarrow \mathbb{R}$, $i = 1, 2, \ldots, m$ are random Caratheodory.

(H_2) There exist measurable functions $p, b_i \;\; I :\longrightarrow L^\infty(\Omega, \mathbb{R}_+)$, $i = 1, 2, \ldots, m$

$$|f(t, u_1, w) - f(t, u_2, w)| \le p(t, w)\|u_1 - u_2\|_{[-r,\beta]},$$

and

$$|g(t, u_1, w) - g(t, u_2, w)| \le b_i(t, w)\|u_1 - u_2\|_{[-r,\beta]},$$

for $t \in I$, $w \in \Omega$ and each $u_i, v_i \in \mathbb{R}$, $\;i = 1, 2$

(H_3) There exist measurable functions $p, k_i \;\; I :\longrightarrow L^\infty(\Omega, \mathbb{R}_+)$, $i = 1, 2, \ldots, m$ such that

$$|f(t, x, w)| \le p(t, w)(\|x\|_{[-r,\beta]} + 1), t \in I, x \in C([-r, \beta], \mathbb{R}) \;\; w \in \Omega.$$

and

$$|g_i(t, x, w)| \le k_i(t, w)\|x\|_{[-r,\beta]}, t \in I, x \in C([-r, \beta], \mathbb{R}) and \;\; w \in \Omega.$$

Set

$$p^* = ess \sup_{t\in I} p(t)$$

$$\widetilde{G} = \sup\left\{\int_a^T |G(t, s)|ds, t \in I\right\}.$$

Now, we state and prove our existence result for Equations (4.1)-(4.3) based on the Banach contraction principle.

Theorem 4.3 *Assume (H_1) and (H_2) hold. If*

$$\sum_{i=1}^{m} \frac{b_i(w)}{\Gamma(\nu_i + 1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu_i} + \widetilde{G}p^*(\cdot, w) < 1, \tag{4.12}$$

then the problem (4.1)-(4.3) has a unique solution.

Proof: Let the operator $T : \mathcal{C} \longrightarrow \mathcal{C}$ defined by

$$(Tx)(t, w) = \begin{cases} \phi(t), \text{ if } \;\; t \in [a - r, a], \\ \\ \sum_{i=1}^m ({}^\rho I_a^{\nu_i} g_i(t, x^t(w)) - \displaystyle\int_a^T G(t, s)h_x(s, w)ds, \text{ if } \;\; t \in I \\ \\ \psi(t), \text{ if } \;\; t \in [T, T + \beta]. \end{cases} \tag{4.13}$$

By Lemma 4.2 it is clear that the fixed points of T are solutions (4.1)-(4.3) .
Let $x_1, x_2 \in \Omega$. If $t \in [a-r,a]$ or $t \in [T, T+\beta]$ then

$$|(Tx_1)(t,w) - (Tx_2)(t,w)| = 0.$$

For $t \in I$, we have

$$|(Tx_1)(t,w)-(Tx_2)(t,w)| \le \sum_{i=1}^{m}({}^\rho I_a^{\nu_i}|g_i(t,x_1^t(w))-g_i(t,x_2^t(w))|+\int_a^T |G(t,s)||h_{x_1}(s)-h_{x_2}(s)|ds, \tag{4.14}$$

by (H_2) we obtain,

$$\begin{aligned} |(Tx_1)(t,w) - (Tx_2)(t,w)| &\le \sum_{i=1}^{m}({}^\rho I_a^{\nu_i} b_i(t,w)\|x_1 - x_2\|_{[-r,\beta]}) + p^*(\cdot,w)\int_a^T |G(t,s)|\|x_1 - x_2\|_{[-r,\beta]}ds \\ &\le \left(\sum_{i=1}^{m}\frac{b_i^*(\cdot,w)}{\Gamma(\nu_i+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu_i} + \widetilde{G}p^*(\cdot,w)\right)\|x_1 - x_2\|_{[-r,\beta]}. \end{aligned}$$

Therefore, for each $t \in I$, we have

$$|(Tx_1)(t,w) - (Tx_2)(t,w)| \le \left(\sum_{i=1}^{m}\frac{b_i^*(\cdot,w)}{\Gamma(\nu_i+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu_i} + \widetilde{G}p^*(\cdot,w)\right)\|x_1 - x_2\|_C.$$

Thus

$$\|Tx_1(\cdot,w) - Tx_2(\cdot,w)\|_C \le \left(\sum_{i=1}^{m}\frac{b_i(w)}{\Gamma(\nu_i+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu_i} + \widetilde{G}p^*(\cdot,w)\right)\|x_1 - x_2\|_C.$$

Hence, by the Banach contraction principle, T has a unique fixed point which is a unique random solution of the problem (4.1)-(4.3).

We now prove an existence result for (4.1)-(4.3) by using the Schauder's fixed point theorem.

Theorem 4.4 *Assume that the hypotheses (H_1) and (H_3) hold. Then problem (4.1)-(4.3) has at least one solution.*

Step 1. T is continuous. Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in $\mathcal{C}$. If $t \in [a-r,a]$ or $t \in [T, T+\beta]$ then

$$|(Tx_n)(t,w) - (Tx)(t,w)| = 0.$$

For $t \in I$, we have

$$\begin{aligned} |(Tx_n)(t,w) - (Tx)(t,w)| \le &\sum_{i=1}^{m}({}^\rho I_a^{\nu_i}|g_i(t,x_n^t(w),w) - g_i(t,x^t(w),w)|) \\ &+\int_a^T |G(t,s)||h_n(s,w) - h(s,w)|ds, \end{aligned} \tag{4.15}$$

where

$$h_n(t) = f(t, x_n^t, w),$$

and

$$h(t) = f(t, x^t, w).$$

Since $x_n \longrightarrow x$, and by (H_1), we get $h_n(t) \longrightarrow h(t)$ and $g_i(t, x_n^t, w) \longrightarrow g_i(t, x^t, w)$, $i = 1, 2, \ldots, m$ as $n \longrightarrow \infty$ for each $t \in I$.
By (H_3) we have for each $t \in I$,

$$|h_n(t)| \leq l_1, \quad |g_n(t)| \leq l_2. \tag{4.16}$$

Then,

$$\begin{aligned} |G(t,s)||h_n(t) - h(t)| &\leq |G(t,s)|\left[|h_n(t)| + |h(t)|\right] \\ &\leq 2l|G(t,s)|, \end{aligned}$$

and

$$\begin{aligned} \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu_i}\right| |g_i(t, x_n^t(w), w) - g_i(t, x^t(w), w)| &\leq \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu_i}\right| \left[|g_i(t, x_n^t(w), w)|\right. \\ &+ \left.|g_i(t, x^t(w), w)|\right] \\ &\leq 2l_2 \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu_i}\right|. \end{aligned}$$

For each $t \in I$ the functions $s \longmapsto 2l|G(t,s)|$ and $s \longmapsto 2l_2 \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\nu_i}\right|$ are integrable on $[a, t]$, then by Lebesgue dominated convergence theorem and (H_1), equation (4.15) implies

$$|(Tx_n)(t, w) - (Tx)(t, w)| \longrightarrow 0 \; as \; n \longrightarrow \infty,$$

and hence

$$\|T(x_n) - T(x)\|_{\mathcal{C}} \longrightarrow 0 \; as \; n \longrightarrow \infty.$$

Consequently, T is continuous.

Let the constant $R(w)$ be such that:

$$R(x) \geq \max \left\{ \frac{p^* \widetilde{G}}{1 - \left(\sum_{i=1}^m \frac{k_i(w)}{\Gamma(\nu_i + 1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu_i} + p^* \widetilde{G} \right)}, \|\phi(\cdot, w)\|_{[a-r,a]}, \|\psi(\cdot, w)\|_{[T,T+\beta]} \right\}, \tag{4.17}$$

and define

$$B_{R(w)} = \{x \in \Omega : \|x(\cdot, w)\|_{\mathcal{C}} \leq R(w)\}.$$

It is clear that $D_R(w)$ is a bounded, closed and convex subset of $\mathcal{C}$.

Step 2. $T(B_R(w)) \subset B_R(w)$.

Let $x \in B_R(w)$ we show that $Tx \in B_R(w)$.
If $t \in [a-r, a]$, then

$$|T(x)(t,w)| \leq \|\phi(\cdot,w)\|_{[a-r,a]} \leq R(w),$$

and if $t \in [T, T+\beta]$, then

$$|T(x)(t,w)| \leq \|\psi(\cdot,w)\|_{[T,T+\beta]} \leq R.$$

For each $t \in I$, we have

$$|(Tx)(t,w)| \leq \sum_{i=1}^{m}({}^\rho I_a^{\nu_i}|g_i(t,x^t(w),w)|) + \int_a^T |G(t,s)||h(s,w)|ds.$$

By (H_3), we have

$$\begin{aligned} |(Tx)(t,w)| &\leq \sum_{i=1}^{m}({}^\rho I_a^{\nu_i}k_iR(w)) + p^*(R(w)+1)\int_a^T |G(t,s)|ds \\ &\leq R(w)\left(\sum_{i=1}^{m}\frac{k_i(w)}{\Gamma(\nu_i+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\nu_i} + p^*\widetilde{G}\right) + p^*\widetilde{G} \\ &\leq R(w), \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $|Tx(t)| \leq R(w)$, which implies that $\|Tx\|_{\mathcal{C}} \leq R(w)$. Consequently,

$$T(B_R(w)) \subset B_R(w).$$

Step 3: $T(B_R(w))$ is bounded and equicontinuous.
By Step 2 we have $T(B_R(w))$ is bounded.

Let $\tau_1, \tau_2 \in I = [a, T], \tau_1 < \tau_2,\ and\ x(\cdot, w) \in B_R(w)$ then

$$\begin{aligned}
|(Tx)(\tau_2, w) - (Tx)(\tau_1, w)| &\leq \sum_{i=1}^{m} |{}^{\rho}I_a^{\nu_i}(g_i(\tau_2, x^{\tau_2}(w), w) - g_i(\tau_1, x^{\tau_1}(w), w))| \\
&+ \int_a^T |G(\tau_2, s) - G(\tau_1, s)||h(s, w)|ds \\
&\leq R(w) \sum_{i=1}^{m} \left| \frac{k_i(w)}{\Gamma(\nu_i + 1)} \int_a^{\tau_2} \left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\nu_i - 1} s^{\rho-1} ds \right. \\
&- \left. \frac{k_i(w)}{\Gamma(\nu_i + 1)} \int_a^{\tau_1} \left(\frac{\tau_1^{\rho} - s^{\rho}}{\rho}\right)^{\nu_i - 1} s^{\rho-1} ds \right| \\
&+ p^*(R(w) + 1) \int_a^T |G(\tau_2, s) - G(\tau_1, s)|ds \\
&\leq R(w) \sum_{i=1}^{m} \left| \frac{k_i(w)}{\Gamma(\nu_i + 1)} \int_a^{\tau_1} \left(\left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\nu_i - 1} - \left(\frac{\tau_1^{\rho} - s^{\rho}}{\rho}\right)^{\nu_i - 1} \right) s^{\rho-1} ds \right. \\
&+ \left. \frac{k_i(w)}{\Gamma(\nu_i + 1)} \int_{\tau_1}^{\tau_2} \left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\nu_i - 1} s^{\rho-1} ds \right| \\
&+ p^*(R(w) + 1) \int_a^T |G(\tau_2, s) - G(\tau_1, s)|ds \\
&= R(w) \sum_{i=1}^{m} \left| \frac{k_i(w)}{\Gamma(\nu_i + 1)} \left(2 \left(\frac{\tau_2^{\rho} - \tau_1^{\rho}}{\rho}\right)^{\nu_i} + \left(\frac{\tau_1^{\rho} - a^{\rho}}{\rho}\right)^{\nu_i} \right. \right. \\
&- \left. \left. \left(\frac{\tau_2^{\rho} - a^{\rho}}{\rho}\right)^{\nu_i} \right) \right| \\
&+ p^*(R(w) + 1) \int_a^T |G(\tau_2, s) - G(\tau_1, s)|ds
\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$ the right hand side of the above inequality tends to zero. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that T is continuous and completely continuous. From Schauder's theorem, we conclude that T has a fixed point with is a random solution of the problem (4.1)-(4.3).

Example :
We equip the space $\mathbb{R}_-^* := (-\infty, 0)$ with the usual σ-algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Consider the boundary value problem of involving both generalized

Caputo and generalized Riemann-Liouville fractional differential equation:

$$\begin{cases} x(t,w) = \frac{1}{1+w^2}(t^2-1), \quad t \in [1,2], \\ {}^{LR}D_{0^+}^{\frac{3}{4},\rho}({}^{C}D_{0^+}^{\frac{3}{4},\rho}(x(t,w) - \sum_{i=1}^{2}({}^{\rho}I_a^{\nu_i} g_i(t,x^t(w))))) = \dfrac{\sin(t)(x^t(w)+1)}{100(w^2+1)} \quad t \in I = [0,1] \\ x(t) = \frac{t}{1+w}, \quad t \in [-1,0]. \end{cases} \tag{4.18}$$

Set

$$f(t,x^t(w)) = \frac{\sin(t)(x^t(w)+1)}{100(w^2+1)}, \quad t \in [0,1], u \in C([-r,\beta]),$$

and

$$g_i(t,x^t(w)) = \frac{\cos(t)x^t(w)}{10i(w^2+1)}, \quad t \in [0,1], u \in C([-r,\beta]), i = 1,2.$$

And $\nu_i = \frac{2i+1}{2}, \alpha = \frac{3}{4} = \delta, \rho = 1, r = 1, \beta = 1$. For each $x_1, x_2 \in C([-r,\beta])$, $v, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\begin{aligned} |f(t,x,w) - f(t,\bar{x},w)| &\leq \left| \frac{\sin(t)(x+1)}{100(w^2+1)} - \frac{\sin(t)(\bar{x}+1)}{100(w+1)} \right| \\ &\leq \frac{\sin(t)}{100(w^2+1)} \|x - \bar{x}\|_{[-1,1]}, \end{aligned}$$

$$\begin{aligned} |g_i(t,x,w) - g_i(t,\bar{x},w)| &\leq \left| \frac{\cos(t)(x)}{10i(w^2+1)} - \frac{\cos(t)(\bar{x})}{10i(w^2+1)} \right| \\ &\leq \frac{\cos(t)}{10i(w^2+1)} \|x - \bar{x}\|_{[-1,1]}, i = 1,2. \end{aligned}$$

Therefore, (H_2) is verified with $p^*(w) = \dfrac{1}{100(w^2+1)}$, $b_i(w) = \dfrac{1}{10i(w^2+1)}$

For each $t \in I$ we have

$$\int_a^T |G(t,s)| ds \leq \frac{1}{\Gamma(\alpha+\beta)} \left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho} \right)^{\alpha+\beta-1} \int_a^T \left| \left(\frac{T^\rho - s^\rho}{\rho} \right)^{\beta+\alpha-1} s^{\rho-1} \right| ds$$

$$+ \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \left| \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha+\beta-1} s^{\rho-1} \right| ds.$$

Then

$$\widetilde{G} = \int_a^T |G(t,s)| ds \leq \frac{2}{\Gamma(\alpha+\beta+1)} \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\beta}.$$

The condition

$$\begin{aligned}\sum_{i=1}^{m}\frac{b_i(w)}{\Gamma(\nu_i)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\nu_i}+\widetilde{G}p^{*}(\cdot,w) &\leq \frac{7}{50(w^2+1)\Gamma(\frac{5}{2})}\\ &= \frac{14}{75(w^2+1)\sqrt{\pi}}\\ &< 1,\end{aligned}$$

is satisfied with $T=1, a=0$ and $\alpha=\frac{1}{2}, \beta=1$. Hence all conditions of Theorem 4.3 are satisfied, it follows that the problem (4.18) admit a unique solution defined on I.

4.3 Conclusions and Perspective

In this work, we have presented some results to the theory of the existence of solutions, random solutions and uniqueness of fractional implicit differential equations with the derivatives of generalized-Caputo. The problem studied implicit fractional differential equations, involving both retarded and advanced arguments. The results obtained are based on some fixed point theorems and the measure of non-compactness. In future research, we plan to study some fractional differential and inclusions with impulses (instantaneous and not instantaneous) in frchet spaces

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