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Mönch−Krasnoselski fixed point theorem in Banach spaces and its applications to certain nonlinear problems

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Dedication

This dissertation is sincerely dedicated to my parents

who have been my source of inspiration, guide and give me strength,

when i thought of giving up, who continually provide their moral, spiritual, emotional, and financial support.

I also dedicate my dissertation to my siblings,

and all my family and friends for their love, support and prayers.

Zahia

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I also dedicate my dissertation to my brothers, and all my family and friends for their love, support and prayers.

Imane

Abstract

In this memory, we present the Mönch-Krasnoselski fixed point theorem in Banach space which is a generalisation of the classical Krasnoselski fixed point theorem. We also give some applications to classes of nonlinear integral equations in the space of continuous functions using the technique of noncompactness measures. Examples are provided to illustrate the obtained results.

Contents

Introduction

The fixed point theorems are important mathematical tools for demonstrating the existence and uniqueness of solution in various types of equations, they have played a major role in various fields, including functional analysis, dynamics, topology and differential equations.

Fixed point theorems concern maps f of a set X into itself. That under certain conditions admit a fixed point that is, a point $x \in X$ such that $f(x) = x$ see [12].

The fixed point theory is at the heart of nonlinear analysis because it provides the necessary tools for existence theorems in many nonlinear problems. It uses analysis and topology tools for this reason we have the classification "fixed point and topological theory" and "fixed point and metric theory" we will mentions some fixed point theorems like Banach, Schauder, Schaefer and Krasnoselski.

The development of fixed point theory is the most important branch of nonlinear analysis, it has had a profound impact on the progress of nonlinear analysis. In fact, many natural phenomena in chemistry, physics, mechanics, economics and biology exhibit nonlinear behavior. Mathematically, these problems are often expressed as nonlinear differential equations, making nonlinear analysis an ongoing active and relevant field of research. It directly addresses real-world problems and applications.

Nonlinear differential equations and integrodifferential equations, along with general optimization problems are prominent subjects in nonlinear analysis. There are many questions related to the existence and uniqueness of solutions for certain types of equations (differential equations, partial integrodifferential equations) can be reduced to the existence and uniqueness of a fixed point for an appropriate mapping defined on a Banach space. One of the most significant existence tools in nonlinear analysis is Krasnoselski theorem established in 1955 by Krasnoselski. His result is captivating and possesses a very wide range of applications see [14].

In this memory, we study the Mönch-Krasnoselski fixed point theorem and some of their applications (to neutral partial integrodifferential equations) see [19],[21], and it is composed of three chapters.

In the first chapter, we introduce the notations, definitions, lemmas, and theorems that will be used throughout this memory.

In the second chapter, we introduce the measures of noncompactness and condensing operators and we state and prove the Mönch and Mönch-Krasnoselski theorems.

In the third chapter, we study the existence of mild solution for two types of neutral partial integrodifferential equations by applying the Mönch-Krasnoselski fixed point theorem. This analysis will be based on the Mönch-Krasnoselski fixed point theorem in Banach spaces.

Main notations used

- Ω: An open bounded of $\mathbb{R}^n$.
- $\|\cdot\|$: A norm on a vector space.
- $\overline{A}$: denote the adherence of A.
- (E, d): A metric space.
- $\mathbb{K}$: real or complex numbers.
- $:=$: Equality by definition.
- $\|\cdot\|_2$: The norm in space $L^2(\Omega)$ defined by: $\|x\|_2 = \left(\int_\Omega |x|^2\right)^{\frac{1}{2}}$.
- ∇u : Gradient operator $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right)$.
- $(F, \|\cdot\|)$: A normed vector space (a Banach space).
- M_F : denote the bounded subsets of F.
- MNC_s: denote the Measures of Noncompactness.
- $\overline{co}(\Omega)$: closed convex hull of Ω.
- $C_0(\mathbb{N})$: The set of all suites.
- $C([a, b], \mathbb{R}^n)$: The set of all continuous functions.
- $B(a, r)$: The open ball.
- $B_c(a, r)$: The closed ball.
- $\mathring{A}$: The interior of subset A.
- $\mathcal{L}$: The space of linear operator.

Chapter 1

Preliminary notions

1.1 Topology of normed vector spaces

Definition 1.1.1. *[2] Let F be a vector space on $\mathbb{K}$ a norm on F is an application $\| \cdot \| : E \longrightarrow \mathbb{R}^+$ verifying the following properties :*

1. $\| x \| = 0 \Leftrightarrow x = 0$;
2. $\| x + y \| \leq \| x \| + \| y \|$ *for all* $x, y \in F$;
3. $\| \lambda x \| = |\lambda| \| x \|$ *for all* $x \in F$ *and* $\lambda \in \mathbb{K}$.

A normed vector space $(F, \| \cdot \|)$ is a vector space F provided with a norm $\| \cdot \|$.

Remark 1.1.1. *A normed vector space is a metric space. The distance between two vectors is defined by the norm:*

$$d(x, y) = \| x - y \| .$$

Proposition 1.1.1. *[2] If $\| \cdot \|$ is a norm on a vector space F then we define on $F, d(a, b) = \| b - a \|$ with d is the distance associated with the norm $\| \cdot \|$ which verifies the two additional properties :*

1. *It is invariant by translation:*

$$\| b + p - a - p \| = \| b - a \| \quad \forall\, p \in F. \tag{1.1}$$

2. *It is homogeneous:*

$$\| \lambda b - \lambda a \| = |\lambda| \, \| b - a \| . \tag{1.2}$$

Example 1.1.1. *[1] The set of real numbers equipped with the distance* $d(x, y) = |x-y|$ *is a metric space.*

Example 1.1.2. *[2]*

1. *The absolute value is a norm on* $\mathbb{R}$.

2. *The module is a norm on* $\mathbb{C}$.

3. *If* $p \in [1, +\infty[$ *we define a norm* $\| \, . \, \|_p$ *on* $\mathbb{K}^d$ *:*

$$\| x \|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}}. \tag{1.3}$$

4. *If* $p = 2$ *the norm* $\| \, . \, \|_2$ *on* $\mathbb{R}^d$ *is the euclidean norm on* $\mathbb{R}^d$ *defined by:*

$$\| x \|_2 = \left(\sum_{i=1}^{d} |x_i|^2 \right)^{\frac{1}{2}}. \tag{1.4}$$

5. *The norm* $\| \, . \, \|_\infty$ *on* $\mathbb{K}^d$ *is defined by:*

$$\| x \|_\infty = \max\{|x_i|; 1 \leq i \leq n\}. \tag{1.5}$$

Definition 1.1.2. *[2]* • *Let* $a \in F$, *and let* $r \geq 0$*: The open ball with center a and radius r is the set:*

$$B(a, r) := \{x \in F; \| x - a \| < r\}. \tag{1.6}$$

• *The closed ball with center a and radius r is the set :*

$$B_c(a, r) := \{x \in F; \| x - a \| \leq r\}. \tag{1.7}$$

Examples 1.1.1. *[2]*

(1) In $\mathbb{R}$, *the open ball B(a, r) is the interval :* $]a - r; a + r[$ *; the closed ball is the interval :* $[a - r; a + r]$.

(2) In $\mathbb{C} = \mathbb{R}^2$ *with the usual distance, the balls are disks.*

(3) In $\mathbb{R}^2$ *with the norm* $\| \, . \, \|_\infty$, *the balls are squares to the axes of contact details.*

Definition 1.1.3. *A neighborhood of* $a \in F$ *is any subset* $V \subset F$ *such that there exists an* $r > 0$ *such that* $B(a, r) \subset V$.

Definition 1.1.4. *A subset* $U \subset F$ *is an open set of* F *if it is a neighborhood of each of its points, i.e.*

$$\forall a \in U, \exists r > 0; B(a, r) \subset U. \tag{1.8}$$

Remark 1.1.2. $\emptyset$ *and* F *are always open sets of* F.

Proposition 1.1.2. *Let* $(F, \|\cdot\|)$ *be a normed* $\mathbb{K}$*-vector space.*

1. *For all* $x \in F$*, the singleton* $\{x\}$ *is a closed set.*
2. *A finite subset of* F *(finite union of singletons) is a closed set of* F.
3. *Closed balls in* F *are closed sets of* F.
4. *Spheres are closed sets of* F.

Definition 1.1.5. *[1] For a subset* A *of* F*, the adherence of* A*, denoted by* $\overline{A}$*, is the set of all adherent points of* A.

Examples 1.1.2. *[2]*

(1) In $\mathbb{R}$*, the closure of an interval is the corresponding closed interval.*

(2) In $(\mathbb{R}^2, \| . \|_\infty)$*, the closure of the half-plane* $\{(x, y); x > 0\}$ *is the half-plane* $\{(x, y); x \geq 0\}$.

Definition 1.1.6. *[1] Let* A *be a subset of* F*. A point* x *of* A *is called an interior point of* A *if* A *is a neighborhood of* x *in* F*,* $A \in V(x)$.
The interior of a subset A *of* F*, denoted by* $\mathring{A}$*, is the set of all interior points of* A.

Definition 1.1.7. *A subset* A *of a normed vector space* F *is said to be dense in* F *if* $\overline{A} = F$.

Example 1.1.3. *[2] A part* A *of normed vector space* $(F, \| . \|)$ *is said to be bounded if there exists a constant* $c < \infty$ *such that* $\| x \| \leq c$ *for all* $x \in A$*, a function* $f : F \longrightarrow F$ *is said to be bounded if its image is a bounded part of* F.

If L is an arbitrary set, we denote by $\mathcal{L}^\infty(L, \mathbb{K})$ the vector space consisting of all bounded functions $f : L \to \mathbb{K}$. We define a norm on $\mathcal{L}^\infty(L, \mathbb{K})$ by setting

$$\| f \|_\infty = \sup\{|f(t)|; t \in L\}.$$

If F is a norm vector space, we note $\mathcal{L}^\infty(L,F)$the vector space constituted by all bounded function $f : L \longrightarrow F$. The natural norm on the space $\mathcal{L}^\infty(L,F)$ is the norm $\| \cdot \|_\infty$ defined by:

$$\| f \|_\infty = \sup\{\| f(t) \|; t \in L\}. \tag{1.9}$$

1.1.1 Continuous applications

Definition 1.1.8. *[4] Suppose that (E,d) and (Y,ϕ) are two metric spaces. A function $f : E \to Y$ is continuous at $x \in E$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\| f(x) - f(x_0) \| < \varepsilon \textit{ provided that } \| x - x_0 \| < \delta.$$

The function f is said to be continuous if f is continuous at all points $x \in E$.

Remark 1.1.3. *[1] Let (E,d) and (E',d') be two metric spaces. The global continuity of a function $f : E \longrightarrow E'$ is written precisely:*

$$\forall x \in E\ , \exists \alpha_{x,\varepsilon} > 0,\ \forall y \in E,\ \| y - x \| \leq \alpha_{x,\varepsilon} \Rightarrow \| f(y) - f(x) \| \leq \varepsilon.$$

Definition 1.1.9. *[1] We say that $f \in C(E,E')$ is uniformly continuous if it verifies:*

$$\forall \varepsilon > 0, \exists \alpha_\varepsilon > 0, \forall x, y \in E, \| x - y \| \leq \alpha_\varepsilon \Rightarrow \| f(x) - f(y) \| \leq \varepsilon.$$

Definition 1.1.10. *[12] Let E be a metric space equipped with a distance d. A map $f : E \to E$ is said to be Lipschitz continuous if there is $k \geq 0$ such that:*

$$d(f(x_1), f(x_2)) \leq kd(x_1, x_2), \quad \forall x_1, x_2 \in E.$$

The smallest k for which the above inequality holds is the Lipschitz constant of f.
If $k < 1$ f is said to be a contraction.

Proposition 1.1.3. *[1] (f Lipschitzian) $\Rightarrow$(f uniformly continuous) $\Rightarrow$ (f is continuous).*

Example 1.1.4. *[1] On a metric space (E,d). For any $x_0 \in E$ the function $d(x_0,\cdot) : x \in E \to d(x_0,x) \in \mathbb{R}$ is Lipschitzian of ratio 1 since:*

$$\forall x, y \in E, | d(x_0, y) - d(x_0, x) | \leq d(x, y).$$

Example 1.1.5. *[2] On $E = K^d$ provided with the norm $\| \, . \, \|_\infty$, coordinated applications are continuous .*

Theorem 1.1.1. *[5] An application $g : E \to E'$ is continuous if and only if the inverse image of any open set (resp. closed) is open (resp. closed).*

Corollary 1.1.1. *[5] if $g : E \longrightarrow E'$ is continuous, then we have $g(\overline{B}) \subset \overline{g(B)}$ for any $B \subset E$.*

Theorem 1.1.2. *[5] The composition of continuous applications is continuous. In other words, if $g : E \longrightarrow E'$ and $h : E' \to Z$ are continuous, then $g \circ h : E \to Z$ is also continuous.*

Example 1.1.6. *$C([a, b], \mathbb{R}^n)$ with $a, b \in \mathbb{R}$ is the space of all continuous y functions define of $[a, b]$ in $\mathbb{R}^n$. the number $\| \, y \, \|_\infty = \sup_{t\in[a,b]} \| \, y(t) \, \|$ define a norm and $(C([a, b], \mathbb{R}^n), \| \cdot \|_\infty)$ a Banach space.*

1.1.2 Convexity

Definition 1.1.11. *[6] A subset C of $\mathbb{R}^n$ is called convex if:*

$$\alpha x + (1 - \alpha)y \in C, \quad \forall x, y \in C, \forall \alpha \in [0, 1].$$

Proposition 1.1.4. *[6]*

(a) The intersection $\bigcap_{i\in I} C_i$ of any collection $\{C_i \mid i \in I\}$ of convex sets is convex.

(b) The vector sum $C_1 + C_2$ of two convex sets C_1 and C_2 is convex.

(c) The set λC is convex for any convex set C and scalar λ. Furthermore, if C is a convex set and λ_1, λ_2 are positive scalars,

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C. \tag{1.10}$$

(d) The closure and the interior of a convex set are convex.

(e) The image and the inverse image of a convex set under an affine function are convex.

Definition 1.1.12. *[6] Let C be a convex subset of $\mathbb{R}^n$. We say that a function $f : C \to \mathbb{R}$ is convex if :*

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \quad \forall x, y \in C, \quad \forall \alpha \in [0, 1]. \tag{1.11}$$

Definition 1.1.13. *Let F be a $\mathbb{R}$–vector space and A be a subset of F. The convex hull of A is defined by:*

$$conv(A) = \{x \in A, x = \sum_{i=1}^{n} \lambda_i x_i; \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1\}. \tag{1.12}$$

1.2 Complete spaces

Definition 1.2.1. *[1] A sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space (E,d) is called a Cauchy sequence if it satisfies :*

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall m, n \geqslant N_\varepsilon, d(x_m, x_n) \leq \varepsilon. \tag{1.13}$$

Definition 1.2.2. *[3] Let F be a normed vector space. A sequence $(U_n)_{n\in\mathbb{N}}$ is bounded if $\| u_n \| \leq M$ for some $M \geq 0$ and all n.*

Definition 1.2.3. *[2] Given a metric space (E,d), a sequence $(x_n)_{n\in\mathbb{N}}$ of points in E converges to a point $a \in E$ if the distance $d(x_n, a)$ tends to 0 as n tends to infinity. In other words,$(x_n)_{n\in\mathbb{N}}$ converges to a if and only if the following property holds:*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geqslant N,\ d(x_n, a) < \varepsilon. \tag{1.14}$$

Remark 1.2.1. *[2] In a normed vector space, every convergent sequence is bounded. The converse is false.*

Example 1.2.1. *[2] If we endow $\mathbb{K}^m$ with the norm $\| \,.\, \|_\infty$ then a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}^m$ converges in $\mathbb{K}^m$ if and only if it converges "coordinate by coordinate".*

Example 1.2.2. *[2] Let $C^1([0,1])$ be the set of $f : [0,1] \longrightarrow \mathbb{R}$ functions of class C^1. We define a norm $\| \,.\, \|_{C^1}$ on $C^1([0,1])$ by setting*

$$\| f \|_{C^1} = \| f \|_\infty + \| f' \|_\infty \,. \tag{1.15}$$

Then, a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C^1([0,1])$ converges to a function $f \in C^1([0,1])$ in the sense of the norm $\| \,.\, \|_{C^1}$ if and only if $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f and (f_n') converges uniformly to f'.

Proposition 1.2.1. *[1] A Cauchy sequence is always bounded.*

Proposition 1.2.2. *[1] Every Cauchy sequence admitting a convergent sub-sequence converge.*

Definition 1.2.4. *[1] The metric space (E, d) is said to be complete if every Cauchy sequence in (E, d) converges.*

Definition 1.2.5. *A Banach space is a normed vector space that is complete.*

Remark 1.2.2. *[2] Let F be a vector space. If* $\| \,.\, \|_1$ *and* $\| \,.\, \|_2$ *are two equivalent norms on F, then* $\| \,.\, \|_1$ *and* $\| \,.\, \|_2$ *have the same Cauchy sequences. Therefore, F is complete for* $\| \,.\, \|_1$ *if and only if it is for* $\| \,.\, \|_2$.

Remark 1.2.3. *[2] If* d_1 *and* d_2 *are two equivalent distances on the same set E, then E can be complete for* d_1 *without being complete for* d_2.

Examples 1.2.1. *[2]*

(1) $\mathbb{R}$ *is complete.*

(2) $\mathbb{Q}$ *is not complete.*

(3) Every finite-dimensional normed vector space is complete.

Proposition 1.2.3. *[2]*

Let (E,d) be a metric space, and let A be a subset of E.

(1) If A is complete for d, then A is closed in E.

(2) If (E,d) is assumed complete, then A is complete for d if and only if A is a closed subset of E.

Corollary 1.2.1. *[2] If* $[a, b]$ *is a closed bounded interval, then the space* $(C([a, b]), \| \,.\, \|_\infty)$ *is complete. More generally, if E is a metric space, F a Banach space, and if we denote* $C_b(E, F)$ *the set of all continuous, bounded functions* $f : E \longrightarrow F$, *then* $C_b(E, F)$ *is complete for the norm* $\| \,.\, \|_\infty$.

Corollary 1.2.2. *[2] The space* $(C_0(\mathbb{N}), \| \,.\, \|_\infty)$ *is a Banach space.*

Corollary 1.2.3. *[2] If F is a normed vector space, then every finite-dimensional subspace of F is closed in F.*

Theorem 1.2.1. *[2] If F is a Banach space, then every absolutely convergent series with terms in F converges in F.*

Proposition 1.2.4. *[2] Let F be a normed vector space. If every absolutely convergent series with terms in F converges, then F is complete.*

Some properties of complete spaces :

Proposition 1.2.5. *[1] In a complete metric space (E, d), the complete subspace are the closed ones.*

Corollary 1.2.4. *[1] Let (E, d) be a metric space. Any intersection of complete subspace is complete.*

Proposition 1.2.6. *[1] Let (E, d) be a metric space. Any finite union of complete subspace of (E, d) is complete.*

Proposition 1.2.7. *[1] A finite or countable product of complete metric spaces is complete.*

Theorem 1.2.2. *[2] Let (E, d) be a complete metric space, and let $(O_n)_{n\in\mathbb{N}}$ be a sequence of open E. In particular, we have $\bigcap_n O_n \neq \emptyset$.*

Corollary 1.2.5. *[2] If (E, d) is a complete metric space, then any closed space of E is a space of Baire, and all open from E as well.*

Corollary 1.2.6. *[2] Let (E, d) be a complete metric space. If $(F_n)_{n\in\mathbb{N}}$ is a sequence of closed of E such as $\bigcup_{n\in\mathbb{N}} F_n = E$ then $\Omega := \bigcup_n \mathring{F}_n$ is a dense open in E.*

1.3 Compactness

Definition 1.3.1. *[7] Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of subsets of the space X and $B \subseteq X$. We say that the family $\{A_\alpha : \alpha \in \Lambda\}$ is a cover of B (or that the family $\{A_\alpha : \alpha \in \Lambda\}$ covers B) if and only if $B \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$.If Λ is finite and $\{A_\alpha : \alpha \in \Lambda\}$ covers B,then $\{A_\alpha : \alpha \in \Lambda\}$ is called an open cover(closed cover)of B.*

Definition 1.3.2. *[7] Let $\{A_\alpha : \alpha \in \Lambda\}$ be a cover of a subset B of a space X. Let $\Omega \subseteq \Lambda$. Then the family $\{A_\alpha : \alpha \in \Omega\}$ is called a sub-cover of the cover $\{A_\alpha : \alpha \in \Lambda\}$ for B if and only if $\{A_\alpha : \alpha \in \Omega\}$ is a cover of B.*

Definition 1.3.3. *[7] A topological space X is called compact if and only if any open cover for X has a finite sub-cover for X. A subset B of a space X is compact if and only if B is a compact topological space with the subspace topology.*

Example 1.3.1. *[7] Let X be any infinite set. Then (X, C), the infinite topology, is compact.*

Definition 1.3.4. *[1] A topological space $(X, \mathcal{T})$ is said to be compact if it is separated and if every open covering admits a finite sub-covering:*

$$\Big(X = \cup_{i \in I} O_i\Big) \Rightarrow \Big(\exists J \subset I, J finite, X = \cup_{i \in J} O_i\Big). \tag{1.16}$$

Theorem 1.3.1. *[8] For any $a, b \in \mathbb{R}$ with $a < b$, the interval $[a, b]$ is compact.*

Proposition 1.3.1. *[1] Every metric space that is compact is also complete.*

Lemma 1.3.1. *[8] Every closed subspace of a compact space is compact.*

Lemma 1.3.2. *[8] Every compact subspace of a Hausdorff space is closed.*

Theorem 1.3.2. *[8] Let X and Y be compact topological spaces. Then $X \times Y$ is also compact.*

Theorem 1.3.3. *[8] A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Corollary 1.3.1. *[8] Every quotient of a compact space is compact.*

1.3.1 Compact metric spaces

Definition 1.3.5. *[2] Let (E, d) be a metric space.*

(1) A point $x \in E$ is said to be an adhesion value of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ if there exists a sub-sequence of $(x_n)_{n\in\mathbb{N}}$ that converges to x.

(2)The metric space E is said to be compact if any sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ has at least one value adhesion in E.

(3) A set $A \subseteq E$ is said to be a compact of E if the metric space (A, d) is compact.

Example 1.3.2. *[2] Every closed bounded interval $[a, b] \subseteq \mathbb{R}$ is compact: this is the Bolzano-Weierstrass theorem.*

Remark 1.3.1. *[2] Let E be a metric space.*

(1) Every compact set $A \subseteq E$ is closed.

(2) If E is compact, then every closed subset of E is compact.

(3) If E is a normed vector space, then every compact subset of E is closed and bounded.

Remark 1.3.2. *In a finite-dimensional normed vector space, compact sets are exactly the closed bounded sets.*

Remark 1.3.3. *[2] Let* (E, d) *be a compact metric space, and let* $a \in E$*. If* a *is the only value of possible adhesion of a sequence* $(x_n)_{n\in\mathbb{N}} \subseteq E$*, then* $(x_n)_{n\in\mathbb{N}}$ *converges to* a*.*

Remark 1.3.4. *[2] Any finite union of compact sets is compact. In particular, any finite set is compact.*

Definition 1.3.6. *[2] Let* (E, d) *be a metric space. It is said that a set* $A \subseteq E$ *is relatively compact in E if* $\bar{A}$ *is a compact of E.*

Remark 1.3.5. *[2] Let* (E, d) *be a metric space. A set* $A \subseteq E$ *is relatively compact in E if and only if any sequence* $(x_n)_{n\in\mathbb{N}} \subseteq A$ *has at least one adhesion value in E.*

Corollary 1.3.2. *[2] Let* (E, d) *be a metric space. If* $(K_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of non-empty compact sets in E, then* $\bigcap_n K_n \neq \emptyset$*.*

Remark 1.3.6. *[2] Let* (E, d) *be a metric space. If every decreasing sequence of non-empty closed sets in E has a non-empty intersection, then E is compact.*

Proposition 1.3.2. *[8]*

- *A sequentially compact metric space is totally bounded.*
- *A sequentially compact metric space is compact.*
- *A sequentially compact metric space is complete.*

Continuous functions on a compact

Theorem 1.3.4. *[2] Let C be a compact metric space, and let E be a metric space. If* $f : C \longrightarrow E$ *is continuous, then f(C) is a compact subset of E.*

Example 1.3.3. *[2] Let C and E be two metric spaces, with E compact. If* $f : C \longrightarrow E$ *is a continuous bijection, then* $f^{-1} : E \longrightarrow C$ *is continuous.*

Theorem 1.3.5. *[2] If E is a compact metric space, then every continuous function* $f : E \longrightarrow \mathbb{R}$ *is bounded and attains its bounds.*

Theorem 1.3.6. *[2] Let C and E be two metric spaces. If C is compact, then every continuous function* $f : C \longrightarrow E$ *is uniformly continuous.*

Corollary 1.3.3. *[2] Let F be a finite-dimensional normed vector space. If* $f : F \to \mathbb{R}$ *is a continuous function satisfying* $\lim_{\|x\|\to\infty} f(x) = +\infty$*, then f is bounded below and attains its lower bound.*

1.3.2 Compact operators on normed spaces

Definition 1.3.7. *[9] Let X and Y be normed spaces. An operator $T : X \longrightarrow Y$ is called a compact linear operator if T is linear and if for every bounded subset M of X, the image T(M) is relatively compact, that is, the closure $\overline{T(M)}$ is compact.*

Lemma 1.3.3. *[9] Let X and Y be normed spaces. Then:*

(a) Every compact linear operator $T : X \longrightarrow Y$ is bounded, hence continuous.

(b)) If dim $X = \infty$, the identity operator $I : X \longrightarrow X$ (which is continuous) is not compact.

Theorem 1.3.7. *[2] Let $T \in \mathcal{L}(X, Y)$. Then T is compact if and only if T^* is compact.*

Proposition 1.3.3. *[2] If $T \in \mathcal{L}(X, Y)$ is compact, then T maps weakly convergent sequences to strongly convergent sequences. The converse is true if X is a Hilbert space.*

Proposition 1.3.4. *[2] Let $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(Y, Z)$. If S or T is compact, then TS is also compact.*

Theorem 1.3.8. *[23](Ascoli-Arzela Theorem) Let $A \subset C([0, b], \mathbb{R}^n)$. A is relatively compact if:*

1. A is bounded, i.e. there exists $M > 0$:

$$\| y(t) \| \leq M, \quad \forall t \in [0, b] \quad and \quad y \in A,$$

2. A is equicontinuous, i.e. for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$

$$\forall t_1, t_2 \in [0, b], |t_1 - t_2| < \delta \Rightarrow \| y(t_1) - y(t_2) \| < \varepsilon, \quad \forall y \in A.$$

Example 1.3.4. *[2] Every finite-rank operator is compact.*

1.4 Some fixed point theorems

Definition 1.4.1. *Let (E, d) be complete metric space and let a map $T : E \longrightarrow E$, we say that $x \in E$ is a fixed point of T if $T(x) = x$.*

1.4.1 Banach fixed point theorem

Definition 1.4.2. *[11] A mapping T from a metric space E into itself is said to be a contraction if: $d(T(x), T(y)) \leq Kd(x, y)$, for all x, y in E and $0 \leq K < 1$.*

A contraction mapping is continuous but not conversely.

Theorem 1.4.1. *[12] Let T be a contraction on a complete metric space E. Then T has a unique fixed point $x \in E$.*

1.4.2 Schauder fixed point theorem

Theorem 1.4.2. *[13] Let C be a closed and convex subset in a Banach space F, and let $T : C \longrightarrow C$ be a continuous mapping such that T(C) is relatively compact. Then T has a fixed point.*

1.4.3 Schaefer's fixed-point theorem

Theorem 1.4.3. *[13] Assume that $(F, \| . \|)$ is a Banach space and that $T : F \longrightarrow F$ is a continuous compact mapping. Moreover assume that the set:*

$$\bigcup_{0\leq\lambda\leq 1} \{x \in F : x = \lambda T(x)\}$$

is bounded. Then T has a fixed point.

1.4.4 Krasnoselski fixed-point theorem

In 1955, Krasnoselski observed that in a large number of problems, the integration of a differential operator gives rise to a sum of two applications, a contraction and a compact application. He then declares:
Principle: The integral of a differential operator can produce a sum of two applications, a contraction, and a compact operator.

Theorem 1.4.4. *[14] Let M be a non-empty closed convex subset of a Banach space $(F, \| . \|)$. Suppose that A and B map M into F such that:*

- $Ax + By \in M \quad (\forall x, y \in M)$,
- *A is continuous and AM is contained in a compact set,*
- *B is a contraction with constant $k < 1$.*

Then there is a $y \in M$ with $Ay + By = y$.

Chapter 2

Mönch-Krasnoselski fixed point theorem

2.1 Measures of noncompactness and condensing operators

In this section we consider the notions connected with measures of noncompactness (MNCs for brivity) and condensing operators.

2.1.1 Notion of a measure of noncompactness

Definition 2.1.1. *[15] A function φ, defined on the set of all subsets of a Banach space F with values in some partially ordered set $(Q, \leq)$, is called a measure of noncompactness if*

$$\varphi(\overline{co}\Omega) = \varphi(\Omega) \ for \ all \ \Omega \subset F. \tag{2.1}$$

Definition 2.1.2. *[15] Let $(E, \| \, . \, \|)$ be a complete metric space. A map*

$$\varphi : M_F \longrightarrow [0, +\infty[$$

is called a measure of noncompactness (MNC) defined on E if it satisfies the following properties:

(a) Regularity : $\varphi(B) = 0 \Leftrightarrow B$ is a precompact set.

(b) Invariant under closure : $\varphi(B) = \varphi(B), \forall B \in M$.

(c) Semi-additivity : $\varphi(B_1 U B_2) = max\{\varphi(B_1), \varphi(B_2)\}, \forall B_1 \in M, \forall B_2 \in M$.

From these axioms, we can immediately deduce the following properties:

(1) Monotonicity : $B_1 \subset M \Rightarrow \varphi(B_1) \leq \varphi(B)$.

(2) $\varphi(B_1 \cap B_2) \leq min\{\varphi(B_1), \varphi(B_2)\}, \forall B_1 \in M, \forall B_2 \in M$.

(3) Non-singularity : If B *is a finite set, then* $\varphi(B) = O$.

(4) Generalized Cantor's intersection theorem : If $\{B_n\}$ *is a decreasing sequence of nonempty, closed and bounded subsets of* E *and* $\lim_{n \longrightarrow +\infty} \varphi(B_n) = 0$, *then the intersection* B_∞ *of all* B_n *is nonempty and compact.*

- *If* E *is a Banach space, the measure of noncompactness* φ *can enjoy some additional properties. Let us list some of them :*

(5) Semi-homogeneity : $\varphi(tB) =\mid t \mid \varphi(B)$ *for any number* t *and* $B \in M$.

(6) Algebraic semi-additive : $\varphi(B_1 + B_2) \leq \varphi(B_1) + \varphi(B_2), \forall B_1 \in M, \forall B_2 \in M$.

(7) In variance under translations : $\varphi(x_0 + B) = \varphi(B)$*for any* $x_0 \in E$ *and* $B \in M$.

(8) Lipschitzianity : $\mid \varphi(B_1) - \varphi(B_2) \mid \leq L_\varphi \rho(B_1, B_2)$, *where* ρ *denotes the Hausdorff semi-metric* $\rho(B_1, B_2) = inf\{\varepsilon > 0 : B_2 \subset B_1 + \varepsilon B(O,1), B_1 \subset B_2 + \varepsilon \bar{B}(O,1)\}$.

(9) Continuity : For every $B \in M$ *and for all* $\varepsilon > 0$, *there is* $\delta > 0$ *such that*

$\mid \varphi(B) - \varphi(B_1) \mid < \varepsilon$ *for all* B_1 *satisfying* $\rho(B, B_1) < \delta$.

(10) Invariance under passage to the convex hull : $\varphi(co(B)) = \varphi(B)$ *for all* $B \in M$.

The Kuratowski and Hausdorff measures of noncompactness

Definition 2.1.3. *[15] The kuratowski measure of noncompactness is a mapping* $\alpha(\Omega) : M_F \longrightarrow \mathbb{R}_+$ *defined by:*

$\alpha(\Omega) = inf\{\varepsilon > 0; \Omega \ admets\ a\ finits\ covering\ by\ sets\ of\ diameter\ smaller\ then\ \varepsilon\}$.

The diameter of a set A *means the number:*

$$diam(A) = sup\{\| x - y \| : x, y \in A\}.$$

In other words:

$$\alpha(\Omega) = inf\{\varepsilon > 0; \exists k \in \mathbb{N}, \Omega = \bigcup_{i=1}^{k} O_i, diam(O_i) < \varepsilon, for\ all\ i = 1....k\}.$$

Definition 2.1.4. *[15] The Hausdorff measure of noncompactness is an application* $\chi(\Omega) : M_F \longrightarrow \mathbb{R}_+$ *defined by :*

$$\chi(\Omega) = inf\{\varepsilon > 0, \Omega \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in F, r_i \in \mathbb{R}^+, for\ all\ i = 1....n\}.$$

In other words:

$$\chi(\Omega) = inf\{\varepsilon > 0, \Omega\ has\ a\ finite\ \varepsilon - net \in F\}.$$

Remark 2.1.1. *The kuratowski and Hausdorff* MNC_s *its satisfy all the above properties of 2.1.2.*

Remark 2.1.2. *[15] Based on the above definition*

$$diam(O_i) < \varepsilon \Rightarrow diam(O_i) \geq \alpha(\Omega), for\ all\ \Omega \in M_F.$$

Which implies that $diam(\Omega)$ *is a condidate to be* $\alpha(\Omega)$ *.*

Definition 2.1.5. *[17] The Kuratowski and Hausdorff MNCs are invariant under passage to the convex hull:* $\phi(M) = \phi(co(M))$*.*

Theorem 2.1.1. *[17] The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(M) \leq \alpha(M) \leq 2\chi(M).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Theorem 2.1.2. *[15] Let* B *be the unit ball in* E*. Then* $\alpha(B) = \chi(B) = 0$ *if* E *is finite-dimensional, and* $\alpha(B) = 2, \chi(B) = 1$ *in the opposite case.*

Remark 2.1.3. *[17] Though in general* α *and* χ *are different MNCs, in some Banach spaces we can find a direct relation between them.*

2.1.2 Condensing operators

In this section we introduce the condensing operators and study some properties.

Definition 2.1.6. *[15] Let* F_1 *and* F_2 *be Banach spaces and let* φ *and* ψ *be MNCs in* F_1 *and* F_2*. A continuous operator* $f : D(f) \subset F_1 \longrightarrow F_2$ *is said to be* $(\varphi, \psi)-condensing$ *if* $\Omega \subset D(f)), \psi[f(\Omega)] \geq \varphi(\Omega) \Rightarrow \Omega$ *is relatively compact. The operator* f *is said to be*

$(\varphi,\psi)-$condensing in the proper sense if $\psi[f(\Omega)] < \varphi(\Omega)$ for any set $\Omega \subset D(f))$ with compact closure. If the set Q is linearly ordered , then the two notions of condensing operator coincide. A continuous operator f is said to be $(q,\varphi,\psi)-$ bounded if

$$\psi[f(\Omega)] \leq q\varphi(\Omega)$$

for any set $\Omega \subset D(f))$. Whenever $F_1 = F_2$ and $\varphi = \psi$ we shall simply say $\psi-$ condensing and $(q,\psi)-$ boundell. In the case $q < 1$, $(q,\psi)-$ bounded operators are sometimes referred to as $\psi-$ condensing with constant q.

Proposition 2.1.1. *[15]*

(a) If the MNC φ_1 is regular, then any $(q,\varphi_1,\varphi_2)-$bounded operator with $q < 1$ is $(\varphi_1,\varphi_2)-$condensing.

(b) If f_1 is a $(\varphi_1,\varphi_2)-$condensing operator and f_2 is a $(\varphi_2,\varphi_3)-$condensing operator that maps totally bounded sets into totally bounded ones, φ_1 and φ_3 are regular MNCs, and $Q = [0,\infty)$, then the composition $f_2 \circ f_1$ is a $(\varphi_1,\varphi_3)-$condensing operator.

(c) If $Q = [0,\infty)$ and φ_2 is semi-additive, then the set of all $(\varphi_1,\varphi_3)-$condensing operators is convex.

Example 2.1.1. *[15] Suppose the operators $g_0, g_1 : X \subset F_1 \longrightarrow F_2$ are $(\gamma,\beta)-$condensing, the set where the MNCs γ and β take their values is linearly ordered (as a consequence of which f_0 and f_1 are $(\gamma,\beta)-$condensing in the proper sense), and γ is semi-additive. Then the family of operators $g = \{g_\lambda : \lambda \in [0,1]\}$, where $g_\lambda(x) = (1-\lambda)g_0(x)+\lambda g_1(x)$, is $(\gamma,\beta)-$condensing.*

Corollary 2.1.1. *[15]*

(a) If the set M is bounded and $q < 1$, then the family f is $\chi-$condensing.

(b) The sum $f + g$ of a compact operator $f : F_1 \longrightarrow F_2$ and a contractive operator $g : F_1 \longrightarrow F_2$ is a $\chi-$condensing operator on any bounded set $M \subset F_1$.

Definition 2.1.7. *[17] If E and Y are metric spaces, ϕ and A measures of noncompactness defined on E and Y respectively, and $T : D \subset E \longrightarrow Y$ a mapping, then*

(a) T is a $(\phi,\lambda)-$contractive operator with constant $k > 0$ (or simply $k-(\phi,\lambda)-$contractive) if T is continuous and verifies that for every bounded subset A of D we have $\lambda(T(A)) \leq k\phi-(A)$. In the particular case when $E = Y$ and $\lambda = \phi$ we simply say that T is a $(k-\phi)$-contractive operator.

(b) T is a $(\phi,\lambda)-$condensing operator with constant $k>0$ (or simply $k-(\phi,\lambda)-$condensing) if T is continuous and verifies that for every bounded and non precompact subset A of D we have $\lambda(T(A)) < k\phi(A)$. In the particular case when $E=Y$ and $\lambda=\phi$ we simply say that T is a $(k-\phi)$-condensing operator. Moreover, if $k=1$ we say that T is a ϕ-condensing operator.

Remark 2.1.4. *[17]*

(a) If $\phi=\alpha$, the $k-\alpha-$contractive (or $(k-\alpha)$-condensing) operators are usually called $k-$set-contractive (or $k-$set-condensing) operators.

(b) If $\phi=\chi$, the $(k-\chi)$-contractive (or $(k-\chi)$-condensing) operators are usually called $k-$ball-contractive (or $k-$ball-condensing) operators.

(c) Every compact operator is $k-(\phi,\lambda)$-contractive and $k-(\phi,\lambda)$-condensing for all $k>0$.

(d) Every $k-(\phi,\lambda)$-condensing operator is $k-(\phi,\lambda)$-contractive.

2.1.3 Fixed point theorems

Theorem 2.1.3. *[18](Mönch) Let D be a bounded, closed and convex subset of a Banach space such that $0\in D$, and let G be a continuous mapping of D into itself. If the implication*

$$V = convG(V) \text{ or } V = G(V)\cup\{0\} \Rightarrow \alpha(V)=0 \tag{2.2}$$

holds for every subset V of D, then G has a fixed point.

Proof. We define a sequence (y_n) by

$$\begin{cases} y_{n+1} = G(y_n)\ (n=0,1,2,...) \\ y_0 = 0 \end{cases} \tag{2.3}$$

Let $Y=\{y_n : n=0,1,2,...\}$. As $Y=G(Y)\cup\{0\}$, from 2.3 it follows that Y is relatively compact in D. Denote by Z the set of all limit points of (y_n). It can be easily verified that $Z=G(Z)$. Let us put

$$R(X) = convG(X)\ for\ X\subset D,$$

and let Ω denote the family of all subsets X of D such that $Z \subset X$ and $R(X) \subset X$. Clearly $D \in \Omega$. Denote by V the intersection of all sets of the family Ω. As $Z \subset V$,V is nonempty and

$$Z = G(Z) \subset R(Z) \subset R(V).$$

Since

$$R(V) \subset R(X) \subset X \ for \ all \ X \in \Omega, R(V) \subset V \ and \ therefore \ V \subset \Omega.$$

Moreover,

$$R(R(V)) \subset R(V), \ and \ hence \ R(V) \in \Omega.$$

Consequently

$$V = R(V) \ i.e. \ V = convG(V).$$

In view of 2.3, this implies that $\overline{V}$ is a compact subset of D. Applying now the Schauder fixed point theorem to the mapping $G \mid \overline{V}$. We conclude that G has a fixed point.

□

Theorem 2.1.4. *(Darbo)[16] Let C be a nonempty, bounded closed and convex subset of a Banach space F and let $T : C \longrightarrow C$ be a continuous mapping. Assume that there exists a constant $K \in [0, 1)$ for any nonempty subset S of C such that:*

$$\alpha(TS) \leq K\alpha(S).$$

Where α denotes the Kuratowski measure of noncompactness defined in F. Then T has a fixed point in the set C.

2.2 Measure of noncompactness in spaces of functions

2.2.1 The Hausdorff MNC in the space $C[a, b]$

In the space $C[a, b]$ of continuous real-valued functions on the segment $[a, b]$ the value of the set-function χ on a bounded set Ω can be computed by means of the formula [15]

$$\chi(\Omega) = \frac{1}{2} \lim_{\delta \longrightarrow 0} \sup_{x \in \Omega} \max_{0 \leq \tau \leq \delta} \| x - x_\tau \| \tag{2.4}$$

where x_τ denotes the τ-translate of the function x:

$$x_\tau(t) = \begin{cases} x(t+\tau), \ if \ a \leq t \leq b-\tau, \\ x(b), \qquad if \ b-\tau \leq t \leq b. \end{cases} \tag{2.5}$$

2.2.2 MNC in the space $C([0,+\infty[,\mathbb{R})$

We will use a measure of noncompactness in the space $C([0,+\infty[,\mathbb{R})$ [20]. In order to define this measure let us fix a nonempty bounded subset X of the space $C([0,+\infty[,\mathbb{R})$ and a positive number T. For $x \in X$ and $\varepsilon \geq 0$ denote by $\omega^T(x,\varepsilon)$ the modulus of the function x on the interval $[0,T]$, i.e.

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0,T], \mid t - s \mid \leq 0\}.$$

Further, let us put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X,\varepsilon), \ \omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

If t is a fixed number of $\mathbb{R}_+$, let us denote $X(t) = \{x(t) : x \in X\}$ and

$$diam X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Finally, consider the function μ is a measure of non compactness in the space $C([0,+\infty],\mathbb{R})$ defined on the family $M_{C([0,+\infty],\mathbb{R})}$ by the formula

$$\mu(X) = \omega_0(X) + \lim_{t \to \infty} \sup diam X(t).$$

2.2.3 MNC in the space $C([a,b],F)$

The space $C([a,b],F)$ is furnished with the standard sup-norm $\|x\|_\infty = sup\{\|x(t)\|_F : t \in [a,b]\}$. We use χ to denote the Hausdorff measure of noncompactness in the Banach space F. We also define the function ψ_C on the family of bounded subsets in $C([a,b],F)$ by taking:

$$\psi_C(\Omega) = \chi_\infty(\Omega(t)) + mod_C(\Omega),$$

where

$$mod_C(\Omega(t)) = \lim_{\delta \to 0} \sup_{x \in \Omega}\{sup\{\mid x(t_2) - x(t_1) \mid : t_1, t_2 \in (t-\delta, t+\delta)\}\},$$

$$mod_C(\Omega), = sup\{mod_C(\Omega(t)) : t \in [a,b]\},$$

and

$$\chi_\infty(\Omega) = sup\{\chi(\Omega(t)) : t \in [a,b]\}.$$

Then, ψ_C is a full monotone and nonsingular MNC on the space $C([a,b],F)$.

2.2.4 MNC in $C^1([a,b],F)$

Let $C^1([a,b],F)$ denote the Banach space of the continuously differentiable functions $x : [a,b] \longrightarrow F$, equipped with the norm $\|x\|_{c^1} = \|x\|_c + \|x'\|_c$. The function ψ_{C^1}, defined on the bounded subsets of $C^1([a,b],F)$ by the formula [15]

$$\psi_{C^1}(\Omega) = \psi_C(\Omega) + \psi_C(\Omega')$$

is an MNC, where $\Omega'(t) = \{x'(t) : x \in \Omega\}$. If the set $\Omega' = \{x' : x \in \Omega\}$ is equicontinuous and the MNC ψ_{C^n} is continuous, then $\psi_{C^1}(\Omega) \in C[a,b]$.

2.2.5 MNC in $C^n([a,b],F)$

Let $C^n([a,b],F)$ denote the Banach space of the n-times continuously differentiable functions $x : [a,b] \longrightarrow F$, endowed with the norm $\|x\|_{C^n} = \sum_{i=0}^n \|x\|_C$. Then each MNC ψ on $C([a,b],F)$ generates an MNC ψ_{C^n} on $C^n([a,b],F)$ by the rule

$$\psi_{C^n}(\Omega) = \psi_C(\Omega) + \psi_C(\Omega') + \cdots + \psi_C(\Omega^n),$$

where $\Omega^{(n)} = \{x^{(n)} :\ x \in \Omega\}$.

2.2.6 MNC in $L_p([a,b])$

Let Ω be a bounded subset of the space $L_p([a,b])$ of equivalence classes of measurable functions $x : [a,b] \longrightarrow \mathbb{R}$ which are p-integrable, endowed with the norm $\|x\| = \left(\int_a^b (|x(t)|^p dt\right)^{\frac{1}{p}}$. Then

$$\frac{1}{2}\mu(\Omega) \leq \chi(\Omega) \leq \mu(\Omega).$$

The function μ appearing above is defined by the formula

$$\mu(\Omega) = \lim_{\varepsilon \longrightarrow 0} \{\sup_{x\in\Omega}[\max_{0 \leq h \leq \varepsilon} \| x - x_h \|]\},$$

where x_h denotes the steklov mean of the function x defined as

$$x_h(t) = \frac{1}{2h}\int_{t-h}^{t+h} x(s)ds.$$

Proposition 2.2.1. *[22] Let F be a Banach space and $T \subset C([a,b],F)$ equicontinuous with $T(t)$ bounded for each $t \in [a,b]$. Then*

1. *For all $t, s \in [a,b]$, one has*

$$|\alpha(T(t)) - \alpha(T(s))| \leq 2\omega(T,\delta),$$

where $\omega(T;\delta)$ is the modulus of continuity of T, namely

$$\omega(T,\delta) = sup\{|u(t) - u(s)|; t, s \in [a,b], |t-s| \leq \delta, u \in T\}.$$

2. *We have*

$$\alpha\left(\int_0^1 T(t)dt\right) \leq \int_0^1 \alpha(T(t))dt,$$

where

$$\int_0^1 T(t)dt = \left\{\int_0^1 u(t)dt; u \in T\right\}.$$

2.3 Mönch-Krasnoselski fixed point theorem

Lemma 2.3.1. *[19] Let $S : F \longrightarrow F$ be a strict contraction with constant $K \in [0,1)$. Then, $I - S$ is bijective and $(I-S)^{-1} : F \longrightarrow F$ is continuous Lipschitzian with constant $(1-K)^{-1}$.*

Proof. Notice for $x, y \in F$, we have:

$$\| (I-S)x - (I-S)y \| \geq (1-K) \| x - y \| . \tag{2.6}$$

Thus, $I - S$ is one-to-one. Now, let $y \in F$ be fixed. The map which assigns to each $x \in F$ the value $Sx + y$ is a strict contraction from F into itself, and so has a unique fixed point $x_0 \in F$, by the contraction mapping principle. Hence, $x_0 = Sx_0 + y$, and therefore, $y = (I-S)x_0$. Consequently, $F = (I-S)F$. The second assertion follows from 2.6. □

The following theorem is a basic result in topological fixed point theory. It generalizes, in some sense, Mönch fixed point theorem as well as Krasnoselski fixed point theorem.

Theorem 2.3.1. *[19] Let F be a Banach space and M be a nonempty closed convex subset of F. Let $T : M \longrightarrow F$ and $S : F \longrightarrow F$ be two continuous mappings satisfying the following conditions:*

(i) There is some $x_0 \in M$ and a positive integer n_0, such that for all countable $C \subset M$, we have:

$$\overline{C} = \mathcal{F}^{(n_0,x_0)}(T, S, C) \textit{ implies that } C \textit{ is relatively compact.} \tag{2.7}$$

(ii) S is a strict contraction.

(iii) $(x = Sx + Ty, y \in M)$ implies $x \in M$.

Then, $T + S$ has at least one fixed point in M.

Proof. [19] Referring to Lemma 2.3.1 , we see that $\tau = (I - S)^{-1}T : M \longrightarrow F$ is well defined and continuous. Moreover, from our assumptions, we know that τ maps M into itself and we have $\mathcal{F}(T, S, \Omega) = \tau(\Omega)$, for any $\Omega \subset M$. Now, we consider the iterative sequence (D_n) of sets:

$$D_0 = \{x_0\}, \quad D_n = \mathcal{F}^{(1,x_0)}(T, S, D_{n-1}), n \in \mathbb{N}.$$

By mathematical induction, it is easily seen that for all $n \in \mathbb{N}$, we have:

$$D_n \subset D_{n+1}. \tag{2.8}$$

We will use mathematical induction to prove that the statement P(n) given by "for all $n \in \mathbb{N} \cup \{0\}$, the set D_n is compact" is true. Observing that D_0 is compact, we obtain that the base case P(0) is true. Next, we perform the inductive step. Assume that P(n) is true for some integer $n \geq 0$; that is, D_n is compact. Using the continuity of τ together with the Krein–Milman theorem, we infer that:

$$D_{n+1} = \mathcal{F}^{(1,x_0)}(T, S, D_n) = \overline{co}(\{x_0\} \cup \tau(D_n)) \tag{2.9}$$

is compact. Consequently, we have shown our statement $P(n + 1)$ to be true, and thus, our inductive step is complete. Let us put $D = \cup_{n\in\mathbb{N}} D_n$. Easy considerations lead us to infer that:

$$\overline{D} = \mathcal{F}^{(1,x_0)}(T, S, D). \tag{2.10}$$

Notice for every $n \in \mathbb{N}$, D_n is compact, and so, it is separable. Thus, for each n, there exists a countable set $C_n \subset D_n$, such that $D_n = \overline{C_n}$. Let us consider $C = \bigcup_{n\in\mathbb{N}} C_n$. It is easy to check that:

$$\overline{D} = \overline{C}. \tag{2.11}$$

Linking 2.10 and 2.11, we arrive at:

$$\begin{aligned}\overline{C} = \overline{D} &= \overline{co}(\{x_0\} \cup \mathcal{F}(T, S, D)) = \overline{co}(\{x_0\} \cup \mathcal{F}(T, S, \overline{D})) \\ &= \overline{co}(\{x_0\} \cup \mathcal{F}(T, S, \overline{C})) = \overline{co}(\{x_0\} \cup \mathcal{F}(T, S, C)) = \mathcal{F}^{(1,x_0)}(T, S, C).\end{aligned}$$

As a result:

$$\overline{C} = \mathcal{F}^{(1,x_0)}(T, S, C). \tag{2.12}$$

Using a simple mathematical induction, we obtain:

$$\overline{C} = \mathcal{F}^{(n_0,x_0)}(T, S, C). \tag{2.13}$$

From our hypotheses, we know that C is relatively compact. In view of 2.12 , we have $\tau(\overline{C}) \subset \overline{C}$. The Schauder fixed point theorem ensures the existence of a fixed point for τ which, in turn, is a fixed point for $T + S$. □

Chapter 3

Applications

3.1 Neutral partial integrodifferential equation without compactness

We will investigate the existence of mild solutions for neutral partial integrodifferential equations of the following form [19]

$$\begin{cases} \frac{d}{dt}\mathcal{G}(u)(t) = A\mathcal{G}(u)(t) + \int_0^t B(t-s)\mathcal{G}(u)(s)ds + \mathcal{M}(t,u(t)) \ for\ t \in [0,a], \\ u(0) = u_0 \in F, \end{cases} \tag{3.1}$$

where $A : D(A) \subset F \longrightarrow F$ is a closed linear operator on a Banach space $(F, \| . \|_F)$, $(B(t))_{t\geq 0}$ is a family of closed linear operators on F having the same domain $D(B) \supset D(A)$ which is independent of t and

$$\mathcal{G}(u)(t) = u(t) - G(t, u(t)) \quad for\ t \in [0,a],$$

where G, $\mathcal{M}$ are given functions to be specified later. Equation 3.1 is known as abstract neutral integrodifferential equations.

3.1.1 Resolvent operators and measure of noncompactness

Let $(F, \| . \|_F)$ be a Banach space and let $C([0,a], F)$ denote the Banach space of all continuous functions defined on $[0,a]$ with values in F equipped with the standard sup-norm. Also, for any closed linear operator (A, D(A)) on F, we denote by Y the Banach space D(A) equipped with the graph norm

$$\| x \|_G := \| Ax \|_F + \| x \|_F \ .$$

We need the following results on the resolvent operator theory. Let us consider the following integrodifferential equation:

$$\begin{cases} y'(t) = Ay(t) + \displaystyle\int_0^t B(t-s)y(s)ds \ for \ t \geq 0 \\ y(0) = y_0 \in F. \end{cases} \tag{3.2}$$

We start by defining the resolvent operator for Eq. 3.2.

Definition 3.1.1. *[19] A resolvent operator for Eq. 3.2 is a bounded linear operator valued function $R(t) \in \mathcal{L}(F)$ for $t \geq 0$ having the following properties:*

(a) $R(0) = I$, the identity map on F and $\| R(t) \|_{\mathcal{L}(F)} \leq Me^{\beta t}$ for some constants $M \leq 1$ and $\beta \in \mathbb{R}$.

(b) For each $x \in F, R(t)x$ is strongly continuous for $t \geq 0$.

(c) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y, R(.)x \in C^1(\mathbb{R}_+, F) \cap C(\mathbb{R}_+, Y)$ and:

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \ for \ t \geq 0 \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \ for \ t \geq 0. \end{aligned} \tag{3.3}$$

In the sequel, we provide sufficient conditions ensuring the existence of the resolvent operator. For this purpose, we consider the following assumptions:

(I) A is a closed densely defined linear operator on a Banach space $(F, \| \, . \, \|_F)$.

(II) $(B(t))_{t\geq 0}$ is a family of linear operators on F, such that B(t) is continuous from Y to F for almost all $t \geq 0$. Moreover, there is a locally integrable function $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, such that B(t)y is measurable and

$$\| B(t)y \|_F \leq b(t) \| y \|_G \ \ for \ all \ y \in Y \ and \ t \geq 0.$$

(III) For any $y \in Y$, the map $t \longrightarrow B(t)y$ belongs to $W^{1,1}_{loc}(\mathbb{R}^+, F)$ and:

$$\| \frac{d}{dt} B(t)y \| \leq b(t) \| y \|_G; \ for \ y \in Y \ and \ a.e. \ t \in \mathbb{R}^+;$$

here, $W^{1,1}_{loc}(\mathbb{R}^+, F)$ stands for the set of all functions $u \in L^1_{loc}(\mathbb{R}^+, F)$ which admits a distributional derivative $u' \in L^1_{loc}(\mathbb{R}^+, F)$.

We point out that resolvent operators do not verify semigroup property.
For example, if we take $F = \mathbb{R}$, $Ay = y$ and $B(t) = -2y$ in Eq. (3.2), then, we have:

$$R(t)x = e^{t}(cost + sint)x \;\; and \;\; T(t)x = e^{2t}x.$$

However, the following significant result ensures the existence a resolvent operator for Eq. 3.2 provided that A generates a strongly continuous semigroup.

Theorem 3.1.1. *[19] Assume that $(I) - (III)$ hold. Then, Eq. (3.2) admits a resolvent operator if and only if A generates a $C_0 -$ semigroup.*

We work in the space $C := C(I, F)$ consisting of all functions defined and continuous on $I = [0, a]$ with values in the Banach space F. The space $C(I, F)$ is furnished with the standard sup-norm $\| x \|_\infty = sup\{\| x(t) \|_F : t \in I\}$. We use χ_C to denote the Hausdorff measure of noncompactness in the space $C(I, F)$. We also define the function ψ_C on the family of bounded subsets in $C(I, F)$ by taking:

$$\psi_C(\Omega) = \chi_\infty(\Omega) + mod_C(\Omega),$$

where

$$mod_C(\Omega(t)) = \lim_{\delta \to 0} \sup_{x\in\Omega}\{sup\{| x(t_2) - x(t_1) | : t_1, t_2 \in (t - \delta, t + \delta)\}\},$$

$$mod_C(\Omega), = sup\{mod_C(\Omega(t)) : t \in I\},$$

and

$$\chi_\infty(\Omega) = sup\{\chi(\Omega(t)) : t \in I\};$$

Then, ψ_C is a full monotone and nonsingular MNC on the space $C(I, F)$. We introduce the following sets. Let M be a nonempty closed convex subset of $F, T, S : M \longrightarrow F$ two nonlinear mappings and $x_0 \in F$. For any $\Omega \subseteq M$, we define:

$$\mathcal{F}(T, S, \Omega) = \{x \in M : x = Sx + Ty, \; for \; some \; y \in \Omega\},$$

$$\mathcal{F}^{(1,x_0)}(T, S, \Omega) = \overline{co}(\{x_0\} \cup \mathcal{F}(T, S, \Omega)),$$

and

$$\mathcal{F}^{(n,x_0)}(T, S, \Omega) = \overline{co}\Big(\{x_0\} \cup \mathcal{F}\Big(T, S, \Big(\mathcal{F}^{(n-1,x_0)}(T, S, \Omega)\Big)\Big)\Big),$$

for $n = 2, 3, ...$
When $S = 0$, we abbreviate the notation $\mathcal{F}^{(n,x_0)}(T, 0, \Omega)$ to $\mathcal{F}^{(n,x_0)}(T, \Omega)$. It may

happen that the set $\mathcal{F}(T,S,\Omega)$ is empty. In such a case, we cannot expect to have a fixed point for the sum $S+T$ in Ω. Therefore, this case is not relevant to our purpose. In the case where S is a strict contraction,we are absolutely sure that all the sets $\mathcal{F}(T,S,\Omega)$ are nonempty.

Proposition 3.1.1. *[19] Assume that:*

(a) $S : F \to F$ *is a strict contraction with constant* $k \in [0,1)$ *and*

(b) $(x = Sx + Ty, y \in M)$ *implies* $x \in M$*.Then:*

(i) $\mathcal{F}^{(n,x_0)}(T,S,\Omega)$ *is a nonempty subset of M for any* $\Omega \subset M$ *and any positive integer* $n \geq 1$.

(ii) $\mathcal{F}^{(n,x_0)}(T,S,\Omega) = \mathcal{F}^{(n,x_0)}((I-S)^{-1}T,\Omega)$ *for any* $\Omega \subset M$ *and any positive integer* $n \geq 1$.

(iii) $\Omega_1 \subset \Omega_2 \subset M$ *implies* $\mathcal{F}^{(n,x_0)}(T,S,\Omega_1) \subset \mathcal{F}^{(n,x_0)}(T,S,\Omega_2) \subset M$ *for any* $n \geq 1$.

Lemma 3.1.1. *[19] Let* $\mathcal{H} \subseteq C([0,a],F)$ *be equicontinuous and* $x_0 \in C([0,a],F)$*. Then,* $\overline{co}(\mathcal{H} \cup \{x_0\})$ *is also equicontinuous in* $C([0,a],F)$*.*

Lemma 3.1.2. *[19] Let* $\mathcal{H} \subset C([0,a];F)$ *be a bounded set. Then,* $\chi(\mathcal{H}(t)) \leq \chi_C(\mathcal{H})$ *for any* $t \in [0,a]$*, where* $\mathcal{H}(t) = \{u(t) : u \in \mathcal{H}\}$*. Furthermore, if* $\mathcal{H}$ *is equicontinuous on* $[0,a]$*, then* $t \to \chi(\mathcal{H}(t))$ *is continuous on* $[0,a]$ *and:*

$$\chi_C(\mathcal{H}) = \chi_\infty(\mathcal{H}),$$

where

$$\chi_\infty(\mathcal{H}) = sup\{\chi(\mathcal{H}(t)) : t \in [0,a]\}.$$

Lemma 3.1.3. *[19] Let* $\mathcal{H}$ *be a bounded subset of F. Then, there exists a sequence* $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$*, such that:*

$$\chi(\mathcal{H}) = \chi((u_n)_{n\geq 1}).$$

We also need the following elementary result.

Lemma 3.1.4. *[19] For all* $0 \leq m \leq n$*, we denote* $C_n^m = \binom{n}{m}$*. Let* $0 < \varepsilon < 1, h > 0$*, and:*

$$S_n = \varepsilon^n + C_n^1 \varepsilon^{n-1} h + C_n^2 \varepsilon^{n-2} \frac{h^2}{2!} + ... + \frac{h^n}{n!}, n \in \mathbb{N}^*.$$

Then, $\lim_{n\longrightarrow\infty} S_n = 0$.

3.1.2 New Mönch–Krasnoselski type fixed point theorems

We establish a new fixed point theorem for the sum of two operators. This fixed point result is really interesting and may have several applications. It will serve as a key tool for the development of our existence theory. Before making a formal statement of our fixed point result, we need to recall the following more or less well-known result. Referring to lemma 2.3.1 and theorem 2.3.1.
We describe the case where $S = 0$.

Corollary 3.1.1. *[19] Let F be a Banach space and M be a nonempty closed convex subset of F. Let $T : M \longrightarrow M$ be a continuous mapping. Assume that there are a vector $x_0 \in M$ and a positive integer n_0, such that for any countable subset C of M, we have:*

$$\overline{C} = \mathcal{F}^{(n_0,x_0)}(T, C) \text{ implies that } C \text{ is relatively compact.} \tag{3.4}$$

Then, T has at least one fixed point in M.

In view of Theorem 2.3.1 , we promptly deduce the following interesting results.

Corollary 3.1.2. *[19] Let F be a Banach space, M be a nonempty bounded closed convex subset of F, and ψ be a nonsingular measure of noncompactness on F. Let $T : M \longrightarrow F$ and $S : F \longrightarrow F$ be two continuous mappings satisfying:*

(i) there exist a vector x_0 and a positive integer n_0, such that for any countable subset Ω of M with $\psi(\Omega) > 0$, we have:

$$\psi(\mathcal{F}^{(n_0,x_0)}(T, S, \Omega)) < \psi(\Omega); \tag{3.5}$$

(ii) S is a strict contraction;

(iii) $(x = Sx + Ty, y \in M)$ implies $x \in M$.

Then, $T + S$ has at least one fixed point in M.

Corollary 3.1.3. *[19] Let F be a Banach space, M be a nonempty bounded closed convex subset of F, and ψ be a nonsingular measure of noncompactness on F. Let $T : M \to M$ be a continuous mapping, such that there exist a vector x_0 and a positive integer n_0, such that for any countable subset Ω of M with $\psi(\Omega) > 0$, we have:*

$$\psi(\mathcal{F}^{(n_0,x_0)}(T, \Omega)) < \psi(\Omega). \tag{3.6}$$

Then, T has at least one fixed point in M.

As a consequence of Corollary 3.1.3, we obtain the following statement which is a sharpening of Daher's theorem

Corollary 3.1.4. *[19] Let F be a Banach space, M be a nonempty bounded closed convex subset of F, and ψ be a nonsingular measure of noncompactness on F. Let $T : M \to M$ be a continuous mapping, such that for any countable subset Ω of M with $\psi(\Omega) > 0$, we have:*

$$\psi(T(\Omega)) < \psi(\Omega). \tag{3.7}$$

Then, T has at least one fixed point in M.

3.1.3 Existence results

We will prove the existence of a mild solution for the neutral equation 3.1. But before that, we need to recall some results regarding the estimation of the Hausdorff measure of noncompactness for integral operators and related results. We start with the following interesting results.

Theorem 3.1.2. *[19] Let $\mathcal{F}$ be a function from $[0, +\infty)$ into $\mathcal{L}(F)$. Suppose that $\mathcal{F}$ is continuous for the strong operator topology. Let Ω be a bounded subset of F and $\mathcal{F} = \{\mathcal{F}(.)x, x \in \Omega\} \subset C(\mathbb{R}^+, F)$. Then, for any $t \geq 0$, we have:*

$$mod_C(\mathcal{F}(t)) \leq \omega(\mathcal{F}(t))\chi(\Omega).$$

In particular, for any $t \in [0, a]$, we have:

$$mod_C(\mathcal{F}(t)) \leq 2M_a\chi(\Omega),$$

where:

$$\omega(\mathcal{F}(t)) = \lim_{\delta\to 0} \sup_{\|x\|\leq 1} \{\| \mathcal{F}(t_2)x - \mathcal{F}(t_1)x \|_F : t_1, t_2 \in (t - \delta, t + \delta)\}$$

and

$$M_a = \sup_{t\in[0,a]} \| \mathcal{F}(t) \|_{\mathcal{L}(F)} .$$

Proposition 3.1.2. *[19] Let $\mathcal{F}$ be a function from $[0, +\infty)$ into $\mathcal{L}(F)$. Suppose that $\mathcal{F}$ is continuous for the strong operator topology. Then, for any compact $\mathcal{K} \subset F$, we have:*

$$\sup_{y\in\mathcal{K}} \| \mathcal{F}(t)y - \mathcal{F}(t_0)y \|_F \to 0 \ as\ t \to t_0.$$

Let $\mathcal{V}$ be an operator defined on $L^1([0,a];F)$ with values in $C([0,a];F)$ satisfying the following conditions:

(S_1) *There exists $d > 0$, such that:*

$$\| \mathcal{V}f(t) - \mathcal{V}g(t) \|_F \leq d \int_0^t \| f(s) - g(s) \|_F \, ds,$$

for every $f, g \in L^1([0,a];F)$ and $t \in [0,a]$.

(S_2) *For any compact $\mathcal{K} \subset F$ and any sequence $(f_n)_{n\geq 1} \subset L^1((0,a);F)$, such that $(f_n(t))_{n\geq 1} \subset \mathcal{K}$ for a .e. $t \in [0,a]$, we have:*

$$f_n \rightharpoonup f_0 \text{ implies } \mathcal{V}f_n \to \mathcal{V}f_0.$$

The following fundamental theorems are crucial for our further work.

Theorem 3.1.3. *[19] Assume that the operator $\mathcal{V}$ satisfies (S_1) and (S_2). Let $(f_n)_{n\geq 1} \subset L^1((0,a);F)$ be integrably bounded, namely:*

$$\| f_n(t) \| \leq v(t) \text{ for all } n \geq 1 \text{ and a.e } t \in [0,a], \tag{3.8}$$

where $v \in L^1(0,a)$. Assume that:

$$\chi((f_n(t))_{n\geq 1}) \leq q(t) \text{ for a.e } t \in [0,a], \tag{3.9}$$

where $q \in L^1(0,a)$. Then:

$$\chi((\mathcal{V}f_n(t))_{n\geq 1}) \leq 2d \int_0^t q(s)ds,$$

for all $t \in [0,a]$, where $d > 0$ is given in (S_1).

Theorem 3.1.4. *[19] Assume that the operator $\mathcal{V}$ satisfies (S_1) and (S_2). Let $(f_n)_{n=1}^{\infty} \subset L^1([0,a];F)$ be as in 3.8. Assume that 3.9 holds. Then, for every $t \in [0,a]$, we have:*

$$mod_C((\mathcal{V}f_n(t))_{n\geq 1}) \leq 4d \int_0^t q(s)ds, \tag{3.10}$$

where $d \geq 0$ is the constant in (S_1).
For later use, we consider the integral operator:

$$(\mathcal{V}_0 f)(t) = \int_0^t R(t-s)f(s)ds, \text{ for } t \in [0,a],$$

where $f \in C([0,a];F)$.
The operator $\mathcal{V}_0$ enjoys some interesting and useful properties given by the following.

Theorem 3.1.5. *[19] Let $(f_n)_{n=1}^{\infty} \subset L^1([0,a];F)$ be such that the set $(f_n(t))_{n=1}^{\infty}$ resides in a compact set $\mathcal{K}$, for almost every $t \in [0,a]$. Then, the sequence $(\mathcal{V}_0 f_n)_{n=1}^{\infty}$ is relatively compact in $C([0,a];F)$.*

Theorem 3.1.6. *[19] The integral operator $\mathcal{V}_0$ satisfies (S_1) and (S_2).*
From now on, we assume that the assumptions (I)–(III) hold true and the operator A generates a strongly continuous semigroup. Our main purpose in the immediate sequel is to show the existence of solutions to Eq.3.1. Before doing so, it is appropriate to clarify the definition of solution which we will consider.

Definition 3.1.2. *[19] A continuous function $u : [0,a] \to F$ is said to be a mild solution of Eq. 3.1 if:*

$$u(t) = R(t)\mathcal{G}(0,u_0) + G(t,u(t)) + \int_0^t R(t-s)\mathcal{M}(s,u(s))ds \ for \ t \in [0,a]. \quad (3.11)$$

To obtain the existence of mild solutions to 3.1 , we assume the following assumptions.

$(H1)$ The function $\mathcal{M} : [0,a] \times F \to F$ satisfies the Carathéodory conditions; that is, $\mathcal{M}(.,z)$ is measurable for all $z \in F$ and $\mathcal{M}(t,.)$ is continuous for almost all $t \in [0,a]$.

$(H2)$ There exist a function $\rho \in L^1((0,a);\mathbb{R}^+)$ and a nondecreasing continuous function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$, such that:

$$\| f(t,z) \| \leq \rho(t)\Omega(\| z \|_F) \ for \ all \ t \in [0,a] \ and \ z \in F.$$

$(H3)$ There exists a function $\theta \in L^1([0,a];\mathbb{R}^+)$, such that for any bounded set $\Omega \subseteq F$:

$$\chi(f(t,\Omega)) \leq \theta(t)\chi(\Omega).$$

$(H4)$ There is a $k_0 \in [0,1)$, such that for any $t_1, t_2 \in [0,a]$ and any $z_1, z_2 \in F$, we have:

$$\| G(t_1,z_1) - G(t_2,z_2) \|_F \leq k_0(| t_1 - t_2 | + \| z_1 - z_2 \|_F).$$

$(H5)$

$$k_0 + M_a \liminf_{r\to\infty} \frac{\Omega(r)}{r} \int_0^a \rho(s)ds < 1.$$

To allow the abstract formulation of our problem, we define the following operators as follows:

$$(Su)(t) = R(t)\mathcal{G}(0, u_0) + G(t, u(t)) \ for \ t \in [0, a], \ u \in C([0, a]; F)$$

and

$$T = \mathcal{V}_0 \circ N_f,$$

where

$$N_f u = f(., u(.)) \ for \ u \in C([0, a]; F).$$

It is plainly visible that u is a mild solution of Eq. 3.1 if and only if u is a fixed point of $T + S$. With this in mind, we shall show that operators T,S satisfy all conditions of Theorem 2.3.1 . This will be achieved in a series of lemmas.

Lemma 3.1.5. *[19] The operator T maps continuously $C([0, a]; F)$ into itself.*

Proof. Let $(u_n)_n$ be a sequence in $C([0, a]; F)$, such that

$$\lim_{n\to\infty} u_n = u \ in \ C([0, a]; F).$$

By $(H1)$, we have:

$$\lim_{n\to\infty} \mathcal{M}(s, u_n(s)) = \mathcal{M}(s, u(s))$$

for a.e. $s \in [0, a]$. Hence:

$$\| Tu_n - Tu \|_\infty \leq M_a \int_0^a \| \mathcal{M}(s, u_n(s)) - \mathcal{M}(s, u(s)) \|_F \, ds.$$

Using the dominated convergence Theorem, we obtain:

$$\lim_{n\to\infty} \| Tu_n - Tu \|_\infty = 0.$$

This completes the proof. □

Lemma 3.1.6. *[19] S is a strict contraction.*

Proof. Let $u, v \in C([0, a]; F)$ and $t \in [0, a]$. Then, by $(H4)$, we have:

$$\begin{aligned} \| (Su)(t) - (Sv)(t) \|_F &\leq \| G(t, u(t)) - G(t, v(t)) \|_F \\ &\leq k_0 \| u(t) - v(t) \|_F . \end{aligned}$$

Consequently:

$$\| Su - Sv \|_\infty \leq k_0 \| u - v \|_\infty .$$

□

Lemma 3.1.7. *[19] There is a $r_0 > 0$, such that:*

$$(u = Su + Tv, v \in B_{r_0}) \text{ implies } u \in B_{r_0},$$

where $B_r = \{u \in C([0,a];F) : \| u \|_\infty \leq r\}$.

Proof. We argue by contradiction. Assume that for all $r > 0$, there are $u \in C([0,a];F)$ and $v \in B_r$, such that $u = Su + Tv$ and $u \notin B_r$. Hence, for any $t \in [0,a]$, we have:

$$\begin{aligned}
&\| (Su)(t) + (Tv)(t) \|_F \\
&\quad \leq \| R(t)(u_0 - G(0,u_0)) + G(t,u(t)) \|_F + \| \int_0^t R(t-s)\mathcal{M}(s,v(s))ds \|_F \\
&\quad \leq M_a \| u_0 - G(0,u_0) \|_F + \| G(t,u(t)) \|_F + M_a\Omega(r) \int_0^a \rho(s)ds \\
&\quad \leq M_a \| u_0 - G(0,u_0) \|_F + \| G(t,0) \|_F + k_0 \| u \|_\infty + M_a\Omega(r) \int_0^a \rho(s)ds.
\end{aligned}$$

Thus:

$$\begin{aligned}
\| u \|_\infty &= \| Su + Tv \|_\infty \\
&\leq M_a \| u_0 - G(0,u_0) \|_F + \| G(.,0) \|_\infty + k_0 \| u \|_\infty + M_a\Omega(r) \int_0^a \rho(s)ds.
\end{aligned}$$

Consequently:

$$r < \| u \|_\infty \leq \frac{1}{1-k_0} \Big(M_a \| u_0 - G(0,u_0) \|_F + \| G(.,0) \|_\infty + M_a\Omega(r) \int_0^a \rho(s)ds \Big),$$

this implies that:

$$1 < \frac{1}{1-k_0} \Big(\frac{M_a \| u_0 - G(0,u_0) \|_F}{r} + \frac{\| G(.,0) \|_\infty}{r} + M_a \frac{\Omega(r)}{r} \int_0^a \rho(s)ds \Big).$$

Taking the lim inf as $r \to \infty$, we obtain that:

$$1 \leq k_0 + M_a \liminf_{r\to\infty} \frac{\Omega(r)}{r} \int_0^a \rho(s)ds,$$

which contradicts $(H5)$. This achieves the proof. □

Lemma 3.1.8. *[19] There is an integer n_0, such that $\chi_\infty(\mathcal{F}^{(n_0,0)}(T,S,D)) < \chi_\infty(D)$, for any bounded subset D of $C([0,a],F)$, with $\chi_\infty(D) > 0$.*

Proof. Let D be a bounded subset of $C([0,a],F)$, such that $\chi_\infty(D) > 0$. Then, for any $t \in [0,a]$, we have that:

$$\begin{aligned}\mathcal{F}^{(1,0)}(T,S,D)(t) &= \{u(t), u \in \mathcal{F}^{(1,0)}(T,S,D)\}\\ &\subseteq \{u(t) - Su(t), u \in \mathcal{F}^{(1,0)}(S,K,D)\} + \{Su(t), u \in \mathcal{F}^{(1,0)}(S,K,D)\}.\end{aligned}$$

Using the properties of the Hausdorff measure of noncompactness, we get:

$$\chi(\mathcal{F}^{(1,0)}(T,S,D)(t)) \le \chi(T(D)(t)) + k_0\chi(\mathcal{F}^{(1,0)}(T,S,D)(t)).$$

As a result:

$$\chi(\mathcal{F}^{(1,0)}(T,S,D)(t)) \le \frac{1}{1-k_0}\chi(T(D)(t)). \tag{3.12}$$

Referring to Lemma 3.1.3, we see that there is a sequence $(u_n)_{n\ge1} \subseteq D$, such that:

$$\chi(T(D)(t)) \le \chi((Tu_n(t))_{n\ge1}) \le \chi\bigg(\Big(\int_0^t R(t-s)\mathcal{M}(s,u_n(s))ds\Big)_{n\ge1}\bigg).$$

Invoking Theorem 3.1.3, we obtain:

$$\chi(T(D)(t)) \le 2M_a\int_0^t C(s)\chi((u_n(s))_{n\ge1})ds \le 2M_a\chi_\infty(D)\int_0^t C(s)ds.$$

Taking into account the density of $C([0,a];\mathbb{R})$ in $L^1([0,a];\mathbb{R})$, we see that for any $\delta < \frac{1-k_0}{2M_a}$, there exists $\varphi \in C([0,a];\mathbb{R})$ satisfying

$$\int_0^a |C(s) - \varphi(s)|ds < \delta.$$

Consequently:

$$\begin{aligned}\chi(T(D)(t)) &\le 2M_a\chi_\infty(D)\Big[\int_0^t |C(s)-\varphi(s)|ds + \int_0^t |\varphi(s)|ds\Big]\\ &\le 2M_a\chi_\infty(D)[\delta + \tau t],\end{aligned}$$

where $\tau = \sup_{0\le s\le a}|\varphi(s)|$. Thus:

$$\chi(T(D)(t)) \le (2M_a\delta + 2M_a\tau t)\chi_\infty(D).$$

This means by 3.12 that:

$$\chi(F^{(1,0)}(T,S,D)(t)) \leq (\lambda+\mu t)\chi_\infty(D), \tag{3.13}$$

where $\lambda = \frac{2M_a\delta}{(1-k_0)}$ and $\mu = \frac{2M_a\tau}{(1-k_0)}$. Furthermore:

$$\begin{aligned}\mathcal{F}^{(2,0)}(T,S,D)(t) &\subseteq \{u(t)-Su(t), u\in\mathcal{F}^{(2,0)}(T,S,D)\} + \{Su(t), u\in\mathcal{F}^{(2,0)}(T,S,D)\}\\ &\subseteq \{Tv(t), v\in\overline{co}(\mathcal{F}^{(1,0)}(T,S,D)\cup\{0\})\} + \{Su(t), u\in\mathcal{F}^{(2,0)}(T,S,D)\}.\end{aligned}$$

Hence, a similar reasoning as above yields:

$$\chi(\mathcal{F}^{(2,0)}(T,S,D)(t)) \leq \chi(T(\overline{co}(\mathcal{F}^{(1,0)}(T,S,D)\cup\{0\}))(t)) + k_0\chi(\mathcal{F}^{(2,0)}(T,S,D)(t)).$$

Thus:

$$\chi(\mathcal{F}^{(2,0)}(T,S,D)(t)) \leq \frac{1}{1-k_0}\chi(S(\overline{co}(\mathcal{F}^{(1,0)}(T,S,D)\cup\{0\}))(t)). \tag{3.14}$$

Referring to Lemma 3.1.3, we see that there exists a sequence

$$(w_n)_{n\geq 1} \subseteq \overline{co}(\mathcal{F}^{(1,0)}(T,S,D)\cup\{0\}),$$

such that:

$$\begin{aligned}\chi(T(\overline{co}(\mathcal{F}^{(1,0)}(T,S,D)\cup\{0\}))(t)) &\leq \chi\Big(\Big(\int_0^t R(t-s)\mathcal{M}(s,w_n(s))ds\Big)_{n\geq 1}\Big)\\ &\leq 2M_a\int_0^t C(s)\chi(\overline{co}(\mathcal{F}^{(1,0)}(S,K,D)\cup\{0\})(s))ds\\ &\leq 2M_a\int_0^t C(s)\chi(\mathcal{F}^{(1,0)}(T,S,D)(s))ds.\end{aligned}$$

Liking 3.13 and 3.14 , we arrive at:

$$\begin{aligned}\chi(\mathcal{F}^{(2,0)}(T,S,D)(t)) &\leq \frac{2M_a}{(1-k_0)}\int_0^t [|C(s)-\varphi(s)|+|\varphi(s)|](\lambda+\mu s)\chi_\infty(D)ds\\ &\leq \frac{2M_a}{(1-k_0)}\Big[(\lambda+\mu t)\int_0^t |C(s)-\varphi(s)|ds + \tau\Big(\lambda t+\mu\frac{t^2}{2}\Big)\Big]\chi_\infty(D)\\ &\leq \Big[\lambda(\lambda+\mu t)+\mu\Big(\lambda t+\mu\frac{t^2}{2}\Big)\Big]\chi_\infty(D)\\ &\leq \Big[\lambda^2+2\lambda\mu t+\frac{(\mu t)^2}{2}\Big]\chi_\infty(D).\end{aligned}$$

Accordingly:

$$\chi(\mathcal{F}^{(2,0)}(T,S,D)(t)) \leq \Big[\lambda^2 + 2\lambda\mu t + \frac{(\mu t)^2}{2}\Big]\chi_\infty(D).$$

By mathematical induction, we obtain for all integer $n \geq 1$ that:

$$\chi(\mathcal{F}^{(n,0)}(T,S,D)(t)) \leq \Big[\lambda^n + C_n^1\lambda^{n-1}\mu t + C_n^2\lambda^{n-2}\frac{(\mu t)^2}{2!} + ... + \frac{(\mu t)^n}{n!}\Big]\chi_\infty(D).$$

Accordingly:

$$\chi_\infty(\mathcal{F}^{(n,0)}(T,S,D)) \leq \Big[\lambda^n + C_n^1\lambda^{n-1}\mu a + C_n^2\lambda^{n-2}\frac{(\mu a)^2}{2!} + ... + \frac{(\mu a)^n}{n!}\Big]\chi_\infty(D).$$

Since $0 < \lambda < 1$ and $\mu a > 0$, then from Lemma 3.1.4, we deduce that there exists $n_0 \in \mathbb{N}^*$, such that:

$$S_{n_0} = \Big[\lambda^{n_0} + C_{n_0}^1\lambda^{n_0-1}\mu a + C_{n_0}^2\lambda^{n_0-2}\frac{(\mu a)^2}{2!} + ... + \frac{(\mu a)^{n_0}}{n_0!}\Big] < 1,$$

which implies that:

$$\chi_\infty(\mathcal{F}^{(n_0,0)}(T,S,D)) < \chi_\infty(D).$$

□

Lemma 3.1.9. *[19] Let D be a bounded subset of $C([0,a],F)$. If $T(D)$ is equicontinuous, then so is $\mathcal{F}^{(n,0)}(T,S,D)$ for any integer $n \geq 1$.*

Proof. Let $u \in \mathcal{F}(T,S,D)$. Then, there exists $v \in D$, such that:

$$u = Su + Tv.$$

Hence, for $t, t' \in [0,a]$, we have that:

$$\begin{aligned} \| u(t) - u(t') \|_F = & \ \| Su(t) + Tv(t) - Su(t') - Tv(t') \|_F \\ \leq & \ \| Su(t) - Su(t') \|_F + \| Tv(t) - Tv(t') \|_F \\ \leq & \ \| (R(t) - R(t'))\mathcal{G}(0,u_0) \|_F + k_0\Big(|t-t'| + \| u(t) - u(t') \|_F\Big) \\ & + \| Tv(t) - Tv(t') \| . \end{aligned}$$

Consequently:

$$\| u(t) - u(t') \| \leq \frac{1}{1-k_0}\Big(\| Tv(t) - Tv(t') \|_F + \| (R(t) - R(t'))\mathcal{G}(0,u_0) \|_F \Big) + \frac{k_0}{1-k_0}|t-t'|.$$

Keeping in mind the fact that T(D) is equicontinuous on $[0, a]$, we deduce that:

$$\parallel u(t) - u(t') \parallel_F \to 0 \ as, \ t \to t', \tag{3.15}$$

uniformly in $u \in \mathcal{F}(T, S, D)$. This implies that $\mathcal{F}(T, S, D)$ is equicontinuous.
In view of Lemma 3.1.1, we conclude that $\mathcal{F}^{(1,0)}(T, S, D) := \overline{co}(\mathcal{F}(T, S, D) \cup \{0\})$ is equicontinuous. By mathematical induction, one can see that $\mathcal{F}^{(n,0)}(T, S, D)$ is equicontinuous for all $n \geq 1$.

□

After these preparations, we are now ready to state the main result of this section.

Theorem 3.1.7. *[19] Assume that* $(H1) - -(H5)$ *hold. Then, the problem 3.1 has at least one mild solution on* $[0, a]$.

Proof. Let C be a countable subset of B_{r_0}, such that:

$$\overline{C} = \mathcal{F}^{(n_0,0)}(T, S, C). \tag{3.16}$$

Referring to Lemma 3.1.8 , we see that $\chi_\infty(C) = 0$. Hence, by Theorem 3.1.5 together with assumption $(H1)$, we deduce that T(C) is compact. Now, we apply Lemma 3.1.9 to conclude that $\mathcal{F}^{(n_0,x_0)}(T, S, C)$ is equicontinuous. Going back to 3.16 , we infer that C is equicontinuous. The use of Lemma 3.1.2 yields $\chi_C(C) + \chi_\infty(C) = 0$ and, therefore, C is relatively compact. Invoking Theorem 2.3.1 together with Lemmas 3.1.5 , 3.1.6 , 3.1.7 , we deduce that $S + T$ has a fixed point in B_{r_0} , which is, in turn, a mild solution to 3.1. □

3.2 Example

[19] Now, we apply our abstract results to investigate the existence of mild solutions for the following neutral integrodifferential equation subjected to some initial data :

$$\begin{cases} \frac{\partial}{\partial t}\big[u(t,x) - g(u(t,x))\big] = \upsilon \cdot \nabla\big[u(t,x) - g(u(t,x))\big] \\ + \int_0^t \beta e^{-(t-s)\mu}\upsilon \cdot \nabla\big[u(s,x) - g(u(s,x))\big]ds \\ +p_1(t)p_2(u(t,x)), \ for \ t \in [0,a] \ and \ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \ for \ x \in \mathbb{R}^d, \end{cases} \tag{3.17}$$

where $\beta > 0$, $\mu \in [0,1]$, $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_d)$ is a fixed element in $\mathbb{R}^d, d \geq 1$, and $\upsilon \cdot \nabla\omega$ is the $\upsilon -$ *directional* distributional derivative of ω, that is:

$$\upsilon \cdot \nabla\omega(x) = \sum_{i=1}^{d} \upsilon_i \frac{\partial\omega}{\partial x_i}(x),$$

for each $\omega \in L^p(\mathbb{R}^d)$ with $\upsilon \cdot \nabla\omega \in L^p(\mathbb{R}^d), 1 \leq p < +\infty$ and a.e. for $x \in \mathbb{R}^d$. Assume that:

(i) $p_1 : [0,a] \longrightarrow \mathbb{R}$ is integrable, $p_2 : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitzian with constant $L_2 > 0$ and $p_2(0) = 0$.

(ii) There exists $k_0 \in [0,1)$ such that:

$$|g(z) - g(z')| \leq k_0|z - z'| \ for \ z, z' \in \mathbb{R}.$$

(iii) $u_0 \in L^p(\mathbb{R}^d)$.

Let $F = L^p(\mathbb{R}^d), d \geq 1$, with $1 \leq p < +\infty$ and let $\upsilon \in \mathbb{R}^d$.Let $u(t) = u(t,.)$ and define the functions $G, \mathcal{M} : [0,a] \times F \to F$ by

$$\mathcal{M}(t,\omega)(x) = p_1(t)p_2(\omega(x))$$

and

$$G(t,\omega)(x) = g(\omega(x)), \ for \ t \in [0,a], x \in \mathbb{R}^d$$

and $\omega \in F$.Hence, 3.17 takes the following form:

$$\begin{cases} \frac{d}{dt}\big[u(t) - G(u(t))\big] = A\big[u(t) - G(u(t))\big] + \int_0^t B(t-s)\big[u(s) - G(u(s))\big]ds \\ +\mathcal{M}(t, u(t)) \ for \ t \in [0,a] \\ u(0) = u_0. \end{cases} \tag{3.18}$$

Where $A : D(A) \subseteq F \longrightarrow F$ is defined by:

$$D(A) = \{u \in L_p\mathbb{R}^d, \upsilon \cdot \nabla u \in L^p(\mathbb{R}^d)\},$$
$$Au = \upsilon \cdot \nabla u,$$

for each $u \in D(A)$, and $B(t) = \beta e^{-t\mu}A = b(t)A$, for $t \geq 0$. We will show that all conditions of Theorem 3.1.7 are satisfied. This will be achieved in a series of lemmas.

Before we do that, the operator A is the infinitesimal generator of the $C_0 - group$ of isometries $\{T(t) : F \longrightarrow F; t \in \mathbb{R}\}$ given by:

$$[T(t)\xi](x) = \xi(x + tv),$$

for each $\xi \in F, t \in \mathbb{R}$ and a.e. for $x \in \mathbb{R}^d$. It should be stressed here that the semigroup $(T(t))_{t\geq 0}$ is neither compact nor equicontinuous. Moreover, for any $t \geq 0$ and any $y \in D(A)$, we have:

$$\| B(t)y \|_F \leq \| b(t)Ay \|_F \leq b(t) \| y(t) \|_G,$$

and

$$\| \frac{d}{dt}B(t)y \|_F \leq \mu b(t) \| Ay \|_F \leq b(t) \| y(t) \|_G .$$

Referring to Theorem 3.1.1 we see that Eq. 3.17 admits a resolvent operator $(R(t))_{t\geq 0}$:

Lemma 3.2.1. *[19]*

$$\| G(x) - G(y) \|_p \leq K_0 \| x - y \|_p .$$

Proof. Let $\omega_1, \omega_2 \in F$. Then, from (ii), we have:

$$| G(\omega_1)(x) - G(\omega_2)(x) |=| g(\omega_1)(x) - g(\omega_2)(x) |\leq K_0 | (\omega_1(x)) - (\omega_2(x)) | .$$

Thus:

$$\| G(\omega_1) - G(\omega_2) \|_p \leq K_0 \| \omega_1 - \omega_2 \|_p .$$

□

Lemma 3.2.2. *[19]*

$$\| \mathcal{M}(t,\omega) \|_p \leq |p_1(t)|\Omega(\| \omega \|_p) \ for\ all\ t \in [0, a]\ and\ \Omega \in F,$$

where $\Omega(r) = L_2 r$.

Proof. Since $p_2(0) = 0$, then:

$$| \mathcal{M}(t,\omega)(x) |\leq| p_1(t) || p_2(\omega(x))| \leq| p_1(t) | L_2 | \omega(x) | .$$

Hence:

$$\| \mathcal{M}(t,\omega) \|_p \leq| p_1(t) | L_2 ||\| \omega \|_p .$$

□

Lemma 3.2.3. *[19] For any bounded set $D \subseteq F$ and any $t \in [0,a]$, we have:*

$$\chi(\mathcal{M}(t,D)) \leq \mid p_1(t) \mid L_2 \chi(D).$$

Proof. Let $t \in [0,a]$, D a subset of F and $\lambda > \chi(D)$. Then, there are $\omega_1, ..., \omega_n \in F$, such that $D \subset \bigcup_{i=1}^{n} B(\omega_i, \lambda)$. Notice that for any $\omega \in D$, there is an $i_0 \in \{1, ..., n\}$, such that $\| \omega - \omega_{i_0} \|_p \leq \lambda$. Hence, for any $x \in F$:

$$\begin{aligned} \mid \mathcal{M}(t,\omega)x - \mathcal{M}(t,\omega_{i_0})x \mid &\leq \mid p_1(t) \mid\mid p_2(\omega(x)) - p_2(\omega_{i_0}(x)) \mid \\ &\leq \mid p_1(t) \mid L_2 \mid \omega(x) - \omega_{i_0}(x) \mid . \end{aligned}$$

This leads to:

$$\| \mathcal{M}(t,\omega) - \mathcal{M}(t,\omega_{i_0}) \|_p \leq \mid p_1(t) \mid L_2 \| \omega - \omega_{i_0} \|_p \leq \mid p_1(t) \mid L_2 \lambda.$$

Therefore:

$$\chi(\mathcal{M}(t,D)) \leq \mid p_1(t) \mid L_2 \lambda.$$

Letting $\lambda \longrightarrow \chi(D)$, we get:

$$\chi(\mathcal{M}(t,D)) \leq \mid p_1(t) \mid L_2 \chi(D).$$

□

Lemma 3.2.4. *[19] Assume that $2\beta\big(\frac{\mu a^2}{2} + 1\big) < 1$. Then:*

$$M_a \leq \frac{1}{1 - 2\beta(\frac{\mu a^2}{2} + 1)}, \tag{3.19}$$

where

$$M_a = \sup_{t \in [0,a]} \| R(t) \| .$$

Proof. We know that:

$$T(t)x = R(t)x + \int_0^t R(t-s)Q(s)xds, \tag{3.20}$$

with

$$Q(s)x = -\int_0^s B'(s-\tau) \int_0^\tau T(\theta)xd\theta d\tau - B(0) \int_0^s T(\tau)xd\tau.$$

It is readily verified that:

$$Q(s)x = \beta\mu \int_0^s e^{-\mu(s-\tau)} A \int_0^\tau T(\theta)xd\theta d\tau - \beta A \int_0^s T(\tau)xd\tau$$
$$= \beta\mu \int_0^s e^{-\mu(s-\tau)}(T(\tau)x - x)d\tau - \beta(T(s)x - x).$$

Therefore, $\| Q(s)x \| \leq 2\beta(\mu s + 1) \| x \|$. Now, we see from 3.20 that:

$$M_a \leq 1 + 2\beta M_a \left(\frac{\mu a^2}{2} + 1\right).$$

Hence, $M_a \leq \frac{1}{1-2\beta(\frac{\mu a^2}{2}+1)}$ as asserted. □

Theorem 3.2.1. *[19] If* $k_0 + \frac{\|p_1\|_1 L_2}{1-2\beta(\frac{\mu a^2}{2}+1)} < 1$, *then Eq. 3.17 has a mild solution on* $[0, a]$.

Proof. [19] This follows from Theorem 3.1.7 on the basis of Lemmas 3.2.1, 3.2.2, 3.2.3, and 3.2.4. □

3.3 Results on neutral partial integrodifferential equations with nonlocal conditions

We establish the solution of the existence of Equations (3.21) and (3.22) with finite delay [21]

$$\frac{d}{dv}\mathcal{D}(v, z_v) = \mathcal{A}\mathcal{D}(v, z_v) + \int_0^v H(v-s)\mathcal{D}(s, z_s)ds + \phi\left(v, z_v, \int_0^v h(v, s, z_s)ds\right), \tag{3.21}$$

$$z_0 = \varphi + g(z) = C([-r, 0], F). \tag{3.22}$$

where $v \in I = [0, b]$, $\mathcal{A}$ is a closed linear operator defined on Banach space $(F, \| \, . \, \|)$ with domain $D(\mathcal{A})$. Let $[H(v)]_{v\geq 0}$ be the set of all closed linear operators on F with domain $D(H) \supset D(\mathcal{A})$ and $C([-r, 0], F)$ denote the set of all continuous functions defined on $[-r, 0]$ into F. Throughout this theory, F will be used as Banach space. The function $\mathcal{D}$ in $\mathbb{R}^+ \times C \longrightarrow F$ is defined as follows

$$\mathcal{D}(v, \varphi) = \varphi(0) - \mathcal{M}(v, \varphi),$$

where the function $\mathcal{M}$ is continuous from $\mathbb{R}^+ \times C$ into F and the function f is also continuous from $\mathbb{R}^+ \times C \times F$ into F. Let $z_v \in C([-r,0],F), \forall v \geq 0$, then the history function $z_v \in C$ is defined by

$$z_v(t) = z(v+t) \ for \ t \in [-r,0].$$

3.3.1 Existence results

Here, to establish the result on the existence of 3.21 and 3.22, we need the following lemmas [21].

Lemma 3.3.1. *[21] Let H be a bounded subset of F, if there is (u_n) in H, then*

$$\psi(H) = \psi(u_n) \ for \ n \geq 1.$$

Lemma 3.3.2. *[21] Let $H : [0,b] \longrightarrow F$ be an equicontinuous map and $x_0 \in [0,b]$, then $\overline{co}(H \cup \{x_0\})$ is also equicontinuous.*

Theorem 3.3.1. *[21] The continuous function $\mathcal{F}$ from $[0,\infty)$ to $\mathcal{L}(F)$ and for some compact set $K \subset F$, then*

$$\sup_{y \in K} \parallel \mathcal{F}(v)y - \mathcal{F}(v_0)y \parallel \rightarrow 0 \ as \ v \rightarrow v_0.$$

The operator V defined on $L^1([0,b];F)$ in $C([0,b];F)$ satisfies,

(S1) For some $d > 0$, we have

$$\parallel Vf_1(v) - Vf_2(v) \parallel_F \leq d \int_0^v \parallel f_1(s) - f_2(s) \parallel_F ds, \ for \ all \ f_1, f_2 \in L^1([0,b],F), v \in [0,b].$$

(S2) The compact set $K \subset F$ and $(f_n)_{n\geq 1} \subset L^1([0,b],F)$ implies $(f_n(v))_{n \geq 1} \subset K$ for all $v \in [0,b]$ we have

$$f_n \longrightarrow f_0 \Rightarrow Vf_n \longrightarrow Vf_0.$$

Theorem 3.3.2. *[21] Suppose the operator V satisfies (S1) and (S2) and $(f_n)_{n\geq 1} \subset L^1([0,b],F)$ is integrable and bounded,*

$$\parallel f_n(v) \parallel \leq \omega(v), \ \forall v \in [0,b], n \geq 1, \ for \ some \ \omega \in L^1(0,b).$$

Assume that for all $v \in [0,b]$ and for some $q \in L^1(0,b)$ such that

$$\psi\Big(\big(f_n(v)\big)_{n\geq 1}\Big) \leq q(v).$$

Then

$$\psi\Big(\big(vf_n(v)\big)_{n\geq 1}\Big) \leq 2d\int_0^v q(s)ds \ for\ all\ v \in [0,b], d \in S_1.$$

Definition 3.3.1. *[21] The continuous function $z : [-r,\infty) \longrightarrow F$ is called a mild solution of Equations 3.21 and 3.22 if the following integral equation is satisfied*

$$z(v) = \mathcal{F}(v, z_v)$$

$$+ R_1(v)\Big[D\Big(0, \varphi(0) + g(z)(0)\Big)\Big] + \int_0^v R_1(v-s)\phi\Big(s, z_s, \int_0^s h(s,\tau,z_\tau)d\tau\Big)ds. \quad (3.23)$$

To establish this result, we need the below hypotheses:

(H1) The mapping $\phi : [0,b] \times C \times F$ satisfied Caratheodary conditions, i.e., $\phi(v,.,.)$ is continuous for all $v \in I$ and $\phi(.,x,y)$ is measurable, for each $(x,y) \in C \times F$.

(H2) There is $m_\phi \in C([0,b],\mathbb{R}^+)$ and the mapping Ω_ϕ from , $\mathbb{R}^+$ into $\mathbb{R}^+$ then

$$\| \phi(v,x,y) \| \leq m_\phi(v)\Omega_\phi(\| x \|_C + \| y \|), \forall v \in I\ and\ (x,y) \in C \times F.$$

(H3) The mapping $h : \mathbb{R}^+ \times \mathbb{R}^+ \times C \longrightarrow F$ is continuous and $m_h : [0,b] \longrightarrow [0,\infty)$ for some continuous function m_h we have

$$\| h(v,s,x) \| \leq m_h(s)\Omega_h(\| x \|_C), \forall x \in C, 0 \leq s \leq v \leq b,$$

where $\Omega_h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the increasing function.

(H4) There exists the functions $p_1, p_2 \in L^1([0,b],\mathbb{R}^+)$ such that

$$\psi(\phi(v,\Omega_1,\Omega_2)) \leq p_1(v)\psi(\Omega_1) + p_2(v)\psi(\Omega_2)\ for\ some\ bounded\ subsets\ \Omega_1,\Omega_2 \subset F.$$

(H5) There is a constant $k \in [0,1)$ for any $x_1, x_2 \in C$ we have

$$\| \mathcal{F}(v,x_1) - \mathcal{F}(v,x_2) \|_F \leq k \| x_1 - x_2 \| \ \ for\ v \geq 0.$$

(H6) For $k_1 > 0$ and there is a $k_1 \in L^1([0,b];\mathbb{R})$ then

$$\sup_{\|x\|_C \le K_1} \| \mathcal{F}(v,x) \| \le \alpha_{k_1}(v) \text{ and } \liminf_{k_1 \longrightarrow \infty} \int_0^b \frac{\alpha_{k_1}}{k_1} = \sigma < \infty, \forall v \in I.$$

(H7) $\sigma + M_a \liminf_{v \longrightarrow \infty} \dfrac{\Omega(r)}{r} \displaystyle\int_0^b m_\phi(s)ds < 1.$

Now we define the following operators as follows:

$$(\mathcal{S}z)(v) = R_1(v)\left[D\Big(0, \varphi(0) + g(z)(0)\Big)\right] + \mathcal{F}(v, z_v)$$
$$(\mathcal{K}z)(v) = \int_0^v R_1(v-s)\phi\left(s, z_s, \int_0^s h(s,\tau,z_\tau)d\tau\right)ds.$$

Then z is a mild solution of 3.21 and 3.22 if and only if z is a fixed point of $\mathcal{K} + \mathcal{S}$. Clearly, the linear operator $\mathcal{K}$ is continuous on $C([0,b],F)$ into itself.

Lemma 3.3.3. *[21] The linear operator $\mathcal{S}$ is a strict contraction.*

Proof. Let $x, y \in C([0,b],F)$ and $v \in [0,b]$, we have

$$\| (\mathcal{S}x)(v) - (\mathcal{S}y)(v) \| \le \| \mathcal{F}(v,x_v) - \mathcal{F}(v,y_v) \| \le k \| x_v - y_v \| = k \| x - y \| .$$

Then

$$\| \mathcal{S}x - \mathcal{S}y \| \le k \| x - y \| .$$

This implies that $\mathcal{S}$ is a contraction. □

Lemma 3.3.4. *[21] There is $r > 0$, such that $z = \mathcal{S}z + \mathcal{K}\omega, \omega \in B_r$ implies that $z \in B_r$. Where $B_r = \{z \in C([0,b],F) : \| z \|_\infty \le r\}$.*

Proof. We prove this by the contradiction method. Suppose $r > 0$ and $z \in C([0,b],F)$ and $\omega \in B_r$, then $z = \mathcal{S}z + \mathcal{K}\omega$ and $z \notin B_r$. Then for any $v \in [0,b]$, we have

$$\begin{aligned}
\| (\mathcal{S}z)(v) + (\mathcal{K}\omega)(v) \| &= \| \mathcal{F}(v,z_v) + R_1(v)\big[D\big(0,\varphi(0) + g(z)(0)\big)\big] \| \\
&\quad + \| \int_0^v R_1(v-s)\phi\left(s, z_s, \int_0^s h(s,\tau,z_\tau)d\tau\right)ds \| \\
&\le \| \mathcal{F}(v,z_v) \| + M_a \| D(0,\varphi(0) + g(z)(0)) \| \\
&\quad + M_a \int_0^b m_\phi(s)\Omega\left[\| z_s \| + \int_0^s m_h(\tau)\Omega_h(\| z_\tau \|)d\tau\right]ds
\end{aligned}$$

$$r < \| z \|_{\infty} \leq M_a \| D(0, \varphi(0) + g(z)(0)) \| + \alpha_r(v) + M_a \int_0^b m_\phi(s)\Omega(r)ds.$$

Dividing r on both sides, we have

$$1 \leq \frac{M_a}{r} \| D(0, \varphi(0) + g(z)(0)) \| + \frac{\alpha_r(v)}{r} + \frac{M_a}{r}\Omega(r) \int_0^b m_\phi(s)ds.$$

This implies that,

$$1 \leq \sigma + M_a \liminf_{r \longrightarrow \infty} \frac{\Omega(r)}{r} \int_0^b m_\phi(s)ds,$$

which contradicts (H7), hence $z \in B_{r_0}$. □

Lemma 3.3.5. *[21] Let M be a bounded subset of $C([0,b], F)$ with $\psi_\infty(M) > 0$, there is an integer n, such that*

$$\psi_\infty\Big(\mathcal{F}^{(n,0)}(\mathcal{K}, \mathcal{S}, M)\Big) < \psi_\infty(M).$$

Proof. For $M \subseteq C([0,b], F)$ is bounded and $\psi_\infty > 0$, we have

$$\begin{aligned}\mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M)(v) &= \{z(v), z \in \mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M)\}\\ &\subseteq \{z(v) - \mathcal{S}z(v), z \in \mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M)\}\\ &+ \{\mathcal{S}z(v), z \in \mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M)\}.\end{aligned}$$

By using properties of Hausdorff measure of noncompactness

$$\psi\Big(\mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M)(v)\Big) \leq \psi(\mathcal{K}(M)(v)) + k\psi\Big(\mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M)(v)\Big) \tag{3.24}$$

$$\psi\Big(\mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M)(v)\Big) \leq \frac{1}{1-k}\psi(\mathcal{K}(M)(v)).$$

Let $\| z \| = \sup_{-r<v<0} z(v)$ and $\int_0^v h(v, \tau, z_\tau)d\tau \in M$ be integrable. There is a function $C(v) \in L^1([0,b], \mathbb{R})$, then bringing Theorem 3.3.2, we have

$$\psi(\mathcal{K}(M)(v)) \leq \psi(\mathcal{K}z(v)) \leq \psi\bigg(\int_0^v R_1(v-s)\phi\Big(s, z_s, \int_0^s h(s, \tau, z_\tau)d\tau\Big)ds\bigg)$$

$$\psi(\mathcal{K}(M)(v)) \leq 2M_a \int_0^v C(s)\psi(z(s))ds \leq 2M_a\psi_\infty(M) \int_0^v C(s)ds.$$

Taking into account the density of $C([0,b], \mathbb{R})$ in $L_1([0,b], \mathbb{R})$. For any $\delta < \frac{1-k}{2M_a}$, there is a function $\mu \in C([0,b],\mathbb{R})$ with $\int_0^b \mid C(s)-\mu(s)\mid ds < \delta$. Equivalently

$$\begin{aligned}\psi(\mathcal{K}(M)(v)) &\leq 2M_a\psi_\infty(M)\Bigg[\int_0^b \mid C(s)-\mu(s)\mid ds + \int_0^b \mid \mu(s)\mid ds\Bigg]\\ &\leq 2M_a\psi_\infty(M)[\delta+\tau v],\end{aligned}$$

where $\tau = \sup_{0\leq s\leq b}\mid h(s)\mid$. Hence, $\psi(\mathcal{K}(M)(v)) \leq (2M_a\delta + 2M_a\tau(v))\psi_\infty(M)$.
Using Equation3.27, we have

$$\psi(\mathcal{K}(M)(v)) \leq (\alpha+\beta v)\psi_\infty(M), \tag{3.25}$$

where $\alpha = \frac{2M_a\delta}{1-k}$ and $\beta = \frac{2M_a\tau}{1-k}$.
Furthermore,

$$\begin{aligned}\mathcal{F}^{(2,0)}(\mathcal{K},\mathcal{S},M) \subseteq &\Big\{\mathcal{K}\omega(v), \omega \in \overline{co}\Big(\mathcal{F}^{(1,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big)\Big\}\\ &+\Big\{\mathcal{S}z(v), z\in \mathcal{F}^{(2,0)}(\mathcal{K},\mathcal{S},M)\Big\}.\end{aligned}$$

This implies that

$$\begin{aligned}\psi\Big(\mathcal{F}^{(2,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big) \leq\ &\psi\Big(\mathcal{K}(\overline{co}\Big(\mathcal{F}^{(1,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big))(v)\Big)\\ &+k\psi\Big(\mathcal{F}^{(2,0)}(\mathcal{K},\mathcal{S},M)\Big).\end{aligned}$$

$$\psi\Big(\mathcal{F}^{(2,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big) \leq \frac{1}{1-k}\psi\Big(\mathcal{K}(\overline{co}\Big(\mathcal{F}^{(1,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big))(v)\Big). \tag{3.26}$$

Using Lemma 3.3.1, there is $\sup_{-r<v<0}\omega(v)$, $\int_0^v h(v,\tau,z_\tau)d\tau \in F$ and
$\omega(v) \subseteq \overline{co}\Big(\mathcal{F}^{(1,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big)$, which implies that

$$\begin{aligned}\psi\Big(\mathcal{K}(\overline{co}\Big(\mathcal{F}^{(1,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big))(v)\Big) &\leq \psi\Big(\int_0^v R_1(v-s)\phi\Big(s,\omega_s,\int_0^s h(s,\tau,z_\tau)d\tau\Big)\Big)\\ &\leq 2M_a\int_0^v C(s)\psi(\overline{co}\Big(\mathcal{F}^{(1,0)}(\mathcal{K},\mathcal{S},M)\cup\{0\}\Big))(s)\Big)ds\\ &\leq 2M_a\int_0^v C(s)\psi\Big(\mathcal{F}^{(1,0)}(\mathcal{K},\mathcal{S},M)(s)\Big)ds.\end{aligned} \tag{3.27}$$

Using3.25 and 3.27 in 3.26, we have

$$\begin{aligned}\psi\Big(\mathcal{F}^{(2,0)}(\mathcal{K},\mathcal{S},M)(v)\Big) &\leq \frac{2(M_a)}{1-k}\int_0^v [\mid C(s)-\mu(s)\mid + \mid \mu(s)\mid](\alpha+\beta s)\psi_\infty(M)ds \\ &\leq \frac{2(M_a)}{1-k}\Big[(\alpha+\beta v)\int_0^v \mid C(s)-\mu(s)\mid ds + \tau(\alpha v+\beta\frac{v^2}{2})\Big]\psi_\infty(M) \\ &\leq \Big[\alpha^2+2\alpha\beta v+\frac{(\beta v)^2}{2}\Big]\psi_\infty(M).\end{aligned}$$

Thus

$$\psi\Big(\mathcal{F}^{(2,0)}(\mathcal{K},\mathcal{S},M)(v)\Big) \leq \Big[\alpha^2+2\alpha\beta v+\frac{(\beta v)^2}{2}\Big]\psi_\infty(M).$$

Using induction,

$$\psi\Big(\mathcal{F}^{(n,0)}(\mathcal{K},\mathcal{S},M)(v)\Big) \leq \Big[\alpha^n + C_n^1\alpha^{n-1}\beta v + C_n^2\alpha^{n-2}\frac{(\beta v)^2}{2!}+...+\frac{(\beta v)^n}{n!}\Big]\psi_\infty(M).$$

Accordingly,

$$\psi_\infty\Big(\mathcal{F}^{(n,0)}(\mathcal{K},\mathcal{S},M)\Big) \leq \Big[\alpha^n + C_n^1\alpha^{n-1}\beta b + C_n^2\alpha^{n-2}\frac{(\beta b)^2}{2!}+...+\frac{(\beta b)^n}{n!}\Big]\psi_\infty(M).$$

Since $0<\alpha<1$ and $\beta b>0$, then from Lemma **??** there is $n_0\in\mathbb{N}$, and we have

$$S_{n_0} = \Big[\alpha^{n_0} + C_{n_0}^1\alpha^{n_0-1}\beta b + C_{n_0}^2\alpha^{n_0-2}\frac{(\beta b)^2}{2!}+...+\frac{(\beta b)^{n_0}}{n_0!}\Big] < 1,$$

then

$$\psi_\infty\Big(\mathcal{F}^{(n,0)}(\mathcal{K},\mathcal{S},M)\Big) < \psi_\infty(M).$$

□

Lemma 3.3.6. *[21] Let M be a bounded subset of $C([0,b],F)$. If $\mathcal{K}(M)$ is equicontinuous, then $\mathcal{F}^{(n,0)}(\mathcal{K},\mathcal{S},M)$ is also equicontinuous for $n>0$.*

Proof. Let $z\in\mathcal{F}(\mathcal{K},\mathcal{S},M)$ and $v\in M$, which implies $z=\mathcal{S}z+\mathcal{K}w$. For $v,v_1\in[0,b]$ such that

$$\begin{aligned}\parallel z(v)-z(v_1)\parallel_F &\leq \parallel \mathcal{S}z(v)-\mathcal{S}z(v_1)\parallel_F + \parallel \mathcal{K}w(v)-\mathcal{K}w(v_1)\parallel_F \\ &= \parallel (R_1(v)-R_1(v_1))[\mathcal{D}(0,\varphi(0)+g(z)(0))]\parallel_F \\ &+ \parallel \mathcal{F}(v,z_v)-\mathcal{F}(v_1,z_{v_1})\parallel \\ &+ \mathcal{K}(|v-v_1| + \parallel z(v)-z(v_1)\parallel_F) + \parallel \mathcal{K}w(v)-\mathcal{K}w(v_1)\parallel_F .\end{aligned}$$

Consequently

$$\begin{aligned}
&\| z(v) - z(v_1) \| \\
&\leq \frac{1}{1-k}\Big(\| \mathcal{K}w(v) \quad -\mathcal{K}w(v_1) \|_F + \| (R_1(v) - R_1(v_1))[\mathcal{D}(0, \varphi(0) + g(z)(0))] \|_F \Big) \\
&+ \frac{k}{1-k}|v - v_1|.
\end{aligned}$$

Hence, $\| z(v) - z(v_1) \|_F \rightarrow 0$ as $v \rightarrow v_1$ and $\mathcal{F}(\mathcal{K}, \mathcal{S}, M)$ is equicontinuous. By Lemma 3.3.2 , $\mathcal{F}^{(1,0)}(\mathcal{K}, \mathcal{S}, M) = \overline{co}(\mathcal{F}(\mathcal{K}, \mathcal{S}, M) \cup \{0\})$ is equicontinuous. Using induction, $\mathcal{F}^{(n,0)}(\mathcal{K}, \mathcal{S}, M)$ is equicontinuous $\forall n \geq 1$. Now in this position, we give the existence result for this work. □

Theorem 3.3.3. *[21] Suppose that (H1) − (H7) hold. Then Equations* 3.21 *and* 3.22 *have at least one mild solution for* $[-r, b]$.

Proof. For $C \subset B_r$ is a countable set, then $\overline{C} = \mathcal{F}^{(n_0,0)}(\mathcal{K}, \mathcal{S}, C)$. By Lemma 3.3.5 ,

$$\psi_\infty\big(C\big) = 0 \Rightarrow \mathcal{K}\big(C\big) \ \textit{is compact.}$$

By Lemma 3.3.6 , $\mathcal{F}^{(n_0,x_0)}(\mathcal{K}, \mathcal{S}, C)$ is equicontinuous and by Lemma **??** ,

$$\psi_C(C) = \psi_\infty(C) = 0,$$

which implies that C is relatively compact. From Theorem **??** and Lemmas 3.3.3 and 3.3.4 , we have $\mathcal{S} + \mathcal{K}$, which have a fixed point in B_r. Hence systems 3.21 and 3.22 have mild solutions for $[-r, b]$. □

3.3.2 Example

[21] Consider the following neutral partial integrodifferential equation of the form

$$\begin{aligned}
\frac{\partial}{\partial t}[p(s, z(y, t-r))] &= \frac{\partial}{\partial y}[p(s, z(y, t-r))] + \int_0^t e^{-(s-t)} p(s, z(y, s-r))ds \\
&\quad + H\Big(t, z(y, t-r), \int_0^t k(t, s, w(x, y-r))ds\Big) \\
&for\ y \in [0, \pi], \quad t \in I = [0, b], \quad z(0, t) = z(\pi, t) = 0, \quad t \geq 0, \\
z_0(y) = \varphi(t, y) &+ \int_0^b m(s) \log(1 + |z(s)(y)|)ds; \quad t \in [-r, 0], y \in [0, \pi],
\end{aligned} \tag{3.28}$$

where φ is continuous.
Let $h(v, s, z_s) = k(t, s, w(x, y - r)), 0 \leq y \leq \pi$ and $D(t, z_t) = p(s, z(y, t - r))$. Take $F = L^2[0, \pi]$ and define $A : F \to F$ as $Aw = w'$ with domain

$$D(A) = \{w \in F : W \textit{ is absolutely continuous } w' \in F, w(y) = w(0) = 0\}.$$

It is clear that A is an infinitesimal generator of semigroup T(t) defined by $T(t)w(s) = w(t + s)$, for each $w \in F$. Thus, $[T(t)]_{t \geq 0}$ is not compact in F and $\beta(T(t)D) \leq \beta(D)$ where β is the Hausdorff measure of noncompactness and $\sup_{t \in I} \parallel T(t) \parallel \leq 1$.
Next, to assume the following, $g : C([0, b]; F) \to F$ is a continuous function defined by $g(z)(y) = \int_0^b m(s) \log(1 + z(s)(y))ds, z \in C([0, b]; F)$. Moreover, for any $v \geq 0$ and $y \in F$, we have

$$\parallel H(v)(y) \parallel_F \leq b(t) \parallel y \parallel \quad \textit{and} \parallel \frac{d}{dt} H(v)y \parallel_F \leq b(t) \parallel y \parallel .$$

We could see that the above system admits a resolvent operator. Further, the functions H and k satisfy all our assumptions. Finally, the above said partial differential system 3.28 has a mild solution of $[-r, b]$.

Conclusion

In this work, we present the Mönch-Krasnoselski fixed point theorem in Banach spaces and some applications of this theorem to prove the existence of solutions to nonlinear problems. We also give examples to illustrate the obtained results. On the other hand, we use the technique of noncompactness measures which is an important tool in nonlinear analysis especially in theory of condensing operators. We can extended those results for another nonlinear problems in more general locally convex spaces.

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