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Option :
« Functional analysis and Differential equation»

Presented by :

Cherif Zahra

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Some integral inequalities via Steklov operator

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رجوت كريما قد وثقت بصنعه

وما كان من يرجو الكريم يخيب

الحمد لله الذي يسر البدء وجمل الختام

تم بحمد الله تخرجي من

قسم الرياضيات

تخصص تحليل دالي ومعادلات تفاضلية

وما كنت سأصل لهذا الا بفضل الله

ثم بفضل والدي ولأجل ذلك سأسعى لدرجات أعلى ومقام أسمى

بتوفيق الله وبركته.

اهداء

الحمد لله حبا وشكرا وامتنان على البدء والختام

{وآخر دعواهم أن الحمد لله رب العالمين }

الى من أمدني باول شمعه علم انارت ظلمه دربي الاولى ورسخ في فكري ان النجاح هو حصاد ارض سقياها الصبر والاصرار الى معلمي الأول

ابي العزيز.

الى الأميرة التي طالما تمنت ان تقر عينها برؤيتي في يوم كهذا الى جنتي التي جعلت الشدائد بردا وسلاما على روحي بدعائها

حبيبتي امي.

الى من شددت عضدي بهم خيرة ارزاق يابي الثاني ايمن عبد السميع وسندي حليمة وضلعي الثابت علياء وقطع من قلبي هناء وخديجه ورميساء.

لكن يا من كن عونا لي في هذا الطريق وجعلن سنين الجامعة تحلو بوجودهن رشا وداد امينه رانيا انوار منال ساميه.

اهديكم نجاحي الذي طالما تمنيته وها انا اليوم اتممت اول ثمراته بفضل الله عز وجل ثم بفضل دعمكم لي.

Contents

Introduction

INTRODUCTION

The purpose of this work is to give some results from two internationally research articles on a class of integral inequalities, [1]-[8]. All integral inequalities are considered in the Lebesgue space L_p, where $p \neq 1$. The techniques employed comprise integral calculus properties, the Fubini theorem, Hölder's classical inequalities, and integration by parts.

We used the derivative formulas of a function defined by an integral since the variable lies at the integral's boundary and is used as the second variable in the integrated function. The articles employ the Hardy, Copson, and Steklov operators.

The memory consists of a preliminary, two chapters and a bibliography.

The **preliminary** part presents certain definitions and properties necessary for this work concerning:

1. **Fubini's theorem:** a definition and an observation are given in the case of a non-constant bounded integral.

2. **Function defined by an integral:** we present the three forms of derivatives for a function defined by an integral, as well as the general case (Leibniz rule).

3. **Weighted function:** we define a weight function and provide several examples.

4. The classical Hölder inequality for $p < 0$, $0 < p < 1$ and $p \geq 1$.

5. **Some several operators** known as:

   (a) Hardy operator, Hardy dual operator and weighted Hardy operator.

   (b) Copson operator, Copson dual operator and weighted Copson operator.

   (c) Steklov operator, Pachpatte operator and weighted Steklov operator.

   (d) General Hardy-Steklov operator.

   (e) Hardy-Steklov operator T and two Hardy-Steklov type operators.

The first chapter contains an academic paper on the generalization of integral inequalities utilizing the Hardy-Steklov and Copson-Steklov operators, with proof given two results associated with two parameters of summation and applications related to these inequalities.

In **the second chapter**, we put our interest in examining some weighted inequalities of the type Hardy-Steklov and Copson-Steklov which depend on integral operators of the Hardy-Steklov and Copson-Steklov, then we give as an application the particular cases depending on the boundary functions r and h.

Chapter I

Preliminary

Chapter 1

Preliminary

1.1 Definitions and properties on L_p spaces

Definition 1.1. *(**The** L_p **spaces**) Let $\Omega \subset \mathbb{R}$, a measurable set with $0 < p < \infty$ and let $f : \Omega \to \mathbb{R}$. We say that $f \in L_p(\Omega)$ if*

1. *f is measurable on Ω.*
2. $\|f\|_{L_p(\Omega)} = \Big(\int |f(x)|^p \, dx \Big)^{\frac{1}{p}} < \infty.$

Definition 1.2. *Let $e \subset \Omega$ such that $mes(e) = 0$, $f : \Omega \to \mathbb{R}$ measurable, the essential sup and the essential inf are defined as a sequence.*

- $ess \sup_{x\in\Omega} f(x) = \inf \sup_{x\in\Omega\setminus e} f(x).$
- $ess \inf_{x\in\Omega} f(x) = \sup \inf_{x\in\Omega\setminus e} f(x).$

Definition 1.3. *Let $\Omega \subset \mathbb{R}$, measurable and let $f : \Omega \to \mathbb{R}$, $\big|\Omega\big| > 0$. We say that $f \in L_\infty(\Omega)$ if f is measurable and*

$$\big\|f\big\|_{L_\infty(\Omega)} = ess \sup_{x\in\Omega} \big|f(x)\big| < \infty. \tag{1.1}$$

1.1.1 Fubini's Theorem

Theorem 1.1. *Let $f(x, y)$ be a measurable function on Ω , and $\Omega = (a, b) \times (c, d) \subseteq \mathbb{R} \times \mathbb{R}$, where $a \le b$ and $c \le d$ hence $f(x, y)$ is integrable on (c, d) for partially all $x \in (a, b)$ and $f(x, y)$*

is integrable on (a,b) for all $y \in (c,d)$:

$$\int_{\Omega} f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

Remark 1.1. *We have the partial case where the integral has a non-constant boundary when using the fubini theorem*

$$\int_0^{\infty} \int_0^{x} \Phi(x,y)dy\,dx = \int_0^{\infty} \int_y^{\infty} \Phi(x,y)dx\,dy.$$

1.1.2 Integration by parts

Definition 1.4. *Let $I \subseteq \mathbb{R}$, μ, υ be two functions of $\mathcal{C}^1(I,\mathbb{R})$ class and $a,b \in I$, then*

$$\int_a^b \mu(x)\upsilon'(x)\,dx = \Big[\mu(x)\upsilon(x)\Big]_a^b - \int_a^b \mu'(x)\upsilon(x)\,dx. \tag{1.2}$$

1.1.3 Weighted function

Definition 1.5. *A function is considered a weight function on an interval $I \subseteq \mathbb{R}$ if and only if it is a positive, measurable function on I.*

Example 1.1. *.*

- *In the field of integral inequality, we generally put $w(x) = 1$ and $I = (0,\infty)$,*

1.1.4 Function defined by an integral

- **Function** $x \longrightarrow F(x) = \int_{r(x)}^{h(x)} f(t)dt$

  Definition 1.6. *Let f be an integrable function on $I \subset \mathbb{R}$ and r,h be two functions of $\mathcal{C}^1(I,\mathbb{R})$ class. Let*

  $$F(x) = \int_{r(x)}^{h(x)} f(t)dt,$$

  then the derivative function of $F(x)$ exists and is given by

  $$F'(x) = h'(x)f(h(x)) - r'(x)f(r(x)).$$

We have the following examples:

Example 1.2. *Let f be an integrable function on*

$$I = [a, b] \subset \mathbb{R}.$$

1. *The function F is defined by the integral $\int_a^x f(t)dt$.*

$$F: \ I \longrightarrow \mathbb{R}$$
$$x \mapsto F(x) = \int_a^x f(t)dt.$$

   (a) *The function F is continuous on I and $F(a) = 0$*

   (b) *F is derivable in all $x \in I$ and $F'(x) = f(x)$*

2. *The function F is defined with the integral $\int_x^b f(t)dt$.*

$$F: \ I \longrightarrow$$
$$\mathbb{R}$$
$$x \mapsto F(x) = \int_x^b f(t)dt.$$

   (a) *The function F is continuous on I and $F(b) = 0$*

   (b) *F is derivable in all $x \in I$ and $F'(x) = -f(x)$*

- **Function** $x \longrightarrow \Phi(x) = \int_a^b f(x,t)dt.$

Theorem 1.2. *Let f be an application of $I \times [a, b]$ where I is an interval of $\mathbb{R}$ and for every $x \in I$, the partial application $\frac{\partial f}{\partial x} : t \mapsto \frac{\partial f}{\partial x}(x, t)$ is integrable on $[a, b]$. If the function*

$$f: \ I \times [a, b] \longrightarrow \mathbb{R},$$
$$(x, t) \mapsto f(x, t).$$

is continuous on $I \times [a, b]$ and the partial derivative function $\frac{\partial f}{\partial x}$ as well continues on $I \times [a, b]$, then the function

$$\Phi: \ I \longrightarrow \mathbb{R}$$
$$x \mapsto \int_a^b f(x,t)dt.$$

is derivable on I and its derivative Φ' verifying

$$\Phi'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)dt, forall \in I,$$

this later relationship can also be written as

$$\frac{d}{dx}\left(\int_a^b f(x,t)dt\right) = \int_a^b \frac{\partial f}{\partial x}(x,t)dt, forallx \in I.$$

- **Leibniz Rule**

  For r, h are in $\mathcal{C}^1(I,\mathbb{R})$ and the function F is defined as

$$F(x) = \int_{r(x)}^{h(x)} f(x,t)dt. \tag{1.3}$$

  We have the derivative function of F exists and it is given by

$$F'(x) = h'(x)f(x,h(x)) - r'(x)f(x,r(x)) + \int_{r(x)}^{h(x)} \frac{\partial f}{\partial x}(x,t)dt. \tag{1.4}$$

1.2 Hölder inequality

Some properties of conjugates

Definition 1.7. *Let $p, q \neq 0$ we say p, q are two conjugates if*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

we conclude that

$$\frac{1}{q} = \frac{p-1}{p} \quad , \quad \frac{1}{p} = \frac{q-1}{q},$$

$$\frac{q}{p} = q-1 \quad , \quad \frac{p}{q} = p-1,$$

$$p.q = q+p.$$

Lemma 1.1. *Let $\Omega \subset \mathbb{R}$ be a measurable set, $f \in L_p(\Omega,\mathbb{R})$ and $g \in L_q(\Omega,\mathbb{R})$*

with $\frac{1}{p} + \frac{1}{q} = 1$, *then*

1. *For* $p \geq 1$ *we get*

$$\int_\Omega | f(x)g(x) | \, dx \leq \| f \|_{L_p(\Omega)} \| g \|_{L_q(\Omega)}$$

$$= \left(\int_\Omega | f(x) |^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega | g(x) |^q \, dx \right)^{\frac{1}{q}}.$$

2. *For* $0 < p < 1$ *we get*

$$\int_\Omega | f(x)g(x) | \, dx \geq \| f \|_{L_p(\Omega)} \| g \|_{L_q(\Omega)}$$

$$= \left(\int_\Omega | f(x) |^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega | g(x) |^q \, dx \right)^{\frac{1}{q}}.$$

3. *For* $p < 0$ *we get*

$$\int_\Omega | f(x)g(x) | \, dx \geq \left(\int_\Omega | f(x) |^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega | g(x) |^q \, dx \right)^{\frac{1}{q}}.$$

1.3 Some several known operators.

1.3.1 Hardy operators

The well-known Hardy operator is defined for all non-negative measurable functions f on $(0, \infty)$.

1. **Hardy operator**:

$$H(f)(x) = \frac{1}{x} \int_0^x f(t)dt.$$

2. **Hardy dual operator**:

$$\tilde{H}(f(x)) = \frac{1}{x} \int_x^\infty f(t)dt.$$

3. **Weighted Hardy operator**:

$$H(f)(x) = \frac{1}{W(x)} \int_0^x f(t)w(t)dt,$$

   where w is a weighted function and

$$W(x) = \int_0^x w(t)dt.$$

4. **Hardy type operator**:

$$F(f)(x) = \int_0^x f(t)dt.$$

1.3.2 Copson operators

Let f be a non-negative measurable function on $(0, \infty)$.

1. **Copson operator**:

$$C(f)(x) = \int_x^\infty \frac{f(t)}{t} dt.$$

2. **Copson dual operator**:

$$\tilde{C}(f)(x) = \int_0^x \frac{f(t)}{t} dt.$$

3. **Weighted Copson operator**:

$$C_w(f)(x) = \int_x^\infty \frac{f(t)w(t)}{W(t)} dt.$$

1.3.3 Steklov operators

1. **Steklov operator**

   The well-known Steklov operator is defined by

$$S(f)(x) = \int_{r(x)}^{h(x)} f(t)dt,$$

where f is a non-negative measurable function on $(0,\infty)$ and the boundary functions r, h satisfying the following conditions :

r, h are non-negative, differentiable, increasing functions on $[0,\infty]$.

$$\text{For all } x \in (0,\infty) : 0 < r(x) < h(x) < \infty. \tag{1.5}$$

$r(0) = h(0) = 0$ and $r(\infty) = h(\infty) = \infty$.

2. **Pachpatte operator** :

$$P(f)(x) = \int_{\frac{x}{2}}^{x} f(t)dt.$$

3. **Weighted Steklov operator**

$$S_w(f)(x) = \int_{r(x)}^{h(x)} f(t)w(t)dt.$$

1.3.4 General Hardy-Steklov operator

The general Hardy-Steklov operator is defined by

$$G_{HS}(f)(x) = g(x)\int_{r(x)}^{h(x)} f(t)dt,$$

where g is a positive measurable function on $(0,\infty)$ and f is a non-negative measurable function on $(0,\infty)$ with the boundary functions r, h satisfying the following conditions :

1. r, h are non-negative, differentiable and increasing functions on $[0,\infty]$.
2. For all $x \in (0,\infty) : 0 < r(x) < h(x) < \infty$.
3. $r(0) = h(0) = 0$ and $r(\infty) = h(\infty) = \infty$.

.

1.3.5 Hardy-Steklov operators

The Hardy-Steklov operator is defined by

1. **Hardy-Steklov operator**:

$$T(f)(x) = \frac{1}{x}\int_{r(x)}^{h(x)} f(t)dt,$$

where the functions f, r and h satisfied the three above conditions of Steklov operator (1.5).

2. **Hardy-Steklov type operator**:

The Hardy-Steklov type operators are defined by

$$T_1(f)(x) = \frac{1}{x}\int_{0}^{h(x)} f(t)dt,$$

with the boundary function h satisfying the following conditions:

(a) h is a non-negative, differentiable and increasing function on $[0, \infty]$.

(b) For all $x \in (0, \infty) : 0 < h(x) < \infty, \ \ h(0) = 0$ and $h(\infty) = \infty$.

$$T_2(f)(x) = \frac{1}{x}\int_{r(x)}^{\infty} f(t)dt,$$

with the boundary function r satisfying the following conditions:

(a) r is a non-negative, differentiable and increasing function on $(0, \infty]$.

(b) For all $x \in (0, \infty) : 0 < r(x) < \infty, \ \ r(0) = 0$ and $r(\infty) = \infty$.

Hardy pachpatte operator

$$HP(f)(x) = \frac{1}{x}\int_{\frac{X}{2}}^{x} f(y)dy.$$

Chapter II

Hardy-Steklov and Copson-Steklov inequalities

Chapter 2

Hardy-Steklov and Copson-Steklov inequalities

2.1 Introduction and Preliminary

In this section Hardy-Steklov and Copson-Steklov integral inequalities are presented. Hardy-Steklov operator is defined by

$$(F_s f)(x) = \int_{r(x)}^{h(x)} f(t)v(t)dt, \tag{2.1}$$

and Copson-Steklov operator is defined by

$$(C_s f)(x) = \int_{r(x)}^{h(x)} \frac{f(t)\phi(t)}{\Phi(t)}dt, \tag{2.2}$$

where v, ϕ are two weighted functions and for $x \in (0, \infty)$

$$\Phi(x) = \int_0^x \phi(t)dt.$$

Now we present the following lemma which is a tool to establish the principal results:

Lemma 2.1. *Let $m \in \mathbb{R}-1$, $1 < p \leq q < \infty$ and f, v be non-negative measurable functions on (a,b), then*

$$\int_a^b \frac{v(x)}{V^m(x)} f^p(x)dx \leq \left(\int_a^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)}{V^{\frac{mq}{p}}(x)} f^q(x)dx\right)^{\frac{p}{q}}, \tag{2.3}$$

where $V(x) = \int_0^x v(t)dt$

Proof. For $1 < p \leq q, \text{then} 1 \leq \frac{q}{p}$.

puting $r = \frac{q}{p}$, we get

$$\frac{1}{r'} = 1 - \frac{1}{r} = 1 - \frac{p}{q}.$$

if $\Phi \in L_r(a,b)$ and $\Psi \in L_{r'}(a,b)$, we obtain

$$\int_a^b \Phi(t)\Psi(t)dt \leq \left(\int_a^b \Psi^{r'}(t)dt\right)^{\frac{1}{r'}} \left(\int_a^b \Phi^r(t)dt\right)^{\frac{1}{r}}. \tag{2.4}$$

Now by choosing $\Psi = v^{\frac{1}{r'}}$, $\Phi = v^{\frac{1}{r}} \frac{f^p}{V^m}$, we get

$$\begin{aligned} \int_a^b \frac{v(t)}{V^m(t)} f^p(t)dt &= \int_a^b v^{\frac{1}{r'}}(t)\big(v^{\frac{1}{r}}(t)\frac{f^p(t)}{V^m(t)}\big)dt \\ &= \int_a^b \Psi(t)\Phi(t)dt. \end{aligned}$$

Applying Hölder‘s inequality (2.4), we get

$$\begin{aligned}\int_a^b \frac{v(t)}{V^m(t)} f^p(t)dt &\leq \int_a^b \left(\Psi^{r'}(t)dt\right)^{\frac{1}{r'}} \left(\int_a^b \Phi^r(t)dt\right)^{\frac{1}{r}} \\ &= \left(\int_a^b \left(v^{\frac{1}{r'}}(t)\right)^{r'} dt\right)^{\frac{1}{r'}} \left(\int_a^b \left(v^{\frac{1}{r}}(t)\frac{f^p(t)}{V^m(t)}\right)^r dt\right)^{\frac{1}{r}} \\ &= \left(\int_a^b v(t)dt\right)^{\frac{1}{r'}} \left(\int_a^b v(t)\frac{f^{rp}(t)}{V^{mr}(t)}dt\right)^{\frac{1}{r}} \\ &= \left(\int_a^b v(t)dt\right)^{1-\frac{p}{q}} \left(\int_a^b v(t)\frac{f^q(t)}{V^{m\frac{q}{p}}(t)}dt\right)^{\frac{p}{q}}.\end{aligned}$$

□

Remark 2.1. *Let $m = p - \alpha$ and $a = 0$, then for $\alpha \neq p - 1$ we deduce that :*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} f^p(x)dx \leq \left(\int_0^b v(t)dt\right)^{1-\frac{p}{q}} \left(\int_0^b v(t)\frac{f^q(t)}{V^{q-\frac{\alpha q}{p}}(t)}dt\right)^{\frac{p}{q}}. \tag{2.5}$$

2.2 Main results

We present the first result involving the Copson-Steklov operator defined as follows :

$$(C_s f)(x) = \int_{r(x)}^{h(x)} \frac{f(t)\phi(t)}{\Phi(t)}dt.$$

Theorem 2.1. *Let f, ϕ be non-negative measurable functions on $(0, \infty)$, $1 < p \leq q < \infty$ and r, h satisfied the conditions (1.5). If $\alpha < p - 1$, then*

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_s f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \times\left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} \mid J(x) \mid^q dx\right)^{\frac{p}{q}}, \tag{2.6}$$

where

$$J(x) = \frac{\Phi(x)}{\phi(x)}\left\{\frac{[\Phi(h(x))]'}{\Phi(h(x))}f(h(x)) - \frac{[\Phi(r(x))]'}{\Phi(r(x))}f(r(x))\right\}.$$

Proof. We know that:

$$\Big(\Phi(h(x))\Big)' = h'(x)\Phi'(h(x)) = h'(x)\phi(h(x)).$$

and

$$\Big(\Phi(r(x))\Big)' = h'(x)\phi(r(x)).$$

By putting $g(t) = \dfrac{f(t)\phi(t)}{\Phi(t)}$ and applying (1.3), we obtain :

$$\begin{aligned}
\Big(C_s f\Big)'(x) &= h'(x)g(h(x)) - r'(x)g(r(x)) \\
&= h'(x)\phi(h(x))\frac{f(h(x))}{\Phi(h(x))} - r'(x)\phi(r(x))\frac{f(r(x))}{\Phi(r(x))} \\
&= \Big(\Phi(h(x))\Big)'\frac{f(h(x))}{\Phi(h(x))} - \Big(\Phi(r(x))\Big)'\frac{f(r(x))}{\Phi(r(x))} \\
&= \frac{\Big(\Phi(h(x))\Big)'}{\Phi(h(x))}f(h(x)) - \frac{\Big(\Phi(r(x))\Big)'}{\Phi(r(x))}f(r(x)) \\
&= \frac{\phi(x)}{\Phi(x)}\left\{\frac{\Phi(x)}{\phi(x)}\left[\frac{\Big(\Phi(h(x))\Big)'}{\Phi(h(x))}f(h(x)) - \frac{\Big(\Phi(r(x))\Big)'}{\Phi(r(x))}f(r(x))\right]\right\} \\
&= \frac{\phi(x)}{\Phi(x)}J(x).
\end{aligned}$$

Integrating by parts in the left-hand side of inequality (2.6),
for $p - \alpha \neq 1$ we get:

$$\begin{cases} u'(x) = \dfrac{\phi(x)}{\Phi^{p-\alpha}(x)} \\ \\ w(x) = (C_s f)^p(x) \end{cases} \Longrightarrow \begin{cases} u(x) = \dfrac{-1}{(p-\alpha-1)\Phi^{p-\alpha-1}(x)} \\ \\ w'(x) = p(C_s f)^{p-1}(x)(C_s f)'(x), \end{cases}$$

and

$$\begin{aligned}
\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_s f)^p(x)dx &= \left[\frac{-1}{(p-\alpha-1)\Phi^{p-\alpha-1}(x)}(C_s f)^p(x)\right]_0^b \\
&- \int_0^b \frac{-1}{(p-\alpha-1)\Phi^{p-\alpha-1}(x)} p(C_s f)^{p-1}(x)(C_s f)'(x)dx \\
&= \left[\frac{-(C_s f)^p(x)}{(p-\alpha-1)\Phi^{p-\alpha-1}(x)}\right]_0^b \\
&+ \frac{p}{(p-\alpha-1)}\int_0^b \frac{1}{\Phi^{p-\alpha-1}(x)}(C_s f)^{p-1}(x)(C_s f)'(x)dx \\
&= \frac{-(C_s f)^p(b)}{(p-\alpha-1)\Phi^{p-\alpha-1}(b)} + \frac{(C_s f)^p(0)}{(p-\alpha-1)\Phi^{p-\alpha-1}(0)} \\
&+ \frac{p}{(p-\alpha-1)}\int_0^b \frac{\phi(x)J(x)(C_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)}dx.
\end{aligned}$$

We have

$$(C_s f)(b) > 0,\ (C_s f)(0) = 0,\ \Phi(b) > 0 \text{ and } \Phi(0) = 0,$$

since $p > \alpha + 1$ we deduce

$$\frac{-(C_s f)^p(b)}{(p-\alpha-1)\Phi^{p-\alpha-1}(b)} < 0 \text{ and } \frac{(C_s f)^p(0)}{(p-\alpha-1)\Phi^{p-\alpha-1}(0)} = 0,$$

we conclude

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_s f)^p(x)dx \le \frac{p}{(p-\alpha-1)}\int_0^b \frac{\phi(x)J(x)(C_s f)^{p-1}(x)}{\Phi^{p-\alpha}(x)}dx.$$

For $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, applying Hölder inequality yields

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_sf)^p(x)dx$$

$$\leq \frac{p}{(p-\alpha-1)}\int_0^b \frac{\phi(x)J(x)(C_sf)^{p-1}(x)}{\Phi^{p-\alpha}(x)}dx$$

$$= \frac{p}{(p-\alpha-1)}\left[\int_0^b \left(\frac{\phi(x)}{\Phi^{p-\alpha}(x)}\right)^{\frac{1}{p}}\left(\frac{\phi(x)}{\Phi^{p-\alpha}(x)}\right)^{\frac{1}{p'}} J(x)(C_sf)^{p-1}(x)dx\right]$$

$$\leq \frac{p}{(p-\alpha-1)}\left(\int_0^b \left(\left(\frac{\phi(x)}{\Phi^{p-\alpha}(x)}\right)^{\frac{1}{p'}}(C_sf)^{\frac{p}{p'}}(x)\right)^{p'} dx\right)^{\frac{1}{p'}}$$

$$\times\left(\int_0^b \left(\left(\frac{\phi(x)}{\Phi^{p-\alpha}(x)}\right)^{\frac{1}{p}} \mid J(x) \mid \right)^{p} dx\right)^{\frac{1}{p}}$$

$$= \frac{p}{(p-\alpha-1)}\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_sf)^p(x)dx\right)^{\frac{1}{p'}}\left(\int_0^b \left(\frac{\phi(x)}{\Phi^{p-\alpha}(x)} \mid J(x) \mid^p dx\right)^{\frac{1}{p}},\right.$$

then

$$\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_sf)^p(x)dx\right)^{1-\frac{1}{p'}} \leq \frac{p}{(p-\alpha-1)}\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} \mid J(x) \mid^p dx\right)^{\frac{1}{p}},$$

it's the same as

$$\left(\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_sf)^p(x)dx\right)^{\frac{1}{p}} \leq \frac{p}{(p-\alpha-1)}\int_0^b \left(\frac{\phi(x)}{\Phi^{p-\alpha}(x)} \mid J(x) \mid^p dx\right)^{\frac{1}{p}},$$

for $p > 1$, we deduce

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_sf)^p(x)dx \leq \left(\frac{p}{(p-\alpha-1)}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} \mid J(x) \mid^p dx. \quad (2.7)$$

From the inequality (2.5), we have

$$\int_a^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} J^p(x)dx \leq \left(\int_0^b \phi(t)dt\right)^{1-\frac{p}{q}} \left(\int_0^b \phi(t)\frac{\mid J(x)\mid^q}{\Phi^{q-\frac{\alpha q}{p}}(t)}dt\right)^{\frac{p}{q}}. \tag{2.8}$$

Applying the inequalities (2.7) and (2.8), we result that

$$\begin{aligned}\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_s f)^p(x)dx \ &\leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}} \\ &\times\left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} \mid J(x)\mid^q dx\right)^{\frac{p}{q}}.\end{aligned}$$

We obtain the desired inequality. □

We present now the second Theorem via the Steklov operator defined by

$$(F_s f)(x) = \int_{r(x)}^{h(x)} f(t)v(t)dt.$$

Theorem 2.2. *Let f, v be non-negative measurable functions on $(0,\infty)$, $1 < p \leq q < \infty$ and r, h satisfied the conditions (1.5). If $\alpha < p-1$, then*

$$\begin{aligned}\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_s f)^p(x)dx \ &\leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \\ &\times\left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} \mid K(x)\mid^q dx\right)^{\frac{p}{q}},\end{aligned} \tag{2.9}$$

where

$$K(x) = \frac{V(x)}{v(x)}\Big\{[V(h(x))]' f(h(x)) - [V(r(x))]' f(r(x))\Big\}.$$

Proof. We already know that:

$$\Big(V(h(x))\Big)' = h'(x)V'(h(x)) = h'(x)v(h(x)),$$

and

$$\Big(V(r(x))\Big)' = h'(x)v(r(x)).$$

Setting $g(t) = f(t)v(t)$ gives $(F_s f)(x) = \displaystyle\int_{r(x)}^{h(x)} g(t)dt$, then

$$\begin{aligned}
\Big(F_s f\Big)'(x) &= h'(x)g(h(x)) - r'(x)g(r(x)) \\
&= h'(x)v(h(x))f(h(x)) - r'(x)v(r(x))f(r(x)) \\
&= \Big(V(h(x))\Big)' f(h(x)) - \Big(V(r(x))\Big)' f(r(x)) \\
&= \frac{v(x)}{V(x)}\left\{\frac{V(x)}{v(x)}\left[\Big(V(h(x))\Big)' f(h(x)) - \Big(V(r(x))\Big)' f(r(x))\right]\right\} \\
&= \frac{v(x)}{V(x)}K(x).
\end{aligned}$$

Integrating by parts in the left-hand side of (2.9), for $p - \alpha \neq 1$, we get

$$\begin{cases} u'(x) = \dfrac{v(x)}{V^{p-\alpha}(x)}, \\ \\ w(x) = (F_s f)^p(x), \end{cases} \Longrightarrow \begin{cases} u(x) = \dfrac{-1}{(p-\alpha-1)V^{p-\alpha-1}(x)}, \\ \\ w'(x) = p(F_s f)^{p-1}(x)(F_s f)'(x). \end{cases}$$

Hence

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_sf)^p(x)dx = \left[\frac{-1}{(p-\alpha-1)V^{p-\alpha-1}(x)}(F_sf)^p(x)\right]_0^b$$

$$-\int_0^b \frac{-1}{(p-\alpha-1)V^{p-\alpha-1}(x)}p(F_sf)^{p-1}(x)(F_sf)'(x)dx$$

$$= \left[\frac{-(F_sf)^p(x)}{(p-\alpha-1)V^{p-\alpha-1}(x)}\right]_0^b$$

$$+\frac{p}{(p-\alpha-1)}\int_0^b \frac{1}{V^{p-\alpha-1}(x)}(F_sf)^{p-1}(x)(F_sf)'(x)dx$$

$$= \frac{-(F_sf)^p(b)}{(p-\alpha-1)V^{p-\alpha-1}(b)} + \frac{(F_sf)^p(0)}{(p-\alpha-1)V^{p-\alpha-1}(0)}$$

$$+\frac{p}{(p-\alpha-1)}\int_0^b \frac{v(x)K(x)(F_sf)^{p-1}(x)}{V^{p-\alpha}(x)}dx.$$

We have

$$(F_sf)(b) > 0,\ (F_sf)(0) = 0,\ V(b) > 0 \text{ and } V(0) = 0,$$

given that $p > \alpha + 1$, we deduce

$$\frac{-(F_sf)^p(b)}{(p-\alpha-1)V^{p-\alpha-1}(b)} < 0 \text{ and } \frac{(F_sf)^p(0)}{(p-\alpha-1)V^{p-\alpha-1}(0)} = 0,$$

we obtain

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_sf)^p(x)dx \leq \frac{p}{(p-\alpha-1)}\int_0^b \frac{v(x)K(x)(F_sf)^{p-1}(x)}{V^{p-\alpha}(x)}dx.$$

By using Hölder inequality for $\dfrac{1}{p}+\dfrac{1}{p'}=1$, we deduce

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_sf)^p(x)dx$$

$$\leq \frac{p}{(p-\alpha-1)}\left(\int_0^b \left(\frac{v(x)}{V^{p-\alpha}(x)}\right)^{\frac{1}{p}} \left(\frac{v(x)}{V^{p-\alpha}(x)}\right)^{\frac{1}{p'}} K(x)(F_sf)^{p-1}(x)dx\right)$$

$$= \frac{p}{(p-\alpha-1)}\left[\int_0^b \left(\left(\frac{v(x)}{V^{p-\alpha}(x)}\right)^{\frac{1}{p'}} (F_sf)^{\frac{p}{p'}}(x)\right)^{p'} dx\right]^{\frac{1}{p'}}$$

$$\times\left[\left(\int_0^b \left(\frac{v(x)}{V^{p-\alpha}(x)}\right)^{\frac{1}{p}} K(x)\right)^{p} dx\right]^{\frac{1}{p}}$$

$$= \frac{p}{(p-\alpha-1)}\left(\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_sf)^p(x)dx\right)^{\frac{1}{p'}} \left(\int_0^b \left(\frac{v(x)}{V^{p-\alpha}(x)} \mid K(x)\mid^p dx\right)^{\frac{1}{p}},$$

then

$$\left(\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_sf)^p(x)dx\right)^{1-\frac{1}{p'}} \leq \frac{p}{(p-\alpha-1)}\left(\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} \mid K(x)\mid^p dx\right)^{\frac{1}{p}},$$

as a result

$$\left(\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_sf)^p(x)dx\right)^{\frac{1}{p}} \leq \frac{p}{(p-\alpha-1)}\int_0^b \left(\frac{v(x)}{V^{p-\alpha}(x)} \mid K(x)\mid^p dx\right)^{\frac{1}{p}},$$

for $p>1$, we deduce

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_sf)^p(x)dx \leq \left(\frac{p}{(p-\alpha-1)}\right)^{p}\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} \mid K(x)\mid^p dx. \quad (2.10)$$

From the inequality (2.5), we have

$$\int_a^b \frac{v(x)}{V^{p-\alpha}(x)} K^p(x)dx \leq \left(\int_0^b v(t)dt\right)^{1-\frac{p}{q}} \left(\int_0^b v(t) \frac{| K(x) |^q}{V^{q-\frac{\alpha q}{p}}(t)} dt\right)^{\frac{p}{q}}. \tag{2.11}$$

Applying inequalities (2.8) and (2.11), it follows

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (F_s f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}}$$

$$\times \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} \mid K(x) \mid^q dx\right)^{\frac{p}{q}}.$$

We obtain the desired inequality. □

2.2.1 particular cases

Some particular cases are established involving the choice of the functions h and r.

- Let $0 < \beta < \lambda < \infty, p > 1$.
  By taking $h(x) = \lambda x$ and $r(x) = \beta x$ in Theorem 2.6 and Theorem 2.2, we get the following corollary.

  Corollary 2.1. *Let f, v, ϕ be non-negative measurable functions on* $(0.\infty)$, $0 < \beta < \lambda < \infty, p > 1$ *and*

$$(F_{s,1} f)(x) = \int_{\beta x}^{\lambda x} f(y)v(y)dy, \qquad x > 0$$

$$(C_{s,1} f)(x) = \int_{\beta x}^{\lambda x} \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0$$

If $\alpha < p-1,$ *then*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_{s,1}f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}}$$

$$\times\left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} \mid K_1(x) \mid^q dx\right)^{\frac{p}{q}},$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_{s,1}f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}}$$

$$\times\left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} \mid J_1(x) \mid^q dx\right)^{\frac{p}{q}}.$$

If $q = p,$ *we get*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_{s,1}f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)}|K_1(x)|^p dx,$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_{s,1}f)^p(x)dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}|J_1(x)|^p dx,$$

where

$$K_1(x) = \frac{V(x)\Big[\lambda v(\lambda x)f(\lambda x) - \beta v(\beta x)f(\beta x)\Big]}{v(x)},$$

$$J_1(x) = \frac{\Phi(x)}{\phi(x)}\left\{\frac{\lambda\phi(\lambda x)}{\Phi(\lambda x)}f(\lambda x) - \frac{\beta\phi(\beta x)}{\Phi(\beta x)}f(\beta x)\right\}.$$

Remark 2.1. *For $\lambda = 1$ and $\beta = \frac{1}{2}$, we get a Pachpatte-type inequality.*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_{s,1}f)^p(x)dx \leq \left(\frac{p}{(p-\alpha-1)}\right)^p \times \int_0^b \frac{v^{1-p}(x)}{V^{-\alpha}(x)}\left(v(x)f(x) - \frac{1}{2}v\left(\frac{1}{2}x\right)f\left(\frac{1}{2}x\right)\right)^p(x)dx, \tag{2.12}$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_{s,1}f)^p(x)dx \leq \left(\frac{p}{(p-\alpha-1)}\right)^p \times \int_0^b \frac{\phi^{1-p}(x)}{\Phi^{-\alpha}(x)}\left(\frac{\phi(x)}{\Phi(x)}f(x) - \frac{\phi\left(\frac{1}{2}x\right)}{2\Phi\left(\frac{1}{2}x\right)}f\left(\frac{1}{2}x\right)\right)^p(x)dx. \tag{2.13}$$

Where

$$(F_{s,1}f)(x) = \int_{\frac{x}{2}}^{x} f(y)v(y)dy, \qquad x > 0,$$

$$(C_{s,1}f)(x) = \int_{\frac{x}{2}}^{x} \frac{f(y)\phi(y)}{\Phi(y)}dy, \qquad x > 0.$$

- Setting $r(x) = 0$ in Theorem 2.6 and Theorem 2.2, we obtain the following results.

Corollary 2.2. *Let f, v, ϕ be non-negative measurable functions on* $(0, \infty)$ *, $p > 1$ and*

$$(F_{s,2}f)(x) = \int_0^{h(x)} f(y)v(y)dy, \qquad x > 0,$$

$$(C_{s,2}f)(x) = \int_0^{h(x)} \frac{f(y)\phi(y)}{\Phi(y)}dy, \qquad x > 0,$$

where

$$\begin{cases} 0 < h(x) < \infty \quad \textit{for all} \quad x \in (0,\infty), \\ h(0) = 0 \quad \textit{and} \quad h(\infty) = \infty. \end{cases}$$

If $\alpha < p-1$, *then*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (F_{s,2}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}}$$

$$\times \left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)} \mid K_2(x) \mid^q dx\right)^{\frac{p}{q}},$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (C_{s,2}f)^p(x)dx \le \left(\frac{p}{p-\alpha-1}\right)^p \left(\int_0^b \phi(x)dx\right)^{1-\frac{p}{q}}$$

$$\times \left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)} \mid J_2(x) \mid^q dx\right)^{\frac{p}{q}}.$$

If $p = q$, *we have*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (F_{s,2}f)^p(x)dx \le \left(\frac{p}{(p-\alpha-1)}\right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)} |K_2(x)|^p dx,$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (C_{s,2}f)^p(x)dx \le \left(\frac{p}{(p-\alpha-1)}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} |J_2(x)|^p dx.$$

Where

$$K_2(x) = \frac{V(x)\big[V(h(x))\big]' f(h(x))}{v(x)},$$

$$J_2(x) = \frac{\Phi(x)}{\phi(x)} \frac{\left[\Phi(h(x))\right]'}{\Phi(h(x))} f(h(x)).$$

Remark 2.2. *If we take $h(x) = x$ in corollary 2.2 we obtain the following weighted Hardy-type inequality and Copson-type inequality.*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)} (F_{s,2}f)^p(x)dx \le \left(\frac{p}{(p-\alpha-1)}\right)^p \int_0^b \frac{v(x)}{V^{-\alpha}(x)} f^p(x)dx, \quad (2.14)$$

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} (C_{s,2}f)^p(x)dx \le \left(\frac{p}{(p-\alpha-1)}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx. \quad (2.15)$$

Where

$$(F_{s,2}f)(x) = \int_0^x f(y)v(y)dy, \qquad x > 0,$$

$$(C_{s,2}f)(x) = \int_0^x \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0.$$

- By setting $h(x) = \infty$ and reasoning a manner analogous to the proof of Theorem 2.6 and Theorem 2.2, we get the following corollary.

Corollary 2.3. *Let f, v, ϕ be non-negative measurable functions on $(0, \infty)$, $1 < p \le q < \infty$ and*

$$(F_{s,3}f)(x) = \int_{r(x)}^\infty f(y)v(y)dy, \qquad x > 0,$$

$$(C_{s,3}f)(x) = \int_{r(x)}^\infty \frac{f(y)\phi(y)}{\Phi(y)} dy, \qquad x > 0,$$

where

$$\begin{cases} 0 < r(x) < \infty \quad \text{for all} \quad x \in (0, \infty), \\ r(0) = 0 \quad \text{and} \quad r(\infty) = \infty. \end{cases}$$

If $\alpha > p-1$, *then*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_{s,3}f)^p(x)dx \;\leq \left(\frac{p}{(\alpha-p+1)}\right)^p \times\Big(\int_0^b v(x)dx\Big)^{1-\frac{p}{q}}\left(\int_0^b \frac{v(x)}{V^{q-\frac{\alpha}{p}q}(x)}|K_3(x)|^p dx\right)^{\frac{p}{q}},$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_{s,3}f)^p(x)dx \;\leq \left(\frac{p}{(\alpha-p+1)}\right)^p \times\Big(\int_0^b \phi(x)dx\Big)^{1-\frac{p}{q}}\left(\int_0^b \frac{\phi(x)}{\Phi^{q-\frac{\alpha}{p}q}(x)}|J_3(x)|^p dx\right)^{\frac{p}{q}}.$$

If $p = q$, *we get*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_{s,3}f)^p(x)dx \;\leq \left(\frac{p}{(\alpha-p+1)}\right)^p \int_0^b \frac{v(x)}{V^{p-\alpha}(x)}|K_3(x)|^p dx,$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_{s,3}f)^p(x)dx \;\leq \left(\frac{p}{(\alpha-p+1)}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha(x)}}|J_3(x)|^p dx.$$

Where

$$K_3(x) = -\frac{V(x)\big[V(r(x))\big]' f(r(x))}{v(x)},$$

$$J_3(x) = -\frac{\Phi(x)}{\phi(x)}\frac{\big[\Phi(r(x))\big]'}{\Phi(r(x))}f(r(x)).$$

Remark 2.3. *The following particular case of corollary 2.3 can be derived* *by taking $r(x) = x$ and $q = p$.*

$$\int_0^b \frac{v(x)}{V^{p-\alpha}(x)}(F_{s,3}f)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \int_0^b \frac{v(x)}{V^{-\alpha}(x)} f^p(x)dx, \quad (2.16)$$

and

$$\int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)}(C_{s,3}f)^p(x)dx \leq \left(\frac{p}{\alpha - p + 1}\right)^p \int_0^b \frac{\phi(x)}{\Phi^{p-\alpha}(x)} f^p(x)dx, \quad (2.17)$$

where

$$(F_{s,3}f)(x) = \int_x^\infty f(y)v(y)dy, \qquad x > 0,$$

$$(C_{s,3}f)(x) = \int_x^\infty \frac{f(y)\phi(y)}{\Phi(y)}dy, \qquad x > 0.$$

Remark 2.4. *We note that if $v(x) = 1$ in inequality (2.16) and inequality (2.14), we get the following Hardy weighted inequalities.*

$$\int_0^b x^\alpha H^p(x)dx \leq \left(\frac{p}{p - \alpha - 1}\right)^p \int_0^b x^\alpha f^p(x)dx,$$

and

$$\int_0^b x^\alpha \tilde{H}^p(x)dx \leq \left(\frac{p}{p - \alpha - 1}\right)^p \int_0^b x^\alpha f^p(x)dx,$$

where

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt,$$

and

$$\tilde{H}(f)(x) = \frac{1}{x}\int_x^\infty f(t)dt.$$

Chapter III

Some estimates for Hardy-steklov type operators

Chapter 3

Some estimates for Hardy-Steklov type operators

3.1 Introduction and Preliminary

It is generally known that the Hardy inequality is satisfied for monotone functions but not for arbitrary non-negative lebesgue measurable functions for L_p spaces with $0 < p < 1$ (see in [6]). A weaker condition than monotonicity was used in 2007 to derive the Hardy type inequality ([19]). Particularly, the following assumptions were validated.

The well-known Hardy-Steklov operator is defined as

$$T(f)(x) = \frac{1}{x} \int_{r(x)}^{h(x)} f(y)dy,$$

with the boundary functions r, h satisfying the following conditions:

1. r, h are non-negative, differentiable and increasing functions on $[0, \infty]$.
2. For all $x \in (0, \infty) : 0 < r(x) < h(x) < \infty$.
3. $r(0) = h(0) = 0$ and $r(\infty) = h(\infty) = \infty$.

where f is a non-negative lebesgue measurable function on $(0, \infty)$.

The classical Hardy operator and its dual are defined as follows:

$$H(f)(x) = \frac{1}{x}\int_0^x f(y)dy, \quad \tilde{H}(f)(x) = \frac{1}{x}\int_x^\infty f(y)dy.$$

The objective of this section is to give more details for some results involving the bellow Hardy-Steklov type operators T_1, T_2 and T_3 defined as follows:

$$T_1(f)(x) = \frac{1}{x}\int_0^{h(x)} f(y)dy, \tag{3.1}$$

with the boundary function h satisfying the following conditions:

1. h is a non-negative, differentiable and increasing function on $[0,\infty]$.
2. For all $x \in (0,\infty) : 0 < h(x) < \infty,\;\; h(0) = 0$ and $h(\infty) = \infty$.

$$T_2(f)(x) = \frac{1}{x}\int_{r(x)}^{\infty} f(y)dy, \tag{3.2}$$

with the boundary function r satisfying the following conditions:

1. r is a non-negative, differentiable and increasing function on $[0,\infty]$.
2. For all $x \in (0,\infty) : 0 < r(x) < \infty,\;\; r(0) = 0$ and $r(\infty) = \infty$.

$$T_3(f)(x) = \frac{1}{x}\int_{r(x)}^{h(x)} f(y)dy, \tag{3.3}$$

with the boundary functions r, h satisfying the following conditions:

1. r, h are non-negative, differentiable and increasing functions on $(0,\infty)$.
2. For all $x \in (0,\infty) : 0 < r(x) < h(x) < \infty$.

3.2 Main results

Throughout the section, we assume that the function f is a non-negative Lebesgue measurable function on $(0,\infty)$.

Lemma 3.1. *Let $0 < p < 1$, $c_1 > 0$ and f be a non-negative measurable function on $(0, \infty)$ such that for all $x > 0$,*

$$f(x) \leq \frac{c_1}{x}\bigg(\int_0^x f^p(y)y^{p-1}dy\bigg)^{\frac{1}{p}}. \tag{3.4}$$

Then

$$\bigg(\int_0^x f(y)dy\bigg)^p \leq c_2\int_0^x f^p(y)y^{p-1}dy, \tag{3.5}$$

where

$$c_2 = c_1^{p(1-p)}.$$

Proof. From the hypotheses (3.4), we have

$$f(x) \leq \frac{c_1}{x}\bigg(\int_0^x f^p(y)y^{p-1}dy\bigg)^{\frac{1}{p}},$$

then

$$xf(x) \leq c_1\bigg(\int_0^x f^p(y)y^{p-1}dy\bigg)^{\frac{1}{p}},$$

for $0 < p < 1$, we get

$$x^p f(x)^p \leq c_1^p\bigg(\int_0^x f^p(y)y^{p-1}dy\bigg). \tag{3.6}$$

Note that

$$f(x) = (f^p(x)x^p)^{\frac{1}{p}-1} f^p(x)x^{p-1}, \tag{3.7}$$

using the inequality (3.6), we result

$$\big(x^p f(x)^p\big)^{\frac{1}{p}-1} \leq \big(c_1^p\big)^{\frac{1}{p}-1}\bigg(\int_0^x f^p(y)y^{p-1}dy\bigg)^{\frac{1}{p}-1} = c_1^{1-p}\bigg(\int_0^x f^p(y)y^{p-1}dy\bigg)^{\frac{1}{p}-1},$$

multiplying by $f^p(x)x^{p-1}$, we get

$$\left(x^p f(x)^p\right)^{\frac{1}{p}-1} f^p(x)x^{p-1} \;\; \leq c_1^{1-p}\left(\int_0^x f^p(y)y^{p-1}dy\right)^{\frac{1}{p}-1} f^p(x)x^{p-1},$$

applying (3.7), we get

$$f(x) \leq c_1^{1-p}\left(\int_0^x f^p(y)y^{p-1}dy\right)^{\frac{1}{p}-1} f^p(x)x^{p-1},$$

for $0 < t \leq y \leq x$, we deduce

$$f(y) \leq c_1^{1-p}\left(\int_0^y f^p(t)t^{p-1}dt\right)^{\frac{1}{p}-1} f^p(y)y^{p-1},$$

integrating the above inequality over $(0, x)$ over y, we get

$$\int_0^x f(y)dy \leq c_1^{1-p}\int_0^x \left(\int_0^y f^p(t)t^{p-1}dt\right)^{\frac{1}{p}-1} f^p(y)y^{p-1}dy. \tag{3.8}$$

we have

$$\int_0^y f^p(t)t^{p-1}dt \leq \int_0^x f^p(t)t^{p-1}dt,$$

since $\dfrac{1}{p} - 1 > 0$, we obtain

$$\left(\int_0^y f^p(t)t^{p-1}dt\right)^{\frac{1}{p}-1} \leq \left(\int_0^x f^p(t)t^{p-1}dt\right)^{\frac{1}{p}-1}, \tag{3.9}$$

adding (3.8) and (3.9), we result

$$\begin{aligned}\int_0^x f(y)dy &\leq c_1^{1-p}\int_0^x\Big(\int_0^y f^p(t)t^{p-1}dt\Big)^{\frac{1}{p}-1} f^p(y)y^{p-1}dy.\\ &\leq c_1^{1-p}\int_0^x\Big(\int_0^x f^p(t)t^{p-1}dt\Big)^{\frac{1}{p}-1} f^p(y)y^{p-1}dy.\\ &= c_1^{1-p}\Big(\int_0^x f^p(t)t^{p-1}dt\Big)^{\frac{1}{p}-1}\int_0^x f^p(y)y^{p-1}dy\\ &= c_1^{1-p}\Big(\int_0^x f^p(y)y^{p-1}dy\Big)^{\frac{1}{p}-1}\int_0^x f^p(y)y^{p-1}dy\\ &= c_1^{1-p}\Big(\int_0^x f^p(y)y^{p-1}dy\Big)^{\frac{1}{p}},\end{aligned}$$

for $0<p<1$, we have the desired inequality

$$\Big(\int_0^x f(y)dy\Big)^p \leq c_2\int_0^x f^p(y)y^{p-1}dy.$$

□

Theorem 3.1. *Let* $0<p<1$, $\alpha<1-\dfrac{1}{p}$ *and* $\dfrac{1}{p}+\dfrac{1}{p'}=1$. *If* f *is non-negative measurable function on* $(0,\infty)$ *and satisfies*

$$f(x)\leq\frac{c_1}{x}\Big(\int_0^x f^p(y)y^{p-1}dy\Big)^{\frac{1}{p}},$$

then

$$\|x^\alpha T_1(f)(x)\|_{L_p(0,\infty)}\leq c_3\|x^{\frac{1}{p'}}(h^{-1}(x))^{\alpha-\frac{1}{p'}}f(x)\|_{L_p(0,\infty)}. \tag{3.10}$$

Where

$$c_3=c_1^{1-p}\big((1-\alpha)p-1\big)^{-\frac{1}{p}},$$

and

$$T_1(f)(x) = \frac{1}{x}\int_0^{h(x)} f(y)dy,$$

with the boundary function h satisfying the following conditions:

1. *h is a non-negative, differentiable and increasing function on $[0, \infty]$.*

2. *For all $x \in (0, \infty) : 0 < h(x) < \infty$, $h(0) = 0$ and $h(\infty) = \infty$.*

Proof. We can rewrite

$$\|x^\alpha T_1(f)(x)\|_{L_p(0,\infty)} = \left(\int_0^\infty \left(x^{\alpha-1}\int_0^{h(x)} f(y)dy\right)^p dx\right)^{\frac{1}{p}}.$$

Given that h is a non-negative, differentiable and increasing function on $[0, \infty]$, we conclude that h be a bijective mapping. Put $t = h(x)$, then $x = h^{-1}(t)$, where h^{-1} is the reciprocal function of h.
Applying inequality (3.4), we get

$$\begin{aligned}\|x^\alpha T_1(f)(x)\|_{L_p(0,\infty)} &= \left(\int_0^\infty \left|\left(h^{-1}(t)\right)^{\alpha-1}\int_0^t f(y)dy\right|^p (h^{-1}(t))'dt\right)^{\frac{1}{p}}\\ &= \left(\int_0^\infty \left(h^{-1}(t)\right)^{p(\alpha-1)}\left(\int_0^t f(y)dy\right)^p (h^{-1}(t))'dt\right)^{\frac{1}{p}},\end{aligned}$$

applying (3.5), we get

$$\|x^{\alpha}T_1(f)(x)\|_{L_p(0,\infty)}$$

$$\leq \left(\int_0^{\infty}(h^{-1}(t))^{p(\alpha-1)}\left(c_2\int_0^t f^p(y)y^{(p-1)}dy\right)(h^{-1}(t))'dt\right)^{\frac{1}{p}}$$

$$= c_2^{\frac{1}{p}}\left(\int_0^{\infty}\left[(h^{-1}(t))^{p(\alpha-1)}\left(\int_0^t f^p(y)y^{(p-1)}dy\right)(h^{-1}(t))'\right]dt\right)^{\frac{1}{p}}$$

$$= c_2^{\frac{1}{p}}\left(\int_0^{\infty}\int_0^t f^p(y)y^{(p-1)}(h^{-1}(t))^{p(\alpha-1)}(h^{-1}(t))'dydt\right)^{\frac{1}{p}}.$$

Applying Fubini's Theorem, we get

$$\|x^{\alpha}T_1(f)(x)\|_{L_p(0,\infty)}$$

$$\leq c_2^{\frac{1}{p}}\left(\int_0^{\infty}\int_y^{\infty} f^p(y)y^{(p-1)}(h^{-1}(t))^{p(\alpha-1)}(h^{-1}(t))'dtdy\right)^{\frac{1}{p}}$$

$$= c_2^{\frac{1}{p}}\left(\int_0^{\infty} f^p(y)y^{(p-1)}\int_y^{\infty}(h^{-1}(t))^{p(\alpha-1)}(h^{-1}(t))'dtdy\right)^{\frac{1}{p}}.$$

Since $\alpha < 1 - \dfrac{1}{p}$, $h^{-1}(\infty) = \infty$ and $\displaystyle\int u'(s)u^{\beta}(s)ds = \frac{u^{\beta+1}(s)}{\beta+1}$, we get

$$\int_y^{\infty}(h^{-1}(t))'(h^{-1}(t))^{p(\alpha-1)}dt = \left[\frac{(h^{-1}(t))^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]_y^{\infty}$$

$$= -\frac{(h^{-1}(y))^{p(\alpha-1)+1}}{p(\alpha-1)+1}$$

$$= \frac{(h^{-1}(y))^{p(\alpha-1)+1}}{p(1-\alpha)-1},$$

consequently,

$$\|x^{\alpha}T_1(f)(x)\|_{L_p(0,\infty)}$$

$$\leq c_2^{\frac{1}{p}}\Bigg(\int_0^{\infty} f^p(y)y^{p-1}\frac{(h^{-1}(y))^{p(\alpha-1)+1}}{p(1-\alpha)-1}dy)\Bigg)^{\frac{1}{p}}$$

$$=\frac{\big(c_1^{p(1-p)}\big)^{\frac{1}{p}}}{\big((1-\alpha)p-1\big)^{\frac{1}{p}}}\left[\int_0^{\infty} f^p(y)y^{p-1}(h^{-1}(y))^{(\alpha-1)p+1}dy\right]^{\frac{1}{p}}$$

$$=c_1^{(1-p)}((1-\alpha)p-1)^{\frac{-1}{p}}\left[\int_0^{\infty}\left(f(y)y^{1-\frac{1}{p}}(h^{-1}(y))^{(\alpha-1)+\frac{1}{p}}\right)^p dy\right]^{\frac{1}{p}}$$

$$=c_1^{(1-p)}((1-\alpha)p-1)^{\frac{-1}{p}}\left\|f(y)y^{1-\frac{1}{p}}(h^{-1}(y))^{(\alpha-1)+\frac{1}{p}}\right\|_{L_p(0,\infty)}.$$

Since $\frac{1}{p'}=1-\frac{1}{p}$, we obtain the desired inequality:

$$\begin{aligned}\|x^{\alpha}T_1(f)(x)\|_{L_p(0,\infty)} &\leq c_1^{(1-p)}((1-\alpha)p-1)^{\frac{-1}{p}}\left\|x^{\frac{1}{p'}}(h^{-1})^{\alpha-\frac{1}{p'}}(x)f(x)\right\|_{L_p(0,\infty)}\\ &= c_3\left\|x^{\frac{1}{p'}}(h^{-1})^{\alpha-\frac{1}{p'}}(x)f(x)\right\|_{L_p(0,\infty)}.\end{aligned}$$

□

By taking $h(x)=\lambda x$ where $\lambda>0$ in the above Theorem, we get $h^{-1}(x)=\frac{x}{\lambda}$, then

$$x^{\frac{1}{p'}}(h^{-1})^{\alpha-\frac{1}{p'}}(x)=\frac{x^{\alpha}}{\lambda^{\alpha-\frac{1}{p'}}}.$$

Consequently, we obtain the following corollary involving the Hardy-Steklov operator.

Corollary 3.1. *Let f satisfy the assumptions of the Theorem 3.1 and*

$$S_1(f)(x) = \frac{1}{x}\int_0^{\lambda x} f(y)dy \ for \ x > 0,$$

then the inequality

$$\|x^\alpha S_1(f)\|(x) \leq \left(\frac{1}{\lambda}\right)^{\alpha-\frac{1}{p'}} c_3 \|x^\alpha f(x)\|_{L_p(0,\infty)}.$$

Remark 3.1. *Taking $\lambda = 1$ in the above corollary, we get the following inequality via Hardy operator H.*

$$\|x^\alpha H(f)(x)\| \leq c_3 \|x^\alpha f(x)\|_{L_p(0,\infty)},$$

where

$$H(f)(x) = \frac{1}{x}\int_0^x f(y)dy.$$

Lemma 3.2. *Let $0 < p < 1$. Suppose that a non-negative function f satisfies the condition: there is a positive constant c_4 such that for all $x > 0$,*

$$f(x) \leq \frac{c_4}{x}\left(\int_x^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}}. \tag{3.11}$$

Then

$$\left(\int_x^\infty f(y)dy\right)^p \leq c_5 \int_x^\infty f^p(y)y^{p-1}dy, \tag{3.12}$$

where

$$c_5 = c_4^{p(1-p)}.$$

Proof. From the hypotheses (3.11), we have

$$f(x) \leq \frac{c_4}{x}\left(\int_x^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}},$$

then

$$xf(x) \leq c_4\left(\int_x^\infty f^p(y)y^{p-1}dy\right)^{\frac{1}{p}},$$

for $0 < p < 1$, we get

$$x^p f^p(x) \leq c_4^p \bigg(\int_x^\infty f^p(y) y^{p-1} dy \bigg). \tag{3.13}$$

Note that

$$f(x) = (f^p(x) x^p)^{\frac{1}{p}-1} f^p(x) x^{p-1}, \tag{3.14}$$

using the inequality (3.13), we result

$$\big(x^p f^p(x)\big)^{\frac{1}{p}-1} \leq c_4^{1-p} \bigg(\int_x^\infty f^p(y) y^{p-1} dy \bigg)^{\frac{1}{p}-1},$$

multiplying by $f^p(x) x^{p-1}$, we get

$$\big(x^p f^p(x)\big)^{\frac{1}{p}-1} f^p(x) x^{p-1} \leq c_4^{1-p} \bigg(\int_x^\infty f^p(y) y^{p-1} dy \bigg)^{\frac{1}{p}-1} f^p(x) x^{p-1}.$$

By applying (3.14), we get

$$f(x) \leq c_4^{1-p} \bigg(\int_x^\infty f^p(y) y^{p-1} dy \bigg)^{\frac{1}{p}-1} f^p(x) x^{p-1},$$

for $0 < t \leq x \leq y < \infty$, we deduce

$$\int_x^\infty f^p(y) y^{p-1} dy \leq \int_t^\infty f^p(y) y^{p-1} dy,$$

then

$$f(x) \leq c_4^{1-p} \bigg(\int_t^\infty f^p(y) y^{p-1} dy \bigg)^{\frac{1}{p}-1} f^p(x) x^{p-1},$$

integrating the above inequality over $x \in (t, \infty)$, we get

$$\begin{aligned}
\int_t^\infty f(x)dx &\leq c_4^{1-p} \int_t^\infty \left(\int_t^\infty f^p(y) y^{p-1} dy \right)^{\frac{1}{p}-1} f^p(x) x^{p-1} dx \\
&= c_4^{1-p} \left(\int_t^\infty f^p(y) y^{p-1} dy \right)^{\frac{1}{p}-1} \int_t^\infty f^p(x) x^{p-1} dx \\
&= c_4^{1-p} \left(\int_t^\infty f^p(x) x^{p-1} dx \right)^{\frac{1}{p}-1} \int_t^\infty f^p(x) x^{p-1} dy \\
&= c_4^{1-p} \left(\int_t^\infty f^p(x) x^{p-1} dx \right)^{\frac{1}{p}},
\end{aligned}$$

for $0 < p < 1$, we have the desired inequality

$$\left(\int_t^\infty f(x)dx \right)^p \leq c_5 \int_t^\infty f^p(x) x^{p-1} dx.$$

□

Theorem 3.2. *Let* $0 < p < 1$, $\alpha > 1 - \frac{1}{p}$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *If* f *is non-negative measurable function on* $(0, \infty)$ *and satisfies*

$$f(x) \leq \frac{c_4}{x} \left(\int_x^\infty f^p(y) y^{p-1} dy \right)^{\frac{1}{p}},$$

then

$$\|x^\alpha T_2(f)(x)\|_{L_p(0,\infty)} \leq c_6 \, \|x^{\frac{1}{p'}} (r^{-1}(x))^{\alpha - \frac{1}{p'}} f(x)\|_{L_p(0,\infty)}. \tag{3.15}$$

Where

$$c_6 = c_4^{1-p} \big((1-\alpha)p - 1 \big)^{-\frac{1}{p}},$$

and

$$T_2(f)(x) = \frac{1}{x} \int_{r(x)}^\infty f(y) dy,$$

with the boundary function r satisfying the following conditions:

1. *r is a non-negative, differentiable and increasing function on $(0, \infty]$.*

2. *For all $x \in (0, \infty) : 0 < r(x) < \infty$, $r(0) = 0$ and $r(\infty) = \infty$.*

Proof. We can rewrite

$$\|x^\alpha T_2(f)(x)\|_{L_p(0,\infty)} = \left(\int_0^\infty \left(x^{\alpha-1} \int_{r(x)}^\infty f(y)dy \right)^p dx \right)^{\frac{1}{p}}.$$

Given that r is a non-negative, differentiable and increasing function on $[0, \infty]$, we conclude that r be a bijective mapping. Put $t = r(x)$, then $x = r^{-1}(t)$, where r^{-1} is the reciprocal function of r, we get

$$\begin{aligned}
\|x^\alpha T_2(f)(x)\|_{L_p(0,\infty)} &= \left(\int_0^\infty \left| \left(r^{-1}(t)\right)^{\alpha-1} \int_t^\infty f(y)dy \right|^p (r^{-1}(t))'dt \right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left(r^{-1}(t)\right)^{p(\alpha-1)} \left(\int_t^\infty f(y)dy \right)^p (r^{-1}(t))'dt \right)^{\frac{1}{p}}.
\end{aligned}$$

By applying inequality (3.12), we get

$$\begin{aligned}
&\|x^\alpha T_2(f)(x)\|_{L_p(0,\infty)} \\
&\leq \left(\int_0^\infty (r^{-1}(t))^{p(\alpha-1)} \left(c_5 \int_t^\infty f^p(y) y^{(p-1)} dy \right) (r^{-1}(t))'dt \right)^{\frac{1}{p}} \\
&= c_5^{\frac{1}{p}} \left(\int_0^\infty \left[(r^{-1}(t))^{p(\alpha-1)} \left(\int_t^\infty f^p(y) y^{(p-1)} dy \right) (r^{-1}(t))' \right] dt \right)^{\frac{1}{p}} \\
&= c_5^{\frac{1}{p}} \left(\int_0^\infty \int_t^\infty f^p(y) y^{(p-1)} (r^{-1}(t))^{p(\alpha-1)} (r^{-1}(t))'dydt \right)^{\frac{1}{p}},
\end{aligned}$$

by using Fubini's Theorem, we obtain

$$\|x^{\alpha}T_2(f)(x)\|_{L_p(0,\infty)}$$

$$\leq c_5^{\frac{1}{p}}\left(\int_0^{\infty}\int_0^{y} f^p(y)y^{(p-1)}(r^{-1}(t))^{p(\alpha-1)}(r^{-1}(t))'dtdy\right)^{\frac{1}{p}}$$

$$= c_5^{\frac{1}{p}}\left(\int_0^{\infty} f^p(y)y^{(p-1)}\int_0^{y}(r^{-1}(t))^{p(\alpha-1)}(r^{-1}(t))'dtdy\right)^{\frac{1}{p}}.$$

Given that $\alpha > 1 - \dfrac{1}{p}$ and $r^{-1}(0) = 0$, we have

$$\int_0^{y}(r^{-1}(t))'(r^{-1}(t))^{p(\alpha-1)}dt = \left[\frac{(r^{-1}(t))^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]_0^{y}$$

$$= \frac{(r^{-1}(y))^{p(\alpha-1)+1}}{p(\alpha-1)+1},$$

therefore,

$$\|x^{\alpha}T_2(f)(x)\|_{L_p(0,\infty)}$$

$$\leq c_5^{\frac{1}{p}}\left(\int_0^{\infty} f^p(y)y^{p-1}\frac{(r^{-1}(y))^{p(\alpha-1)+1}}{p(1-\alpha)-1}dy\right)^{\frac{1}{p}}$$

$$=\frac{\left(c_4^{p(1-p)}\right)^{\frac{1}{p}}}{\left((\alpha-1)p+1\right)^{\frac{1}{p}}}\left[\int_0^{\infty} f^p(y)y^{p-1}(r^{-1}(y))^{(\alpha-1)p+1}dy\right]^{\frac{1}{p}}$$

$$=c_4^{(1-p)}((\alpha-1)p+1)^{\frac{-1}{p}}\left[\int_0^{\infty}\left(f(y)y^{1-\frac{1}{p}}(r^{-1}(y))^{(\alpha-1)+\frac{1}{p}}\right)^p dy\right]^{\frac{1}{p}}$$

$$=c_4^{(1-p)}((\alpha-1)p+1)^{\frac{-1}{p}}\left\|f(y)y^{1-\frac{1}{p}}(r^{-1}(y))^{(\alpha-1)+\frac{1}{p}}\right\|_{L_p(0,\infty)}$$

$$=c_6\left\|f(y)y^{1-\frac{1}{p}}(r^{-1}(y))^{(\alpha-1)+\frac{1}{p}}\right\|_{L_p(0,\infty)}.$$

Which gives the desired inequality. □

Now, we present some special cases related to the Theorem 3.2.

Taking $r(x)=\beta x$ where $\beta>0$ in the above Theorem, we get $r^{-1}(x)=\dfrac{x}{\beta}$. Then

$$x^{\frac{1}{p'}}(r^{-1})^{\alpha-\frac{1}{p'}}(x)=\frac{x^{\alpha}}{\beta^{\alpha-\frac{1}{p'}}},$$

we obtain the following corollary involving the Hardy-Steklov operator.

Corollary 3.2. *Let f satisfy the assumptions of the Theorem 3.2 and*

$$S_2(f)(x)=\frac{1}{x}\int_{\beta x}^{\infty} f(y)dy \ for\ x>0,$$

then the bellow inequality holds:

$$\|x^{\alpha} S_2 (f)\| (x) \leq \left(\frac{1}{\beta}\right)^{\alpha - \frac{1}{p'}} c_6 \|x^{\alpha} f(x)\|_{L_p(0,\infty)} .$$

Remark 3.2. *Taking $\beta = 1$ in the above corollary, we get the following inequality via Hardy's dual operator $\tilde{H}$.*

$$\left\|x^{\alpha} \tilde{H} (f) (x)\right\| \leq c_6 \|x^{\alpha} f(x)\|_{L_p(0,\infty)} ,$$

where

$$\tilde{H} (f) (x) = \frac{1}{x} \int_x^{\infty} f(y) dy.$$

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