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DEDICATION

I extend heartfelt appreciation to my family for their unwavering support throughout this journey.

To my father, **Bouafia**, your constant encouragement has been a source of strength at every stage of my studies.

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Abir Hameurlain

DEDICATION

This humble work is dedicated

To my dear father **Ali**, whose unwavering support sustains me, and I pray for his continued health and well-being.

I also extend my gratitude to my dear mother, my brother **Mohammed** and my sisters **Fatima**, **Ritaj**, **Khaira**, **Ferail** and **Saida**.

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ABSTRACT

In the analysis and control of linear and nonlinear systems, the most important thing is to study stability, because an unstable system is usually useless. The objective of this memoir is to present the Lyapunov method, and analyze the stability of nonlinear systems. This memoir is concerned with the asymptotic stability analysis of linear time-varying (LTV) systems. With the help of the notion of stable functions, some the differential Lyapunov inequalities based on necessary and sufficient conditions are derived for testing the asymptotic stability, exponential stability, and input-to-state stability of general LTV systems.

CONTENTS

LIST OF FIGURES

THE MAIN NOTATIONS USED

Notation

$\mathbb{R}$	:	The set of all real numbers.
$\mathbb{R}^n$	:	The n-dimensional Euclidean vector space.
$\mathbb{R}^m$	:	The m-dimensional Euclidean vector space.
$\mathbb{R}_+$	:	The set of all non-negative real numbers.
$U(r)$	:	$= \{x \in \mathbb{R}^n / \parallel x \parallel < r\}$.
B_r	:	$= \{x \in \mathbb{R}^n / \parallel x \parallel \leqslant r\}$ a bounded ball.
R	:	The region of attraction of B_r.
L^1_{loc}	:	set of measurable applications, of power 1 integrable.
$\mathcal{PC}(\mathbb{R}_+, \mathbb{R})$	:	The space of piece-wise continuous functions on $\mathbb{R}_+ \to \mathbb{R}$.
$\mathcal{C}(\mathbb{R}_+, \mathbb{R})$	:	The set of continues function on $\mathbb{R}_+ \to \mathbb{R}$.
$\lvert . \rvert$	:	Absolute value of a real number.
$\lvert\lvert . \rvert\rvert$	:	Euclidean distance norm.
$\mathcal{K}$	:	The class of functions $\mathbb{R}_+ \to \mathbb{R}_+$.
$\mathcal{K}_\infty$	:	The subset of $\mathcal{K}$-functions that are unbounded.
$\beta(.,.)$	:	A class $\mathcal{KL}$ function.
$\alpha(.)$	:	A class $\mathcal{K}$ function.
$\dot{V}$	:	$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x)$ the derivative of Lyapunov function $V(t,x)$.
A^T	:	The transpose of the matrix A.
$\mathbb{J}$	:	$= [\alpha, \beta]$.
$sp(A)$	:	The set of all eigenvalues of A.

$\lambda_{\max}(A)$	:	$= \max Real(\lambda) : \lambda \in sp(A)$.
$\lambda_{\min}(A)$	:	$= \min Real(\lambda) : \lambda \in sp(A)$.
I	:	The identity matrix.
$exp(x)$	:	The exponential function e^x.
ln	:	The logarithm function lnx.
$M_{m,n}(\mathbb{R})$	:	The set of real term matrices with m rows and n columns.
$M_{n,n}(\mathbb{R})$	:	The set of square matrices of order n with real terms.
$\mathfrak{L}_{\infty}^{m}(\mathbb{R}_+)$	:	$= \Big\{ f(t) : \mathbb{R}_+ \to \mathbb{R}^m, \sup_{t\in\mathbb{R}_+} \{ \lvert f(t) \rvert \} < \infty \Big\}$.

Abbreviations

(UAS)	:	Uniformly Asymptotically Stable.
(UES)	:	Uniformly Exponentially Stable.
$(UPAS)$	:	Uniformly Practically Asymptotically Stable.
(GUS)	:	Globally Uniformly Stable.
(GUA)	:	Globally Uniformly Attractive.
(GES)	:	Globally Exponentially Stable.
$(GUAS)$	:	Globally Uniformly Asymptotically Stable.
(GPS)	:	Globally Practically Stable.
$(GUES)$	:	Globally Uniformly Exponentially Stable.
$(GPUS)$	:	Globally Practically Uniformly Stable.
$(GUPAS)$	:	Globally Uniformly Practically Asymptotically Stable.
$(GPUES)$	:	Globally Practically Uniformly Exponentially Stable.
(ISS)	:	Input-to-State Stable.
$(iISS)$	:	inInput-to-State Stable.
(USF)	:	Uniformly Stable Function.
$(PUISS)$	:	Practically Uniformly Input-to-State Stable.
(LTV)	:	Linear Time-Varying.
(LTI)	:	Linear Time-Invarying.

GENERAL INTRODUCTION

Stability and stabilization of nonlinear systems have been a central focus in the field of dynamic systems. This memoir aims to explore the various aspects of stability theory, including Lyapunov theory, linear systems, and perturbed systems. We will examine the stabilization techniques for both linear and nonlinear control systems, as well as perturbed systems. Stability theory was developed first for systems of ordinary differential equations, beginning with Lyapunov [12] in 1892. It is characterized by analyzing the response of a dynamical system to small perturbations in the system states. Lyapunov proved that the existence of a Lyapunov function guarantees asymptotic stability and for LTI systems, he also showed the converse statement that asymptotic stability implies the existence of a Lyapunov function. If in addition all solutions of the dynamical system approach the equilibrium point for large values of time, then the equilibrium point is said to be asymptotically stable. According to Lyapunov, one can check the stability of a system by finding some function V, called the Lyapunov function which is definite along every trajectory of the system, and is such that the total derivative $[\frac{\partial V}{\partial t}]$ is semi definite of opposite sign (or identically 0) along every trajectory of the system. If the function V exists with these properties and admits an infinitely small upper bound, and if $[\frac{\partial V}{\partial t}]$ is definite (with sign opposite to that of V), it can be shown further that every perturbed trajectory a that is sufficiently close to the unperturbed motion approaches the latter asymptotically. We use the class $\mathcal{KL}$-function and $\mathcal{K}$-function introduced by [11] in our analysis through the solution of an autonomous and non-autonomous scalar differential equation. As a consequence, we obtain upper and lower bounds on a positive definite function in terms of the class $\mathcal{K}$-function. Lyapunov's indirect method also called the first method of Lyapunov draws conclusions about the local stability of the equilibrium point by examining the stability of the linearized nonlinear system about the equilibrium point in question. Furthermore, we give illustrative examples showing the applicability of the results. To generalize Lyapunov theorems, we establish sufficient conditions for various Lyapunov stability types of perturbed systems to study the problem of uniform practical asymptotic stability and the GUPAS of time-varying perturbed systems. Notice that the system is not assumed to have an equilibrium point.

This memoir is structured into two chapters.

the first chapter aims to give an overview of the theoretical tools and concepts used in this work. So, first we introduce, the concept of dynamical systems. Then, in Section 1.2, we give stability theory (definitions of equilibrium point stability). Section 1.3 presents the Lyapunov theory, a powerful tool in stability analysis. It provides conditions for asymptotic stability based on the comparison between two functions, a Lyapunov function and its derivative, along the system's trajectories. In Section 1.4, Linear Systems and Linearization, which proposes some stability results for LTI systems and describes how those results are extended to LTV systems, we present the indirect-Lyapunov method, which shows the global stability of a nonlinear system by the use of its linearization. Also Section 1.5 deals with the important topic of stability in the presence of a perturbation. It can be represented also as a combination of both of them. It can be classified into two categories, vanishing perturbation, where the disturbance tends towards zero as time approaches infinity, allowing us to analyze stability using traditional methods like Lyapunov's indirect method. In non-vanishing perturbation scenarios, where disturbances do not tend towards zero. Lastly, Section 1.6 is devoted to the stabilization of linear, nonlinear control systems, and perturbed systems.

The second chapter focuses on the input-to-state stable (ISS), In Section 2.1 we introduce the concept of LTV systems. we provided in [24] a new Lyapunov function-based stability analysis approach, which allows the time derivative of the Lyapunov function to be indefinite. Our main results are given in Section 2.2 which contains two subsections dealing with the asymptotic stability of a perturbed system and ISS stability, respectively. The advantages of the proposed theorems over the existing results are pointed out, and their effectiveness are also illustrated by several numerical examples given in Section 2.3. The chapter is concluded in Section 2.4. Finally, some proofs and Lemma are collected in the **ANNEX**.

CHAPTER 1

GENERAL STABILITY AND STABILIZATION

Introduction

In this chapter, we recall the basic concepts used throughout of the memoir. In particular, the definitions and properties of systems dynamic. For the proofs of the propositions and theorems stated in the following sections the reader may refer to, for example, [4], [10], [11].

1.1 Dynamical systems

A dynamical system is one that evolves with time. Mathematically, it consists of the space of states of the system together with a rule for determining the state at a future point in time, when the present state is given. The mathematical formalism of differential equations has proven useful for describing dynamical systems, that is the evolution of the system with respect to time.

All ordinary differential equations can be written as a system of first order derivatives, the state space form

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1.1.1}$$

where $t \geqslant t_0 \geqslant 0$ is the time $\dot{x}$ denotes differentiation of $x(t)$ with respect to time n is called the dimension of the system and x_0, $x(t) \in \mathbb{R}^n$ often referred to as the state contains the information about the underlying system that is important to as. The right hand side of this equality is referred to the mapping $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ which expresses how the state changes in time and is continuous in (t, x) and locally Lipschitz with respect x uniformly on t.

Theorem 1.1.1. ***(Cauchy-Lipschitz Theorem)***
It is assumed that the function $f : I \times V \longrightarrow V$ satisfies the following two assumptions :

1. *f is locally Lipschitz in x, in the sense that*

$$\forall x \in V,\ \exists r > 0,\ \exists \alpha \in L^1_{loc}(I, \mathbb{R}_+)\ /\ B(x,r) \subset V$$

and

$$\forall t \in I,\ \forall y, z \in B(x,r),\ ||f(t,y) - f(t,z)|| \leq \alpha(t)||y - z||.$$

2. *f is locally integrable with respect to t*

$$\forall x \in V,\ \exists \beta \in L^1_{loc}(I, \mathbb{R}_+)\ /\ \forall t \in I,\ ||f(t,x)|| \leq \beta(t).$$

Then for any initial data $(t_0, x_0) \in I \times V$, there exists a unique maximal solution $x(.)$ of the Cauchy problem (1.1.1).

There are different kinds of stability problems that arise in the study of dynamical systems. We are interest in this work on the stability of equilibrium points for some classes of linear and nonlinear systems. This stability is usually characterized in the sense of Lyapunov. In fact, the theory of stability in this sense is well known and widely used in concrete problems in the real world. In the following, we define the concept of Lyapunov-stability.

1.2 Stability theory

We will be interested in the behavior of the solution of the nonlinear non-autonomous system (1.1.1) as time goes to infinity. This is the subject of stability theory. In order to make things more formal, we need to introduce the concept of an equilibrium point and present a rigorous notion of stability.

Definition 1.2.1. ***(Equilibrium point)***
Let $x_0 \in \mathbb{R}^n$, x_0 is an equilibrium point for(1.1.1)if

$$f(t, x_0) = 0,\ \forall t \geq 0. \tag{1.2.1}$$

An equilibrium point has the property that if the state of the system starts at x_0, it will remain there for all future time. Without loss of generality, we can assume that the origin is an equilibrium point of (1.1.1) because any equilibrium point can be shifted to the origin via some change of variables.

Remark 1.1. *If x_0 is an equilibrium point, we let $y = x - x_0$ then*

$$y_0 = x_0 \Rightarrow f(t,x) = f(t, y + x_0) = g(y)$$

$$g(0) = f(t, x_0) = 0$$

then, 0 *is an equilibrium point.*
The origin is the equilibrium point of (1.1.1).
The solution passing through the location x^* *at time* $t = t^*$ *is also denoted* $x(t, t^*, x^*)$ *such that*

$$x(t^*, x^*) = x^*.$$

Definition 1.2.2. ***(Stability)***
The equilibrium point $x = 0$ *is stable if for each* $\epsilon > 0$ *and any* $t_0 \geq 0$ *there is* $\delta = \delta(t_0, \epsilon) > 0$ *such that*

$$|| x_0 ||< \delta \Longrightarrow || x(t) ||< \epsilon, \ \forall t \geq t_0.$$

An equilibrium point is stable if all solutions starting at nearby points stay nearby, otherwise it is unstable.
Let us note that the stability of systems does not involve the convergence of solutions to the origin, which is why the notion of stability alone is not sufficient to study the behavior of solutions.

Definition 1.2.3. ***(Instability)***
The equilibrium point $x = 0$ *is unstable if it is not stable; that is there exists an* $\epsilon > 0$ *such that for every* $\delta > 0$ *there exist an* $x(t_0)$ *with*

$$||x(t_0)|| < \delta, \ t_1 > t_0$$

such that

$$||x(t_1)|| > \epsilon, \ t > t_1.$$

If this holds for every $x(t_0)$ *in* $||x(t_0)|| < \delta$ *this equilibrium is completely unstable.*

Definition 1.2.4. ***(Attractivity)***
The equilibrium point $x = 0$ *is*

i) Attractive if for each $t_0 \in \mathbb{R}_+$*, there exists* $U(r)$*,* $r > 0$ *such that*

$$\forall x_0 \in U(r), \ \lim_{t \longrightarrow +\infty} x(t) = 0.$$

ii) Globally attractive if for each $t_0 \in \mathbb{R}_+$ *and* $x_0 \in \mathbb{R}^n$

$$\lim_{t \longrightarrow +\infty} x(t) = 0.$$

Definition 1.2.5. ***(Asymptotic stability)***
The equilibrium point $x = 0$ *is*

i) Asymptotically stable, if it is stable and attractive.

ii) Globally asymptotically stable, if it is stable and globally attractive.

That is, if all solutions starting at nearby points not only stay nearby, but also t end to the equilibrium point as time approaches infinity. Corresponding to different types of stability, we can define concepts of boundedness.

Definition 1.2.6. ***(Uniform boundedness)***
The solution of (1.1.1) is said to be uniformly bounded if there is a positive constant a, such that for all $b \in]0, a[$, there exists $c = c(b) > 0$ such that for each $t_0 \in \mathbb{R}_+$

$$\| x_0 \| < b \Longrightarrow \| x(t) \| < c(b), \ \forall t \geq t_0.$$

*It is said to be globally uniformly bounded if the previous property is true for all $b > 0$ **ie**, $a = +\infty$.*

Definition 1.2.7. ***(Uniform stability)***
The equilibrium point $x = 0$ is

i) *Uniformly stable, if for all $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that for each $t_0 \in \mathbb{R}_+$*

$$\| x_0 \| < \delta \Longrightarrow \| x(t) \| < \epsilon, \ \forall t \geq t_0.$$

ii) *Globally uniform stable, if it is uniformly stable and the solutions of system (1.1.1) are globally uniformly bounded.*

It is clear that uniform global stability implies the uniform global boundedness.

Definition 1.2.8. ***(Uniform attractivity)***
The equilibrium point $x = 0$ is

i) *Uniformly attractive, if there exist $r > 0$ such that $\forall \epsilon > 0$ there is $T = T(\epsilon) > 0$ such that for each $t_0 \in \mathbb{R}_+$ and any $x_0 \in U(r)$*

$$t \geq T + t_0, \ \| x(t) \| < \epsilon.$$

ii) *Globally uniformly attractive, if $\forall \epsilon > 0$ there is $T = T(\epsilon) > 0$ such that for each $t_0 \in \mathbb{R}_+$ and any $x_0 \in \mathbb{R}^n$*

$$\forall t \geq T + t_0, \ \| x(t) \| < \epsilon.$$

Definition 1.2.9. ***(Uniform asymptotic stability)***
The equilibrium point $x = 0$ is

i) *Uniformly asymptotically stable, if it is uniformly stable and uniformly attractive.*

ii) *Globally uniformly asymptotically stable, if it is globally uniformly stable and globally uniformly attractive.*

It is instructive to note that the definitions of asymptotic stability do not quantify the speed of convergence of trajectories to the origin. Consequently, we use exponential stability.

Definition 1.2.10. ***(Exponential stability)***
The equilibrium point $x = 0$ is

i) Exponentially stable, if it is stable and there exist $r, \lambda_1, \lambda_2 > 0$ such that for each $x_0 \in U(r)$ and any $t_0 \in \mathbb{R}_+$

$$|| x(t) || \leqslant \lambda_1 || x_0 || e^{-\lambda_2(t-t_0)}, \ \forall t \geq t_0 \geq 0.$$

The constant λ_2 is said the convergence rate.

ii) Globally exponentially stable, if it is stable and there is $\lambda_1, \lambda_2 > 0$ such that for all $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}_+$

$$|| x(t) || \leqslant \lambda_1 || x_0 || e^{-\lambda_2(t-t_0)}, \ \forall t \geq t_0 \geq 0.$$

Remark 1.2. *Global exponential stability always implies global uniform asymptotic stability. The converse is true for linear systems but not for nonlinear systems in general.*
In general, the question of determining whether the equilibrium point of a nonlinear systems is GAS can be extremely hard. The main difficulty is that more often that not it is impossible to explicitly write a solution to the differential equation (1.1.1). Nevertheless, in some cases, we are still able to draw conclusions about the stability of nonlinear systems, thanks to the brilliant idea of the famous Russian mathematician Aleksandr Mikhailovich Lyapunov. This method is known as Lyapunov's direct method and was first published in 1892.

1.3 Lyapunov theory

Stability, asymptotic stability and exponential stability can be characterized in terms of special scalar functions know as class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$-functions.

Definition 1.3.1. ***(Class $\mathcal{K}-$function)***
A continuous function α: $[0, a) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(s) \to +\infty$ as $s \to +\infty$.

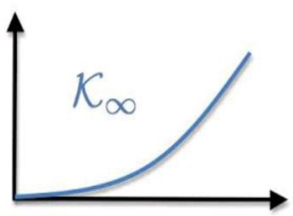

Figure 1.1: Presentation of class $\mathcal{K}_\infty$- function

Definition 1.3.2. ***(Class $\mathcal{KL}$-function)***
A continuous function $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{KL}$, if for each fixed point t, the mapping $\beta(s, t)$ belongs to class $\mathcal{K}$ with respect to s, the mapping $\beta(s, t)$ is decreasing with respect to s and $\beta(s, t) \to 0$ as $s \to +\infty$.

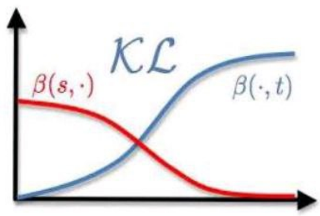

Figure 1.2: Presentation of class $\mathcal{KL}$- function

The following lemma states some obvious properties of these functions.

Lemma 1. *Let $\alpha_1(.)$ and $\alpha_2(.)$ be class $\mathcal{K}$-function on $[0, a), \alpha_3(.), \alpha_4(.)$ be class $\mathcal{K}_\infty$ function and $\beta(., .)$ is a class $\mathcal{KL}$-function on $[0, a) \times [0, +\infty)$. Denote $\lim_{x \to a^{-1}} \alpha_i(x)$ by $\alpha_i(a)$ $(i = 1, 2)$, then*

- $*$ *$\alpha_1^{-1}(.)$ is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- $*$ *$\alpha_3^{-1}(.)$ is defined on $[0, +\infty)$ and belongs to class $\mathcal{K}_\infty$.*
- $*$ *$\alpha_1 \circ \alpha_2(.)$ belongs to class $\mathcal{K}$.*
- $*$ *$\alpha_3 \circ \alpha_4(.)$ belongs to class $\mathcal{K}_\infty$.*

The following result gives equivalent definitions of stability using class $\mathcal{K}$ and $\mathcal{KL}$-functions

Proposition 1.1. *The equilibrium point $x = 0$ of (1.1.1) is*

- *Uniformly stable if and only if there exist a class $\mathcal{K}$-function $\alpha(.)$ and a positive constant c independent of t_0 such that*

$$|| x(t) || \leqslant \alpha(|| x_0 ||), \forall t \geq t_0 \geq 0, \ \forall \, || x_0 || < c.$$

- *Globally uniformly stable if and only if the previous inequality is satisfied for any initial state x_0.*

- *Uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$-function $\beta(., .)$ and a positive constant c independent of t_0 such that*

$$|| x(t) || \leqslant \beta(|| x_0 ||, t - t_0), \ \forall t \geq t_0, \ \forall \, || x_0 || < c. \tag{1.3.1}$$

- *Globally uniformly asymptotically stable if and only if the previous inequality is satisfied for any initial state x_0.*

- *Exponentially (resp. globally exponentially) stable if and only if the inequality (1.3.1) is satisfied with (resp. $c = +\infty$)*

$$\beta(r, s) = kre^{-\gamma s}, \; k > 0, \; \gamma > 0. \tag{1.3.2}$$

Lyapunov's direct method allows us to determine the stability of a system without explicitly integrating the differential equation. This method is a generalization of the idea that if there is an appropriate energy function in a system, then we can study the rate of change of the energy of the system to ascertain stability. To make this precise, we need the following definitions.

Definition 1.3.3.

- ***Positive definite function :*** *A continuous function $V(.,.) : \mathbb{R}_+ \times U(r) \to \mathbb{R}_+$ is said to be positive definite, if there is a class $\mathcal{K}$-function $\alpha_1(.)$ such that $V(t, 0) = 0$ and*

$$V(t, x) \geq \alpha_1(|| \, x \, ||), \; \forall t \geq 0, \; \forall x \in U(r).$$

- ***Proper function :*** *a continuous function $V(.,.) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to be proper (radially unbounded). if there is a class $\mathcal{K}_\infty$-function $\alpha_2(.)$ such that $V(t, 0) = 0$ and*

$$V(t, x) \geq \alpha_2(|| \, x \, ||), \; \forall t \geq 0, \; \forall x \in \mathbb{R}^n.$$

- ***Decrescent function :*** *a continuous function $V(.,.) : \mathbb{R}_+ \times U(r) \to \mathbb{R}_+$ is said to be decrescent, if $V(t, 0) = 0$ and if there is a class $\mathcal{K}$-function $\alpha_3(.)$ such that $V(t, 0) = 0$ and*

$$V(t, x) \leqslant \alpha_3(|| \, x \, ||), \; \forall t \geq 0, \; \forall x \in U(r).$$

- ***Negative definite function*** *a continuous function $V(.,.)$ is said to be negative definite if $-V$ is a positive definite function.*

Next, we introduce the Lyapunov function as a generalization of the idea of the 'energy' of a system. Then the method studies stability by looking at the rate of change of this 'measure of energy'.

Definition 1.3.4. *Let $V(.,.) : \mathbb{R}_+ \times U(r) \to \mathbb{R}_+$ be a continuously differentiable function. Then the time derivative of V along the trajectories of system (1.1.1) is denoted by $\dot{V}(t, x)$ where*

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x).$$

Note that $\dot{V}$ depends not only on the function V but also on the system (1.1.1). The quantity $\dot{V}(t, x)$ can be interpreted as follows:

Suppose a solution trajectory of this system passes through x_0 at time t_0. Then, at the instant t_0, the rate of change of the quantity $V(t, x(t))$ is $\dot{V}(t_0, x_0)$, which can be written

$$\dot{V}(t_0, x_0) = \frac{\partial}{\partial t} V(t, x(t))|_{t=t_0}.$$

Definition 1.3.5. ***(Lyapunov function)***
We consider the system (1.1.1).
Let $r > 0$ and $V : \mathbb{R}_+ \times U(r) \to \mathbb{R}$ a continuously differentiable function. V is said to be a Lyapunov function, if it satisfies the two following properties:

i) $V(.,.)$ *is a positive definite function.*

ii) $\dot{V}(t,x) \leqslant 0 \ , \ \ \forall(t,x) \in \mathbb{R}_+ \times U(r)$.

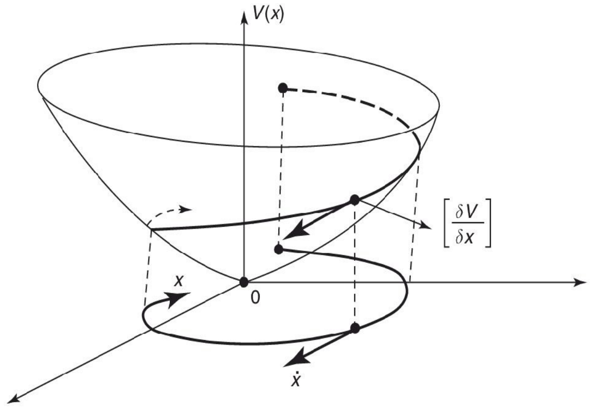

Figure 1.3: Lyapunov function

We use the next theorem to deduce the stability of the equilibrium point when the system has a Lyapunov function. Then the stability theorem of Lyapunov can be stated as follows:

Theorem 1.3.1. ***(Stability)***
Let $x = 0$ be an equilibrium point of (1.1.1), and $r > 0$.
If this system has a Lyapunov function V on $U(r)$ for $r > 0$, then the origin $x = 0$ is an equilibrium point stable.
Moreover, if V is decrescent then $x = 0$ is an equilibrium point uniformly stable.
Finally, If the system (1.1.1) has a Lyapunov function V on $\mathbb{R}^n$, decrescent and radially unbounded, then $x = 0$ is an equilibrium point globally uniformly stable.

Theorem 1.3.2. ***(Asymptotic stability)*** *[10]*
Let $x = 0$ be an equilibrium point of (1.1.1) and $r > 0$.
Let $V : \mathbb{R}_+ \times U(r) \to \mathbb{R}$ be a continuously differentiable function such that there exist class $\mathcal{K}$ -functions, $\alpha_1(.), \alpha_2(.)$ and $\alpha_3(.)$ defined on $[0, r)$ satisfying
$\forall t \geq t_0, \ \ \forall x \in U(r)$

$$\alpha_1(\| x \|) \leq V(x,t) \leqslant \alpha_2(\| x \|) \tag{1.3.3}$$
$$\dot{V}(t,x) \leq -\alpha_3(\| x \|) \tag{1.3.4}$$

Then $x = 0$ is an equilibrium point UAS.
If $U(r) = \mathbb{R}^n$, $\alpha_1(.)$ and $\alpha_2(.)$ are class $\mathcal{K}_\infty$-functions, then $x = 0$ is an equilibrium point GUAS.

Theorem 1.3.3. ***(Exponential stability)*** *[10]*
We consider the system (1.1.1). Assume that the system has a Lyapunov function $V(t,x)$ and there exist constants $c_1, c_2, c_3 > 0$ and $r > 0$, such that $\forall t \geq t_0$, $\forall x \in U(r)$, we have

$$c_1 \parallel x \parallel^2 \leq V(x,t) \leqslant c_2 \parallel x \parallel^2 \quad (1.3.5)$$
$$\dot{V}(t,x) \leq -c_3(\parallel x \parallel)^2. \quad (1.3.6)$$

Then $x = 0$ is an equilibrium point exponentially stable.
If $U(r) = \mathbb{R}^n$, then the origin $x = 0$ is an equilibrium point GES.

Example 1.3.1. *Consider the following system*

$$\begin{cases} \dot{x}_1 = -x_1 - e^{-2t}x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$$

To study the stability of the origin, let

$$V(t,x) = x_1^2 + (1 + e^{-2t})x_2^2$$

Clearly

$$\alpha_1(\parallel x \parallel) = x_1^2 + x_2^2 \leqslant V(t,x) \leqslant x_1^2 + x_2^2 = \alpha_2(\parallel x \parallel).$$

Thus, we have that

* *$V(t,x)$ is positive definite and radially unbounded, since $\alpha_1(\parallel x \parallel) \leqslant V(t,x)$, with $\alpha_1(.)$ is a class $\mathcal{K}_\infty$-function.*

* *$V(t,x)$ is decrescent, since $V(t,x) \leqslant \alpha_2(\parallel x \parallel)$, with $\alpha_2(.)$ is also a class $\mathcal{K}$-function.*

Moreover

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \\ &= -2[x_1^2 - x_1 x_2 + x_2^2(1 + 2e^{-2t})] \\ &\leq -(x_1^2 + x_2^2). \end{aligned}$$

Then the origin is globally exponentially stable (GES).

Lyapunov analysis can be used to find conditions for instability of an equilibrium point, for example $x = 0$ must be unstable if V and $\dot{V}$ are both positive definite. These results are known as instability theorems.
Once again we emphasize that Lyapunov's theorems allow The stability of the system to be verified without explicitly solving the differential equation. Lyapunov's theorems in effect, turn the question of determining stability into a search for The Lyapunov function.
The natural question that immediately arises is if this Lyapunov function always exist ?
In many situations, the answer is positive and it is due to so-called converse theorems.

1.4 Linear systems and linearization

1.4.1 Linear time-invariant systems

Consider the linear time-invariant system

$$\dot{x} = Ax \tag{1.4.1}$$

where $x \in \mathbb{R}^n$ and A is a constant matrix $(n \times n)$. The origin is an equilibrium point.
Stability properties of the origin can be characterized by the locations of the eigenvalues of the matrix A. Recall from linear system theory that the solution of (1.4.1) for a given initial state x_0 is given by

$$x(t) = \exp(A(t))x(0)$$

The following theorem characterizes the origin stability properties.

Theorem 1.4.1. *[10] The origin of (1.4.1) is globally asymptotic stable if and only if all eigenvalues of A have strictly negative real parts, that is A is a Hurwitz[1] matrix asymptotic stability of the origin can be investigated using Lyapunov's method. We can consider as a Lyapunov function candidate $V(x) = x^T P x$ where P is a real symmetric positive definite matrix. The following theorem characterizes the origin asymptotic stability in terms of the Lyapuov equation solution.*

Theorem 1.4.2. *[10] A matrix A is a Hurwitz matrix, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the following Lyapunov equation*

$$PA + A^T P = -Q.$$

Moreover, if A is a Hurwitz matrix, then P is the unique solution of the Lyapunov equation.

1.4.2 Linear time-varying systems

Stability analysis for linear time-varying systems is a theory of increasing interest.
One reason is the growing importance of adaptive controllers for which the underlying closed-loop adaptive system is often time-varying and linear.

Definition 1.4.1. *Consider the linear system*

$$\begin{cases} \dot{x}(t) = A(t)x \\ x(t_0) = x_0 \end{cases} \tag{1.4.2}$$

where $A(t)$ is continuous for any $t \geqslant 0$. The system (1.4.2) is

i) Uniformly stable if, for some $\gamma > 0$

$$||x(t)|| \leqslant \gamma ||x(t_0)||, \ \forall t \geqslant t_0.$$

[1]When any eigenvalue of A has a strictly negative real part $Re(\lambda_i) < 0$, then the matrix A is called the Hurwitz matrix.

ii) Uniformly asymptotically (exponentially) stable if, for some positive constants γ, $\lambda > 0$

$$||x(t)|| \leq \gamma e^{-\lambda(t-t_0)}||x(t_0)||, \ \forall t \geq t_0.$$

The stability behavior of the origin as an equilibrium point of the system (1.4.2) can be characterized in terms of the state transition matrix[2] of the system. For linear system theory, we know that the solution of (1.4.2) is given by

$$x(t) = \phi(t, t_0)x_0.$$

Where $\phi(t, t_0)$ is the state transition matrix. We get some further characterization for asymptotic stability.

Proposition 1.2. ***(Transition matrix and stability)***
The trivial solution of the homogeneous system (1.4.2) is

i) Stable if and only if

$$\forall t_0 \geq 0, \ \exists \beta > 0 \ \forall t \geq t_0, \ \phi(t - t_0) \leq \beta.$$

ii) Asymptotically stable if and only if

$$\forall t > 0, \ \lim_{t \longrightarrow \infty} \phi(t - t_0) = 0.$$

Remark 1.3. *When we turn to a linear time-varying system, the study of stability is much complicated. It might be thought that if all eigenvalues of $A(t)$ have negative real parts for all $t \geq t_0$, then the origin of (1.4.2) would be asymptotically stable.*
Unfortunately, this conjecture is not true. The following examples show that.

Example 1.4.1. *[10] Consider the linear system*

$$\dot{x}(t) = A(t)x$$

where

$$A(t) = \begin{pmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ 1 - 1.5\sin t \cos t & -1 + 1.5\sin^2 t \end{pmatrix}$$

For each t, the eigenvalues of $A(t)$ are given by $\lambda_{1,2} = \frac{-1 \pm \sqrt{7}}{4}$.
Thus, the eigenvalues are independent of t and lie in the open left-half plane. Yet, the origin is unstable. We can verify that

$$\phi(t, 0) = \begin{pmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{pmatrix}$$

Which shows that there are initial states $x(0)$ arbitrarily close to the origin, for which the solution is unbounded and escape to infinity.

[2] Let $\phi(t)$ be a fundamental matrix of (1.4.2). Then

$$\phi(t, t_0) = \phi(t)\phi^{-1}(t_0), \ \forall t \geq t_0$$

is called the state transition matrix of (1.4.2).

Lemma 2. ***(Solution of the Lyapunov-equation)*** *[4]*
Let $A, Q : (0, \infty) \to \mathbb{R}^n$ be continuous. If the integral

$$P(t) = \int_t^{\infty} \phi(s,t)^T Q(s)\phi(s,t)ds$$

exists for all $t > 0$, then the time-varying Lyapunov equation

$$-\dot{P} + P(t)A(t) + A^T(t)P(t) + Q(t). \tag{1.4.3}$$

Has a continuously differentiable solution $P : (0, \infty) \to \mathbb{R}^n$.
The following theorem characterizes uniform asymptotic stability in terms of $\phi(t, t_0)$.

Theorem 1.4.3. *[10] The equilibrium point of system (1.4.2) is globally uniformly asymptotic stable if and only if the state transition matrix satisfies the inequality*

$$||\phi(t, t_0)|| \leq Ke^{-\gamma(t-t_0)},\ \forall t \geq t_0 \geq 0$$

for some positive constants K and γ.

Proof.
Assume that the origin is GUAS then there exists a class $\mathcal{K}$-function $\beta(.,.)$, such that

$$||x(t)|| \leq \beta(x_0, t - t_0),\ \forall t \geq t_0 \geq 0,\ \forall x_0 \in \mathbb{R}^n.$$

From the definition of the matrix norm, we have

$$\phi(t, t_0) = \max_{x=1} \phi(t, t_0)x(t - t_0) \leq \max_{x=1} \beta(x, t - t_0) = \beta(1, t - t_0)$$

Since

$$\lim_{s \longrightarrow +\infty} \beta(1, s) = 0$$

There exists $T > 0$ such that $\beta(1, T) \leq \frac{1}{e}$.
Let N be the smallest positive integer such that $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N - 1)$ equal sub intervals of width T each. Using the transition property of $\phi(t, t_0)$, we can write

$$\phi(t, t_0) = \phi(t, t_0 + (N-1)T)\phi(t_0 + (N-1)T, t_0 + (N-2)T)...\phi(t_0 + T, t_0))$$

Hence

$$\begin{aligned} || \phi(t, t_0) || &\leq \phi(t, t_0 + (N-1)T)|| \prod_{k=1}^{N-1} ||\phi(t_0 + kT, t_0 + (k-1)T) \\ &\leq \beta(1, 0) \prod_{k=1}^{N-1} \beta(1, T) \\ &\leq \beta(1, 0) \prod_{k=1}^{N-1} \frac{1}{e} \\ &\leq e\beta(1, 0)e^{-N} \end{aligned}$$

Since $t \leq t_0 + NT$*, then* $N \geq \frac{t-t_0}{T}$ *which implies that*

$$\phi(t,t_0) \leq e\beta(1,0)e^{\frac{-1}{T}(t-t_0)}$$

It follows that there exists $k = e\beta(1,0)$ *and* $\gamma = \frac{1}{T}$ *such that*

$$\phi(t,t_0) \leq ke^{-\gamma(t-t_0)}, \ \forall t \geq t_0 \geq 0$$

Now, using Lyapunov approach, we suppose that there exists a continuously differentiable bounded, positive definite, symmetric matrix $P(t)$ *that is*

$$0 < c_1 I \leq P(t) \leq c_2 I, \ \forall t \geq 0, (c_1 > 0, c_2 > 0)$$

Which satisfies the Lyapunov-equation (1.4.3) where $Q(t)$ *is continuously, positive definite, symmetric matrix, that is*

$$Q(t) \geq c_3 I > 0, \ \forall t \geq 0, (c_3 > 0).$$

Consider the Lyapunov function candidate

$$V(t,x) = x^T P(t) x$$

this function satisfies

$$c_1 x^2 \leq V(t,x) \leq c_2 x^2.$$

Moreover, this function is proper since $c_1 x^2$ *is a class* $\mathcal{K}_\infty$*-function. The derivative of* $V(t,x)$ *along the trajectories of system (1.4.2) is given by*

$$\begin{aligned}
\dot{V}(t,x) &= x^T \dot{P}(t) x + x^T P(t) \dot{x} + \dot{x}^T P(t) x \\
&= v x^T(\dot{P}(t) + P(t)A(t) + A^T P(t))x \\
&= v - x^T Q(t) x \\
&\leq -c_3 x^2 \\
&\leq -\frac{c_3}{c_2} V(t,x).
\end{aligned}$$

Then $\dot{V}(t,x)$ *is negative definite. All assumptions of theorem (1.4.3) are satisfied with* $\alpha_i(r) = c_i r^2$ *then, the origin is globally uniformly exponentially stable.*

Example 1.4.2. *Consider the linear system*

$$\dot{x}(t) = A(t)x$$

where

$$A(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

We have

$$\phi(t,0) = e^{\lambda_1} \begin{pmatrix} \cos\lambda_2 & \sin\lambda_2 \\ -\sin\lambda_2 & \cos\lambda_2 \end{pmatrix}$$

where

$$\lambda_1 = (\sin t - \cos\tau) \text{ and } \lambda_2 = (\cos\tau - \cos t).$$

All elements are bound for all $t \geq \tau$. The system is uniformly stable. It is not uniformly asymptotic stable because

$$\lim_{t\longrightarrow+\infty}[e^{\sin t}\cos(1-\cos t)] \neq 0.$$

Theorem 1.4.4. *[10] Suppose that the origin is an equilibrium point uniformly asymptotic stable of the system (1.4.2). Suppose also that $A(t)$ is continuous and bounded. Then, for any matrix $Q(t)$ continuous, positive definite, symmetric and bounded, there exists continuously differentiable bounded, positive definite and symmetric matrix $P(t)$ that satisfies (1.4.3).*
It follows that $V(t,x) = x^T P(t)x$ is a Lyapunov function of the system that satisfies the assumptions of the theorem (1.4.3).
The following theorem give necessary and sufficient conditions for the exponential stability of the system (1.4.2).

Theorem 1.4.5. *[4]*

- *Let $A(t)$ be bounded and $Q(t)$ is continuously, bounded and symmetric positive definite matrix. If (1.4.2) is exponentially stable, then there exists a continuously differentiable bounded, positive definite, symmetric matrix $P(t)$, (that is $0 < c_1 I \leq P(t) \leq c_2 I$, $\forall t \geq 0$, $(c_1 > 0, c_2 > 0)$) solution of (1.4.3).*
- *If there exist $P(t)$, $Q(t)$ continuously, bounded and symmetric positive definite matrix, such that $P(t)$ is continuously differentiable, and (1.4.3) holds, then (1.4.2) is exponentially stable.*

The following example illustrates that asymptotic and exponential stabilities are not equivalent, as it is in case when A is constant.

Example 1.4.3. *[10]*
Consider the scalar differential equation

$$\dot{x}(t) = -\frac{1}{t}x, \ t \in (\frac{1}{2}, \infty). \tag{1.4.4}$$

Then, for any $x_0 \in \mathbb{R}^n$, $t_0 > \frac{1}{2}$, the initial value problem (1.4.4), $x(t_0) = x_0$ has a unique global solution

$$x\colon (\tfrac{1}{2}, \infty) \to \mathbb{R}^n$$
$$t \to \tfrac{t_0 x_0}{t}$$

and hence (1.4.4) is asymptotically stable, but not exponentially stable.

1.4.3 Lyapunov's indirect method

The indirect method of Lyapunov uses a linearization of a system to determine the local stability of the original system. So, let us go back to the nonlinear system (1.1.1) where $f : [0, +\infty[\times U(0) \to \mathbb{R}^n$ is a continuously differentiable function and $U(0) = \{x \in \mathbb{R}^n / x < r\}$. Suppose that the origin is an equilibrium point of the system (1.1.1).
Assume also that the Jacobian matrix $[\frac{\partial f}{\partial x}]$ is uniformly bounded and Lipschitz on $U(0)$, that is there exist k and $L > 0$ such that

$$\begin{aligned} \frac{\partial f}{\partial x}(t,x) &\leq k, \ \forall x \in U(0), \ \forall t \geq t_0 \\ \frac{\partial f}{\partial x}(t,x_1) - \frac{\partial f}{\partial x}(t,x_2) &\leq L||x_1 - x_2||, \ \forall x_1, x_2 \in U(0), \ \forall t \geq t_0. \end{aligned}$$

We can write $f(t,x)$ in the form

$$f(t,x) = f(t,0) + \frac{\partial f}{\partial x}(t,z)x$$

where $z \in]0, x[$. Since $f(t,0) = 0$, then

$$\begin{aligned} f(t,x) &= \frac{\partial f}{\partial x}(t,z)x \\ &= \frac{\partial f}{\partial x}(t,0)x + [\frac{\partial f}{\partial x}(t,z) - \frac{\partial f}{\partial x}(t,0)] \\ &= A(t)x + g(t,x) \end{aligned}$$

where

$$A(t) = \frac{\partial f}{\partial x}(t,0) \text{ and } g(t,x) = [\frac{\partial f}{\partial x}(t,z) - \frac{\partial f}{\partial x}(t,0)]x.$$

Therefore, in a small neighborhood of the origin, we may approximate the nonlinear system (1.1.1) by its linearization about the origin. The following theorem states Lyapunov's indirect method for showing the uniform exponential stability of the origin.

Theorem 1.4.6. *[10] Suppose that the origin is an equilibrium point of the nonlinear system (1.1.1) where $f : [0, +\infty[\times U(0) \to \mathbb{R}^n$ is continuously differentiable function.*
$U(0) = \{x \in \mathbb{R}^n / ||x|| < r\}$ and $[\frac{\partial f}{\partial x}]$, is uniformly bounded and Lipschitz on $U(0)$.
Let

$$A(t) = \frac{\partial f(t,x)}{\partial x}|_{x=0} \tag{1.4.5}$$

If the origin is an equilibrium point exponential stable of the nonlinear system (1.1.1) then it is an equilibrium point exponential stable of the system (1.4.2).
The following theorem shows that the exponential stability of the linearized system is a necessary and sufficient assumption for exponential stability of the nonlinear system origin.

Theorem 1.4.7. *[10] Suppose that the origin is an equilibrium point of the nonlinear system (1.1.1) where $f : [0, +\infty[\times U(0) \to \mathbb{R}^n$ is continuously differentiable function.*
$U(0) = \{x \in \mathbb{R}^n / x < r\}$ and $[\frac{\partial f}{\partial x}]$, is uniformly bounded and Lipschitz on $U(0)$. Then the origin is an equilibrium point exponential stable of nonlinear system if and only if it is an equilibrium point exponential stable of the system (1.4.2).

1.5 Perturbed systems

The question addressed in this section is related to the study of the preservation of stability when considering a new system with a perturbation term which could result in general from errors in modeling the nonlinear system, aging of parameters or disturbances. To analyze the stability of perturbed system having the form

$$\dot{x} = f(t, x) + g(t, x) \tag{1.5.1}$$

where $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous on (t, x), locally Lipschitz on x, uniformly on t. such system can be considered as a perturbation of the nominal system

$$\dot{x} = f(t, x). \tag{1.5.2}$$

Here, we represent the perturbation as an additive term on the right-hand side of the state equation. The perturbation term $g(t, x)$ could result from errors in modeling the nonlinear system, aging of parameters, or uncertainties and disturbances which exist in any realistic problem. In a typical situation, we don't know $g(t, x)$, but we know some information about it, like knowing an upper bound on $||g(t, x)||$. The trivial question posed here is if the nominal system present one of type of stability, does perturbed one keep the same behavior or not?.
A natural approach is to use a Lyapunov function for the nominal system as a Lyapunov function for the perturbed system. It is used in [10] to prove the exponential stability of the perturbed system. This approach, called the indirect Lyapunov method, is based on the use of the linearized $A(t) = \frac{\partial f}{\partial x}(t, 0)$ (if the function f is continuously differentiable). But its disadvantage lies the fact the perturbation term could be more general than in the case of linearization. We have two cases :
the first case is when $g(t, 0) = 0$ that is the perturbed system has an equilibrium point at the origin, then we analyze the stability behavior of the origin as an equilibrium point of the perturbed system.
the second case, which is the more general case, is when we don't know that $g(t, 0) = 0$. Therefore, we can no longer study the problem as a question of the stability of equilibria and some new concepts were introduced in [10] to study the exponential stability of the perturbed system.

1.5.1 Vanishing perturbation

Let us start with the case $g(t, 0) = 0$.

Theorem 1.5.1. *Suppose $x = 0$ is exponentially stable equilibrium point of the nominal system (1.5.2) and let $V(t, x)$ be a Lyapunov function that satisfies, for all $(t, x) \in \mathbb{R}^n \times U(r)$*

$$c_1 \mid\mid x \mid\mid^2 \leqslant V(x, t) \leq c_2 \mid\mid x \mid\mid^2 \tag{1.5.3}$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x) \leq -c_3(\mid\mid x \mid\mid)^2 \tag{1.5.4}$$

$$\mid\mid \frac{\partial V}{\partial x}(t, x) \mid\mid \leq c_4(\mid\mid x \mid\mid) \tag{1.5.5}$$

for some positive constants c_1, c_2, c_3 *and* c_4.
If the perturbation term $g(t,x)$ *satisfies the linear growth bound*

$$g(t,x) \leqslant \gamma||x|| \quad (1.5.6)$$
$$\gamma < \frac{c_3}{c_4}. \quad (1.5.7)$$

Then, the origin is exponentially stable equilibrium point of the perturbed system (1.5.1). Moreover, if all assumptions hold globally, then the origin is GES.

Remark 1.4. *One can obtain exponential convergence to zero for system (1.5.1) especially, where*

$$g(t,x) = B(t)x$$

under the conditions $B(t)$ *is continuous and* $B(t) \to 0$ *as* $t \to +\infty$. *Similar conclusions can be obtained where*

$$\int_0^{+\infty} B(t)dt < +\infty$$

Example 1.5.1. *Consider the system*

$$\dot{x} = Ax + g(t,x)$$

where A *is Hurwitz and*

$$||g(t,x)|| \leqslant \gamma||x|| \ \forall t \geq 0, \ \forall x \in \mathbb{R}^n.$$

Let $Q = Q^T > 0$ *and solve the lyapunov equation*

$$PA + A^T P = -Q.$$

There is a unique solution

$$P = P^T > 0.$$

The quadratic Lyapunov function $V(x) = x^T Px$ *satisfies (1.5.3), (1.5.4) and (1.5.5). In particular, Rayleigh's inequality* [3] *allows us to write*

$$\begin{aligned} \lambda_{min}(P)||x||^2 &\leq V(t,x) \leq \lambda_{max}(P)||x||^2 \\ \frac{\partial V}{\partial x}Ax = -x^T Qx &\leq -\lambda_{min(Q)}x^2 \\ \frac{\partial V}{\partial x} = 2x^T||P|| &\leq 2||Px|| \leqslant 2\lambda_{max}(P)||x||. \end{aligned}$$

Hence, from theorem (1.5.1) the origin is globally exponentially stable if $\gamma < \dfrac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$.

Let's study the case when the origin of the nominal system (1.5.2) is uniformly asymptotically stable but not exponentially stable.

[3] Let $P = P^T \in \mathbb{R}^{n\times n}$. Symmetric matrices. Show that : $\lambda_{\min} x^T x \leq x^T Px \leq \lambda_{\max} x^T x$
For all vector x, where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of P, respectively.

The stability analysis of the perturbed system is more involved. Suppose the nominal system has a positive definite, decrescent Lyapunov function $V(t,x)$ that satisfies

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\leqslant -W_3(x),\ \forall (t,x)\in \mathbb{R}_+\times U(r)$$

Where $W_3(x)$ is positive definite and continuous. The derivate of V a long the trajectories of (1.5.1)is given by

$$\begin{aligned}\dot{V}(t,x) &= \frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)f(t,x)+\frac{\partial V}{\partial x}(t,x)g(t,x)\\ &\leq W_3(x)+\frac{\partial V}{\partial x}(t,x)g(t,x)\end{aligned}$$

our task now is to show that

$$\frac{\partial V}{\partial x}g(t,x)<W_3(x),\forall (t,x)\in \mathbb{R}_+\times U(r).$$

This task which cannot be done by a putting a simple order of magnitude bound on $||g(t,x)||$ as we have done in the exponentially stable case. The growth bound on $||g(t,x)||$ will depend on the nature of the Lyapunov function of the nominal system.
One class of Lyapunov functions for which the analysis is almost as simple as in exponential stability is the case when $V(t,x)$ is positive definite, decrescent and satisfies

$$\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)f(t,x) \leq -c_3\phi^2(x) \tag{1.5.8}$$

$$||\frac{\partial V}{\partial x}|| \leq c_4\phi(x) \tag{1.5.9}$$

for all $(t,x)\in \mathbb{R}_+\times U(r)$, for some constants c_3 and c_4.
Where ϕ is positive definite and continuous.
A Lyapunov function satisfying (1.5.8) and (1.5.9) is usually called a quadratic-type Lyapunov function. It is clear that a Lapunov function satisfying (1.5.3) and (1.5.4) is quadratic type, but a quadratic-type Lyapunov function may exist even when the origin is not exponentially stable. In conclusion, if the nominal system (1.5.2) has a quadratic-type Lyapunov function $V(t,x)$, then its derivative along the trajectories of (1.5.1) satisfies

$$\dot{V}(t,x)\leq -c_3\phi^2(x)+c_4\phi(x)g(t,x).$$

Suppose now that the perturbation term satisfies the bound

$$g(t,x)\leq \gamma\phi(x)$$
$$\gamma<\frac{c_3}{c_4}$$

Then

$$\dot{V}(t,x)\leq -(c_3-c_4\gamma)\phi^2(x).$$

Hence, the origin is a globally uniformly asymptotically stable (GUAS) equilibrium point of the perturbed system.

We shall illustrate this point by an example.

Example 1.5.2. *Consider the perturbed system*

$$\dot{x} = -x^3 + g(t,x)$$

the nominal system

$$\dot{x} = -x^3$$

has a global asymptotically stable equilibrium point at the origin but is not exponentially stable. Thus, there is no Lyapunov function that satisfies (1.5.3) and (1.5.4).
The Lyapunov function $V(x) = x^4$ satisfies (1.5.8) and (1.5.9), with $\phi(x) = |x|^3$, $c_3 = 4$ and $c_4 = 4$.
Suppose the perturbation term $g(t,x)$ satisfies the bound $|g(t,x)| \leq \gamma |x|^3$ for all x, with $\gamma < 1$. Then, the derivative of V along the trajectories of the perturbed system satisfies

$$\dot{V}(t,x) \leq -4(1-\gamma)\phi^2(x).$$

Hence, the origin is a GUAS equilibrium point of the perturbed system.

1.5.2 Non vanishing Perturbation

The more interesting case occurs when $g(t,0)$ is not necessarily zero. In this case $x = 0$ is no longer an equilibrium point.

Practical stability

There are some systems that may be unstable and yet these systems may oscillate sufficiently near this state that its performance is acceptable. To deal with this situations, we need a notion of stability which is called practical stability. Unlike Lyapunov stability, the study of the practical stability led back to the study of stability of a ball centered at the origin.
That is why we start by given the definition of the uniformly stability and the uniformly attractiveness of a bounded ball $B_r = \{x \in \mathbb{R}^n / ||x|| \leq r\}$.

Definition 1.5.1. ***(Uniformly stability of** B_r**)***

i) B_r is uniformly stable, if for all $\epsilon > r$, there exist $\delta = \delta(\epsilon)$ such that, for all $t_0 \geq 0$

$$||x_0|| < \delta \Longrightarrow x(t) < \epsilon, \ \forall t \geq t_0.$$

ii) B_r is globally uniformly stable, if it is uniformly stable and the solutions of (1.1.1) are globally uniformly bounded.

Definition 1.5.2. ***(Uniformly attractiveness of** B_r**)***
B_r is uniformly attractive, if for all $\epsilon > r$, there exists $T(\epsilon) > 0$ such that for each $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$

$$x(t) < \epsilon, \ \forall t \geq t_0 + T(\epsilon).$$

Definition 1.5.3. ***(Uniform exponential stability of B_r)***
B_r is globally uniformly exponentially stable if there exist $\gamma > 0$ and $k > 0$, such that for all $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$

$$x(t) \leq k||x_0|| \exp(-\gamma(t - t_0)) + r, \ \forall t \leq t_0. \tag{1.5.10}$$

Definition 1.5.4. ***(Practical stability)***

*i) The system is said uniformly practically asymptotically stable (UPAS) if there exists a ball $B_r \subset \mathbb{R}^n$ such that B_r is uniformly stable and uniformly attractive **ie** UAS.*

*ii) The system is said globally uniformly practically asymptotically stable (GUPAS) if there exists a ball $B_r \subset \mathbb{R}^n$such that B_r is globally uniformly stable and globally uniformly attractive **ie** GUAS.*

iii) The system is said globally practically uniformly exponentially stable (GPUES) if there exist a ball $B_r \subset \mathbb{R}^n$ such that B_r is GUES.

Proposition 1.3. *[13]*
If there exists a class $\mathcal{K}$-function α and a constant $r \geq 0$ such that, given any initial state x_0, the solution satisfies

$$||x(t)|| \leq \alpha(x_0) + r, \ \forall t \geq t_0.$$

Then the system (1.1.1) is GUPS.

Proposition 1.4. *[13]*
If there exists a class $\mathcal{KL}$-function β, and a constant $r \geq 0$ such that, given any initial state x_0, the solution satisfies

$$x(t) \leq \beta(x_0, t - t_0) + r, \ \forall t \geq t_0$$

Then the system (1.1.1) is (GUPAS).

Theorem 1.5.2. *[1]*
Consider the system (1.1.1). Assume there exist a class C^∞ function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, two class $\mathcal{K}_\infty$-functions $\alpha_1(.)$ and $\alpha_2(.)$, a class $\mathcal{K}$-function $\alpha_3(.)$ and a positive real r small enough such that the following inequalities are satisfied for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$\alpha_1(x) \leq V(t,x) \leq \alpha_2(x)\frac{\partial}{\partial t}V(t,x) + \frac{\partial}{\partial x}V(t,x)f(t,x) \leq -\alpha_3(x) + r$$

then the system (1.1.1) is globally practically stable with

$$B_r = \left\{x \in \mathbb{R}^n / ||x|| \leq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(r)\right\}.$$

Theorem 1.5.3. *[1]*
Consider the system (1.1.1). Assume there exist a class C^∞ function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, and some positive constants a, b, c_1, c_2 and c_3 such that the following inequalities are satisfied for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$c_1||x||^2 \leq V(t,x) \leq c_2 x^2 + a \tag{1.5.11}$$
$$\frac{\partial}{\partial t}V(t,x) + \frac{\partial}{\partial x}V(t,x)f(t,x) \leq -c_3(x) + b \tag{1.5.12}$$

Then the ball B_α is GUES with $\alpha = \sqrt{\frac{2ac_3+bc_2}{c_1c_3}}$.

For uniform asymptotic stability of a ball B_r on a ball B_R with $0 \leq r < R < \infty$, in [6], a converse theorem is established when the origin is not an equilibrium but there exists a non negative constant f_0 such that $||f(t,0)|| \leq f_0$, $\forall t \geq 0$.

Theorem 1.5.4. *[1]*
Consider the nonlinear system (1.1.1) and assume there exists a non negative constant f_0 such that $||f(t,0)|| \leq f_0$, $\forall t \geq 0$, and that f is a class C^∞ function and that $[\frac{\partial f}{\partial x}]$ is bounded on $\mathbb{R}^n$, uniformly in t. Assume that the trajectories of the system satisfy (1.5.10) for all $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$, For some positive constants k, γ and r.
Then, there is a function $V(.,.) : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$ that satisfies the inequalities

$$\begin{aligned} c_1 x^2 &\leq V(t,x) \leq c_2 x^2 + a \\ \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,x) &\leq -c_3(x) + \rho \\ \frac{\partial V(t,x)}{\partial x} &\leq c_4 x + b \end{aligned}$$

for some positive constants $c_1, c_2, c_3, c_4, a, \rho$ and b.

1.6 Stabilization

We consider a system whose evolution can be described by the differential system

$$\begin{cases} \dot{x} = f(x,u) \\ x \in \mathbb{R}^n, \;\; u \in \mathbb{R}^m \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ of class C^1 such that $f(0,0) = 0$.

The problem we are interested in is to find a feedback $u = u(x)$ as possible such that $x = 0$ is an asymptotically stable equilibrium position for the looped system $\dot{x} = f(x, u(x))$.
We will address the problem of stabilizing linear control systems and nonlinear.

1.6.1 Stabilization of linear control systems

Autonomous Case

We call a linear control system any system of the form

$$\dot{x} = Ax + Bu \tag{1.6.1}$$

Where the state is $x \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$ and where $A \in M_{n,n}(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$.

We consider the linear system (1.6.1), we then say that it is stabilized by feedback, if there exists a matrix $K \in M_{(m,n)}(\mathbb{R})$ such that the feedback $u = Kx$ stabilizes (1.6.1), that is the closed-loop system $\dot{x} = Ax + BKx$ is asymptotically stable.

Non-autonomous Case

Let the linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.6.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A(t) \in M_{n,n}(\mathbb{R})$ and $B(t) \in M_{n,m}(\mathbb{R})$.

Definition 1.6.1. *we say that the linear system (1.6.2) is stabilized by feedback, if there exists a matrix $K(t) \in \mathbb{R}^{m\times n}$, such as closed-loop system*

$$\dot{x}(t) = A(t)x(t) + B(t)K(t)x(t) \tag{1.6.3}$$

is asymptotically stable.

1.6.2 Stabilization of nonlinear control systems

Let us now go back to our general nonlinear control system

$$\dot{x} = f(x, u) \tag{1.6.4}$$

where the state is $x \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$ and the function f is of class C^1 in a neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ and such that $f(0, 0) = 0$. We now define asymptotic stabilizability by means of a continuous stationary feedback law.

Autonomous case

In the case of nonlinear systems, for local stabilization in the autonomous case necessary conditions by Brockett. The following theorem due to R.PV Brockett gives necessary conditions of asymptotic stabilization of a nonlinear system by a class C^1 control.

Theorem 1.6.1. *[5] If the system (1.6.4) admits a stabilizing feedback of class C^1 in a neighborhood of $0 \in \mathbb{R}^n$ then*

i) *The linearized system does not admit associated uncontrollable modes, to strictly positive eigenvalues.*

ii) *There exists a neighborhood N of $(0,0)$ such that for all $\lambda \in N$ there exists a control $U_\lambda(.)$ defined on $[0, \infty[$ which brings the system from the state $x = \lambda$ at $t = 0$ in state $x = 0$ at $t = \infty$, In other words, if $x(t)$ is a solution of $\dot{x} = f(x, u_\lambda)$ satisfying $x(0) = \lambda$ then $\lim_{t \longrightarrow \infty} x(t) = 0$.*

iii) *The application $\gamma : \mathbb{R}^n \times U \to \mathbb{R}^n$*

$$(x, u) \to f(x, u)$$

is surjective on a neighborhood of the origin of $\mathbb{R}^n$.

Remark 1.5.

- *Condition **(i)** is not necessary if we are satisfied with continuous feedback as shown in the following example*

  Example 1.6.1. ***(Kawski)***
  Consider the system

$$\begin{cases} \dot{x}_1 = x_1 - x_2^3 \\ \dot{x}_2 = u \\ \dot{x} = (x_1, x_2)^T \in \mathbb{R}^2 \end{cases}$$

  is stable with

$$u(x) = -x_2 + x_1 + \left(\frac{4}{3x_1}\right)^{\frac{1}{3}} - x_2^3.$$

- *Condition **(ii)** is a necessary condition of stability, if the regularity required is at least class C^1.*

- *Condition **(iii)** is also a necessary condition for the existence of a continuous stabilizing feedback.*

- *Brockett's conditions are not sufficient for the existence of feedback stabilizer even if the regularity required is only continuity.*

Non-autonomous case

Let the system is controlled

$$\dot{x} = f(t, x, u) \tag{1.6.5}$$

Where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input.

Definition 1.6.2. *We say that the system (1.6.5) is stable, if there exists a function continues $u(t,x)$ such that the origin of the system (1.6.5) in a loop closed by $u(t,x)$ is asymptotically stable. The function $u(t,x)$ is called feedback (or control).*
Note that the theory of stability of linear systems is simple. Also, an approach to the stabilization of nonlinear systems refers to this theory. It involves associating with the system (1.6.5) a simpler system whose study allows us to have information about the initial system. The classic method when $F(t,0,0) = 0$, is to consider the linearized at the origin, that is to say

$$\begin{cases} \dot{x} = A(t)x(t) + B(t)u(t) \\ A(t) = \frac{\partial f}{\partial x}, \;\; B(t) = \frac{\partial f}{\partial u} \end{cases}$$

We then know that if (1.6.2) is stabilized, (1.6.5) is also stable and with the same feedback, but this method only allows us to conclude locally.

1.6.3 Stabilization of perturbed system

We consider perturbed systems of the following form

$$\dot{x} = A(t)x(t) + B(t)u(t) + g(t, x(t)) \tag{1.6.6}$$

Where the state is $x(t) \in \mathbb{R}^n$, the control is $u(t) \in \mathbb{R}$ and $A(t), B(t)$ are unbounded matrices. The function $g(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, locally Lipschitzian in x, uniformly in t. Let the nominal system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.6.7}$$

We will study the stabilization of system (1.6.6) using a state feedback candidate. We find the candidate feedback $K(t)$ in the form

$$u(t) = K(t)x(t) \tag{1.6.8}$$

The closed-loop system (1.6.6) becomes

$$\dot{x} = A_K(t)x(t) \tag{1.6.9}$$

with

$$A_K(t) = A(t) + B(t)K(t)$$

$(\mathcal{H}_1)$ for any positive definite symmetric matrix $Q_1(t)$

$$Q_1(t) \geq c_1 I > 0, \forall t \geq 0$$

there exists a positive definite symmetric matrix $P_1(t)$

$$0 < c_2 I < P_1(t) < c_3 I, \forall t \geq 0$$

which satisfies

$$A_K^T(t)P_1(t) + P_1(t)A_K(t) + \dot{P}_1(t) = -Q_1(t) \tag{1.6.10}$$

$(\mathcal{H}_2)$ we suppose that

there exists $g_0 > 0$, such as

$$||g(t,0)|| \leq g_0, \forall t \geq 0$$

$$||g(t,x) - g(t,y)|| \leq \gamma(t)||x-y|| + \delta(t) + \epsilon, \ \forall t \geq 0, \ \forall x, y \in \mathbb{R}^2$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}$ and $\delta : \mathbb{R}_+ \to \mathbb{R}$ are two positive continuous functions with

$$\int_0^{+\infty} \gamma(s)ds \leq M_\gamma < +\infty$$

and

$$\int_0^{+\infty} \delta^2(s)ds \leq M_\delta < +\infty.$$

Theorem 1.6.2. *Under hypothesis ($\mathcal{H}_1$) and ($\mathcal{H}_2$), the system (1.6.6) in a loop closed by feedback linear $u(t) = K(t)x(t)$ is GPUES.*

Proof.

We consider the candidate Lyapunov function

$$V(t,x) = x^T(t)P_1(t)x(t).$$

The derivative of V along the trajectories of the system (1.6.6) is given by

$$\begin{aligned}
\dot{V}(t,x) &= \dot{x}^T(t)P_1(t)x(t) + x^T(t)\dot{P}_1(t)x(t) + x^T(t)P_1(t)\dot{x}(t) \\
&= (A_k(t)x(t) + g(t,x))^T P_1(t)x(t) + x^T(t)\dot{P}_1(t)x(t) + x^T(t)P_1(t)(A_k(t)x(t) + g(t,x)) \\
&\leq -x^T(t)Q_1(t)x(t) + 2||P_1(t)||||g(t,x)||||x(t)|| \\
&\leq -c_1||x(t)||^2 + 2c_3(\gamma(t)||x|| + \delta(t) + g_0 + \epsilon)||x(t)|| \\
&\leq -\left(\frac{c_1}{c_3} - \frac{2c_3\gamma(t)}{c_2}\right) + 2\frac{c_3}{\sqrt{c_2}}(\delta(t) + g_0 + \epsilon)\sqrt{V(t,x)}
\end{aligned}$$

Either

$$v(t) = \sqrt{V(t,x)}$$

The derivative of v is then

$$\dot{v}(t) = \frac{\dot{V}(t,x)}{2\sqrt{V(t,x)}}$$

Which implies

$$\dot{v}(t) \leq -\left(\frac{c_1}{2c_3} - \frac{c_3\gamma(t)}{c_2}\right)v(t) + \frac{c_3}{\sqrt{c_2}}(\delta(t) + g_0 + \epsilon)$$

Either

$$\alpha(t) = \frac{c_1}{2c_3} - \frac{c_3\gamma(t)}{c_2}$$

So

$$\dot{v} \leq -\alpha(t)v(t) + \frac{c_3}{\sqrt{c_2}}(\delta(t) + g_0 + \epsilon)$$

Either

$$y(t) = v(t)e^{\int_{t_0}^{t} \alpha(s)ds}$$

Which implies

$$\dot{y}(t) = (\dot{v}(t) + \alpha(t)v(t))e^{\int_{t_0}^{t} \alpha(s)ds}$$
$$\leq \frac{c_3}{\sqrt{c_2}}(\delta(t) + g_0 + \epsilon)e^{\int_{t_0}^{t} \alpha(s)ds}$$

Integrating between t_0 and t, we obtain

$$y(t) \leq y(t_0)\frac{c_3}{\sqrt{c_2}} \int_{t_0}^{t} (\delta(s) + g_0 + \epsilon)e^{\int_{t_0}^{t} \alpha(\tau)d\tau} ds, \forall t \geq t_0$$

So

$$v(t) \leq v(t_0)e^{-\int_{t_0}^{t} \alpha(s)ds} + \left(\frac{c_3}{\sqrt{c_2}} \int_{t_0}^{t} (\delta(s) + g_0 + \epsilon)e^{\int_{t_0}^{t} \alpha(\tau)d\tau} ds\right)e^{-\int_{t_0}^{t} \alpha(s)ds}$$

We have

$$\int_{t_0}^{t} \alpha(s)ds = \frac{c_1}{2c_3}(t - t_0) - \frac{c_3}{c_2}\int_{t_0}^{t} \gamma(s)ds$$

From which

$$e^{\int_{t_0}^{t} \alpha(s)ds} \leq e^{\frac{c_3 M_\gamma}{c_2}} e^{-\frac{c_1}{2c_3}(t - t_0)}$$

Furthermore

$$\int_{t_0}^{t} \delta(s)e^{\int_{t_0}^{s} \alpha(\tau)d\tau} ds = \int_{t_0}^{t} \delta(s)e^{\frac{c_1}{2c_3}(s - t_0)}e^{-\frac{c_3}{c_2}\int_{t_0}^{s} \gamma(\tau)d\tau} ds$$
$$\leq \left(\int_{t_0}^{t} \delta^2(s)ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} e^{\frac{c_1}{c_3(s - t_0)}} ds\right)^{\frac{1}{2}}$$
$$\leq \sqrt{\frac{c_3 M_\delta}{c_1}} e^{\frac{c_1}{2c_3}(t - t_0)}$$

and

$$\int_{t_0}^{t} (g_0 + \epsilon)e^{\int_{t_0}^{s} \alpha(\tau)d\tau} ds = (g_0 + \epsilon)\int_{t_0}^{t} e^{\frac{c_1}{2c_3}(s - t_0)}e^{-\frac{c_3}{c_2}\int_{t_0}^{s} \gamma(\tau)d\tau} ds$$
$$\leq 2(g_0 + \epsilon)\frac{c_3}{c_1}e^{\frac{c_1}{2c_3}(t - t_0)}$$

It follows that

$$\left(\int_{t_0}^{t} (\delta(s) + g_0 + \epsilon)e^{\int_{t_0}^{s} \alpha(\tau)d\tau} ds\right)e^{-\int_{t_0}^{t} \alpha(s)ds} \leq \left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(g_0 + \epsilon)\frac{c_3}{c_1}\right)e^{\frac{c_{3M_\gamma}}{c_2}}$$

Therefore

$$v(t) \leq v(t_0)e^{\frac{c_{3M_\gamma}}{c_2}} e^{-\frac{c_1}{2c_3}(t - t_0)} + \frac{c_3}{\sqrt{c_2}}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(g_0 + \epsilon)\frac{c_3}{c_1}\right)e^{\frac{c_{3M_\gamma}}{c_2}}$$

We deduce that

$$||x(t)|| \leq \sqrt{\frac{c_3}{c_2}} e^{\frac{c_{3M_\gamma}}{c_2}} ||x(t_0)|| e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{c_2}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(g_0+\epsilon)\frac{c_3}{c_1}\right) e^{\frac{c_{3M_\gamma}}{c_2}}$$

As a result, the global exponential uniform stability (GEUS) of B_κ *with*

$$\kappa = \frac{c_3}{c_2}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(g_0+\epsilon)\frac{c_3}{c_1}\right) e^{\frac{c_{3M_\gamma}}{c_2}}.$$

Hence, the closed-loop system (1.6.6) with linear feedback $u(t) = K(t)x(t)$ *is generally practically exponentially stable.*

1.7 Conclusion

In this chapter, a review of the study of stability analysis of nonlinear systems using Lyapunov's theory has been proposed. For practical implementation, we have to find the Lyapunov function specifically for each nonlinear system. Asymptotic stability properties and the usefulness of both Lyapunov indirect methods have been presented. Furthermore, we have referred to conditions for uniform stability, uniform asymptotic stability, and the use of the Lyapunov function for the determination of a domain of attraction. Examples have been proposed to illustrate the procedure for studying the stability of nonlinear systems.

CHAPTER 2

INPUT-TO-STATE STABILITY

Introduction

This chapter is concerned with stability analysis of nonlinear time-varying systems by using Lyapunov function based approach. The classical Lyapunov stability theorems are generalized in the sense that the time-derivatives of the Lyapunov functions are allowed to be indefinite. The stability analysis is accomplished with the help of the scalar stable functions introduced in the author's previous study. Both asymptotic stability and input-to-state stability are considered. Particularly, for asymptotic stability, several concepts such as uniform and nonuniform asymptotic stability, and uniform and nonuniform exponential stability are studied. The effectiveness of the proposed theorems is illustrated by several numerical examples.

2.1 stability with indefinite Lyapunov function

Consider the following non-linear time-varying system

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{2.1.1}$$

Where $x(t)$ is the state, the control input $u(t) : \mathbb{R}_+ \to \mathbb{R}^m$ which is assumed to be measurable and locally essentially bounded and $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz in (t, x), to be uniformly continuous in u, and to satisfy $f(t, 0, 0) = 0$ caratheodory conditions ensure that there exists a unique maximal solution $x(t, x(t_0), u)$ for System(2.1.1) for an initial value $x(t_0) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and an initial time $t_0 \geq 0$. In this suction, we are interested in the stability analysis of the system (2.1.1). Throughout this note, for any C^1 function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$.

Definition 2.1.1. ***(PUISS)*** *We say that the nonlinear time-varying (2.1.1) is practical uniform input-to-state stable with respect to u if there exist functions $\sigma(.,.) \in \mathcal{KL}, \gamma(.) \in \mathcal{K}$ and positive scalar ℓ, such that for each initial condition x_0 at any initial time t_0 and each measurable essentially bounded control $u(.) \in \mathfrak{L}^m_\infty(\mathbb{R}_+)$.*

The solution $x(.)$ of the system (2.1.1) exists on $\mathbb{R}_+$ and satisfies

$$|x(t)| \leq \sigma(|x_0|, t - t_0) + \gamma(||u||_{[t_0,t]}) + \ell, \ \forall t \geq t_0. \tag{2.1.2}$$

Remark 2.1. *It is possible to use the 'max' function in an equivalent way instead the plus sign of the addition '+', in the estimation of this bound*

$$|x(t)| \leq \left\{ \max \sigma |x_0|, t - t_0), \gamma(||u||_{[t_0,t]}) \right\} + \ell, \ \forall t \geq t_0. \tag{2.1.3}$$

Remark 2.2. *When $\ell = 0$, then the system (2.1.1) is said to be input-to-state stable (ISS), a notion originally introduced by Sontag [21].*

Definition 2.1.2. *The unforced system*

$$\begin{cases} \dot{x}(t) = f(t, x(t), 0) \\ x(t_0) = x_0 \end{cases} \tag{2.1.4}$$

is said to be GPUAS, if there exist functions $\sigma(.,.) \in \mathcal{KL}$ and positive scalar ℓ, such that for each initial condition x_0 at an initial time t_0, the solution $x(.)$ of the system (2.1.4) exists on $\mathbb{R}_+$ and satisfies

$$|x(t)| \leq \sigma(|x_0|, t - t_0) + \ell, \ \forall t \geq t_0. \tag{2.1.5}$$

We also recall that the system (2.1.4) is said to be 0-GPUAS if the origin is GPUAS.
In addition if

$$\sigma(|x_0|, t - t_0) = \lambda_1 |x_0| e^{-\lambda 2(t-t_0)}$$

with $\lambda_1 > 0$, $\lambda_2 > 0$ we obtain

$$|x(t)| \leq \lambda_1 |x_0| e^{-\lambda 2(t-t_0)} + \ell, \ \forall t \geq t_0 \tag{2.1.6}$$

then the unforced system (2.1.4) is said to be GPUES.
To build our results, we need the following concept of uniformly stable functions which are recalled from [24].

2.1.1 USF and Comparison Principles

Definition 2.1.3. *[24] function $\mu \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ is said to be a uniformly stable function (USF) if the following linear time-varying (LTV) system is GUAS*

$$\dot{y}(t) = \mu(t) y(t), \ t \in \mathbb{R}_+.$$

Hence, $\mu(t)$ is a (USF) if and only if there exist two constants $\epsilon > 0$ and $\delta \geq 0$ such that

$$\int_{t_0}^{t} \mu(s) ds \leq -\epsilon(t - t_0) + \delta, \ t, t_0 \in \mathbb{R}_+, \ t \geq t_0. \tag{2.1.7}$$

Next we introduce the concept of stable functions.
Consider the following scalar LTV system

$$\dot{y}(t) = \mu(t) y(t), \ t \in \mathbb{R}_+ \tag{2.1.8}$$

where $y(t) = \mathbb{R}_+ \to \mathbb{R}$ is the state variable and $\mu(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$. It is not hard to see that the state transition matrix for system (2.1.8) is given by

$$\phi(t, t_0) = \exp\Big(\int_{t_0}^{t} \mu(s)ds\Big), \ \forall t \geq t_0 \in \mathbb{R}_+. \tag{2.1.9}$$

Definition 2.1.4. *[24] The function $\mu(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ is said to be*

1) *Asymptotically stable if the scalar LTV system (2.1.8) is asymptotically stable.*

2) *Exponentially stable if the scalar LTV system (2.1.8) is exponentially stable, namely, there exist constants $k(t_0) > 0$ and $\alpha > 0$ such that*

$$|y(t)| \leq k(t_0)|y(t_0)|e^{-\alpha(t-t_0)}, \ \forall t \geq t_0 \in \mathbb{R}_+. \tag{2.1.10}$$

3) *Uniformly exponentially stable (or UAS) if the scalar LTV system (2.1.8) is uniformly exponentially stable, namely, the constant $k(t_0)$ in (2.1.10) is independent of t_0.*

In the above definition we have noticed that, for linear system, uniformly asymptotic stability and uniformly exponential stability are equivalent. By noting the transition matrix (2.1.9), we can immediately the following fact.

Lemma 3. *The scalar function $\mu(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ is*

a) *Asymptotically stable if and only if*

$$\lim_{t \longrightarrow \infty} \int_{t_0}^{t} \mu(s)ds = -\infty. \tag{2.1.11}$$

b) *Exponentially stable if and only if there exist $\delta(t_0) \geq 0$ and $\epsilon > 0$ such that*

$$\int_{t_0}^{t} \mu(s)ds \leq -\epsilon(t - t_0) + \delta(t_0), \ \forall t \geq t_0 \in \mathbb{R}_+. \tag{2.1.12}$$

c) *Uniformly exponentially stable if and only if (2.1.12) is satisfied where δ is independent of t_0.*

Of course, if $\mu(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ is a periodic function with period T, then it is easy to see that the three different stability concepts in definition (2.1.4) are equivalent, and moreover they are equivalent to the existence of $c > 0$ such that [24]

$$\int_{t_0}^{t+T} \mu(s)ds \leq -c. \tag{2.1.13}$$

Next, we introduce improved comparison principles for time varying systems. These lemmas in the above will play a critical role in establishing the stability theorems in the next section.

Lemma 4. *[9] **(Generalised Gronwall-Bellman Inequality)***
Assume that $\pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}), \mu \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and $y : \mathbb{R}_+ \to \mathbb{R}_+$ is such that

$$\dot{y}(t) \leq \mu(t)y(t) + \pi(t), \ t \in \mathbb{R}_+. \tag{2.1.14}$$

Then, for any $t \geq s \in \mathbb{R}_+$ the following inequality holds true

$$y(t) \leq y(s) \exp\Big(\int_s^t \mu(r)dr\Big) + \int_s^t \exp\Big(\int_\lambda^t \mu(r)dr\Big)\pi(\lambda)d\lambda. \tag{2.1.15}$$

*The proof of The lemma is included in **Annex** (2.4) for completeness.*

Lemma 5. *[26] Let $y : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that*

$$\dot{y}(t) \leq \mu(t)y(t) + \pi(t) \quad whenever \quad y(t) \leq \psi(t)$$

where $\mu \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ is a UAS function, $\pi(t) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and $\psi(.)$ is a measurable and locally essentially bounded function. Then, for any given $t, t_0 \in \mathbb{R}_+$ with $t \geq t_0$, there holds

$$y(t) \leq \max\big\{y(l(t)\phi(t, t_0), \eta|\psi|_[t_0, t]\big\} + \int_{t_0}^t \phi(t, s)\pi(s)ds. \tag{2.1.16}$$

2.2 Main results

In this section we try to extend the results of [14] and [25] to perturbed systems. The study of the stability of this kind of system is based on taking the Lyapunov function of the nominal system as being a Lyapunov function of the perturbed system [11], based on this approach we try to prove that the indefinite Lyapunov function for the nominal system is also an indefinite Lyapunov function for the perturbed time varying system.
Consider the nonlinear time-varying perturbed system

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) + g(t, x(t)) \\ x(t_0) = 0 \end{cases} \tag{2.2.1}$$

Where $x(t)$ is the state, the control input $u(t) : \mathbb{R}_+ \to \mathbb{R}^m$ which is assumed to be measurable and locally essentially bounded, and $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \ g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are piecewise continuous in t, locally Lipschitz in x, to be uniformly continuous in u, such that

$$f(t, 0, 0) = 0, \ g(t, 0) = 0, \ \forall t \geq t_0.$$

2.2.1 Asymptotic stability analysis of Perturbed system

Theorem 2.2.1. *Assume that there exists a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, functions $\alpha_1(.), \alpha_2(.) \in \mathcal{K}_\infty$, an asymptotically stable function. $\phi(t) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and a scalar function $\pi(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}_+)$ such that, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, we have for the nominal system*

$$\dot{x}(t) = f(t, x(t), u(t))$$

$$\alpha_1(|x|) \leq V(t,x(t)) \leq \alpha_2(|x|) \tag{2.2.2}$$

$$\dot{V}(t,x(t))|_{(2.1.1)\ \textit{where}\ u=0} \leq \phi(t)V(t,x(t)) + \pi(t) \tag{2.2.3}$$

where

$$\int_0^\infty \pi(\tau)d\tau < \infty,\ \forall t \geq t_0.$$

The perturbation term $g(t,x)$ satisfies that

$$||\frac{\partial V}{\partial x}g(t,x(t))|| \leq \gamma(t)V(t,x(t)) + \sigma(t)$$

Where $\gamma(t) \in \mathcal{C}(\mathbb{R}_+,\mathbb{R})$ is an asymptotically stable function, and $\sigma(t) \in \mathcal{PC}(\mathbb{R}_+,\mathbb{R})$, with

$$\int_0^\infty \sigma(\tau)d\tau < \infty,\ \forall t \geq t_0.$$

Then the nonlinear time varying system (2.2.1) with $u = 0$ is practically globally uniformly asymptotically stable.
In addition, if there exist three positive numbers $\epsilon_i, i = 1,2,3$ such that the following conditions are satisfied

$$\alpha_1(s) = \epsilon_1 s^{\epsilon_0}\ ,\ \alpha_2(s) = \epsilon_2 s^{\epsilon_0}.$$

Then the nonlinear time varying system (2.2.1) with $u = 0$ is said to be practically globally uniformly exponentially stable.

Proof.
The derivative of $V(t,x(t))$ along the trajectories of system (2.2.1) is given by

$$\begin{aligned}\dot{V}(t,x(t))|_{(2.2.1)\ \textit{where}\ u=0}(t,x) &\leq \phi(t)V(t,x(t)) + \gamma(t)V(t,x(t)) + \pi(t) + \sigma(t)\\ &\leq (\phi(t)+\gamma(t))V(t,x(t)) + \mu(t)\end{aligned} \tag{2.2.4}$$

with $\mu(t) = \pi(t) + \sigma(t)$.
Applying Lemma (4) on inequality (2.2.4) gives, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$

$$V(t,x(t)) \leq V(t_0,x(t_0))\exp\Big(\int_{t_0}^t(\phi(s)+\gamma(s))ds\Big)+\int_{t_0}^t \exp\Big(\int_s^t(\phi(\lambda)+\gamma(\lambda))d\lambda\Big)\mu(s)ds \tag{2.2.5}$$

By assumption there exist $\epsilon_i > 0$ and $\delta_1 \geq 0$, $i = 1,2$ such that

$$\int_{t_0}^t(\phi(s)+\gamma(s))ds \leq -(\epsilon_1+\epsilon_2)(t-t_0) + (\delta_1+\delta_2),\quad \forall t,t_0 \in \mathbb{R}_+,\ t \geq t_0$$

Then it follows from (2.2.5) that

$$\begin{aligned}V(t,x(t)) &\leq e^{\delta}e^{-\epsilon(t-t_0)}V(t_0,x(t_0)) + e^{\delta}\int_{t_0}^t \mu(s)ds\\ &\leq e^{\delta}e^{-\epsilon(t-t_0)}\alpha_2(||x_0||) + Ce^{\delta}.\end{aligned}$$

where $\epsilon = \epsilon_1+\epsilon_2$ and $\delta = \delta_1+\delta_2$ and $C = \int_0^\infty \mu(s)ds$.
from (2.2.2) we can obtain

$$\begin{aligned}|x(t)| &\leq \alpha_1^{-1}V(t,x_0)\\ &\leq \alpha_1^{-1}\big[e^{\delta}e^{-\epsilon(t-t_0)}\alpha_2(|x_0|) + Ce^{\delta}\big].\end{aligned} \tag{2.2.6}$$

Then from inequality (2.2.6) and the following lemma.

Lemma 6. [21] *Let α be a class $\mathcal{K}$ function. We have for all $a, b \geq 0$*

$$\alpha(a+b) \leq \alpha(2a) + \alpha(2b)$$

Then,

$$|x(t)| \leq \alpha_1^{-1}\Big(2e^{\delta}e^{-\epsilon(t-t_0)}\alpha_2(||x_0||)\Big) + \alpha_1^{-1}(2Ce^{\delta}). \tag{2.2.7}$$

Hence,by (2.2.7) we conclude that the perturbed system (2.2.1) with $u = 0$ is (PGUAS). In addition,

$$\alpha_1(s) = \epsilon_1 s^n \text{ and } \alpha_2(s) = \epsilon_2 s^n$$

We get from (2.2.7) that

$$|x(t)| \leq \sqrt[\eta]{\frac{\epsilon_2}{\epsilon_1}2e^{\delta}|x_0|e^{\frac{-\epsilon}{\eta}(t-t_0)}} + \frac{1}{\epsilon_1}(2Ce^{\delta})^{-\eta}, \quad \forall t, t_0 \in \mathbb{R}_+, \; t \geq t_0 \tag{2.2.8}$$

namely, the perturbed system (2.2.1) with $u = 0$ is (PGUAS).

Example 2.2.1. *Let us consider the following system*

$$\dot{x} = -x + xe^{-t} + \frac{x}{1+t^2} - \frac{1}{x(t^2 + sin(x))} \tag{2.2.9}$$

where

$$f(t, x(t)) = -x + xe^{-t} \quad , \quad g(t, x(t)) = \frac{x}{1+t^2} - \frac{1}{x(t^2 + sin(x))}$$

Let the Lyapunov function $V(t, x) = \frac{1}{2}x^2$ for the nominal system, the derivative of $V(t, x)$ along the trajectory of (2.2.9) is given by

$$\begin{aligned} \dot{V}(t, x) &= 2(-1 + e^{-t})V(t, x) + \frac{2V(t, x)}{1+t^2} - \frac{1}{t^2 + \sin(x)} \\ &\leq 2(-1 + e^{-t})V(t, x) + \frac{2V(t, x)}{1+t^2} - \frac{1}{t^2+1} \end{aligned} \tag{2.2.10}$$

where $\int_{t_0}^{t} 2(-1 + e^{-\tau})d\tau \leq 2(t - t_0) + 2$, $\int_{t_0}^{t} \frac{2}{1+\tau^2}d\tau \leq \pi$ and $\int_{t_0}^{t} -\frac{1}{\tau^2+1}d\tau \leq -\frac{\pi}{2}$, $\forall t \geq t_0$.

Then, the nonlinear system (2.2.9) is PUAGS.

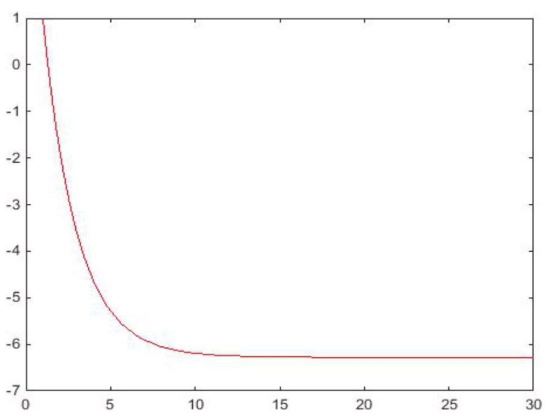

Figure 2.1: State trajectory of system (2.2.9) with $t_0 = 1, x(t_0) = 1$

2.2.2 ISS analysis of perturbed system

Now we will study the practical uniform input-to-state stable (PUISS) of the nonlinear time varying perturbed system (2.2.1) with an indefinite Lyapunov function. Stability of perturbed systems is investigated for instance in [11] practical stability in [2], [1], and PUISS in [7], [8] but with the classical Lyapunov function.

In the literature, the ISS property is frequently characterized by the ISS-Lypunov function [22] and has been studied for both differential equations [14] [22] and functional differential equations ([15], [16], [18] and [17]). As usual, the time-derivative of the ISS-Lyapunov function is required to be negative definite under some additional condition on the input signal u. In this subsection, we will show how to utilize the idea in the above subsection to deal with ISS stability analysis of nonlinear time varying perturbed systems by allowing indefinite time-derivatives for the ISS-Lyapunov functions.

Theorem 2.2.2. *Assume that there exists a continuously differentiable function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, *functions* $\alpha_1(.), \alpha_2(.) \in \mathcal{K}_\infty$, *a* $\mathcal{K}$*-function* ρ, *a uniformly exponentially stable function* $\phi(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ *and a scalar function* $\pi(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}_+)$ *such that, for all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, *we have for the nominal system.*

$$\dot{x}(t) = f(t, x(t), u(t))$$

$$\alpha_1(|x|) \leq V(t, x(t)) \leq \alpha_2(|x|) \tag{2.2.11}$$

$$\dot{V}|_{(2.1.1)}(t, x(t)) \leq \phi(t)V(t, x(t)) + \pi(t), \quad if \quad V(t, x(t)) \geq \rho(||u||_{[}t_0, t]). \tag{2.2.12}$$

where

$$\int_0^\infty \pi(\tau)d\tau < \infty, \ \forall t \geq t_0$$

and the perturbation term $g(t, x)$ *satisfies that*

$$||\frac{\partial V}{\partial x} g(t, x(t))|| \leq \gamma(t)V(t, x(t)) + \sigma(t)$$

where $\gamma(t) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ *is an asymptotically stable function, and* $\sigma(t) \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}_+)$, *with*

$$\int_0^\infty \sigma(\tau)d\tau < \infty, \ \forall t \geq t_0$$

if

$$\dot{V}(t, x(t))|_{(2.2.1)}(t, x) \leq (\phi(t) + \gamma(t))V(t, x(t)) + \mu(t), \ whenever\ V(t, x(t)) \geq \rho(||u(t)||) \tag{2.2.13}$$

Then the nonlinear time varying perturbed system (2.2.1) is practically uniformly input to state stable.

We use the idea in proof of Lemma 1 in [14], the idea in proof of Lemma (3) and Lemma (2.2.1).

Proof.

The derivative of $V(t,x(t))$ along the trajectories of system (2.2.1) is given by

$$\begin{aligned}\dot{V}|_{(2.2.1)}(t,x(t)) &\leq \phi(t)V(t,x(t)) + \gamma(t)V(t,x(t)) + \pi(t) + \sigma(t) + \rho(|u(t)|) \\ &\leq (\phi(t)+\gamma(t))V(t,x(t)) + \mu(t) + \rho(|u(t)|)\end{aligned} \tag{2.2.14}$$

with $\mu(t) = \pi(t) + \sigma(t)$.
Let us consider the inequality

$$V(s,x(s)) \geq \rho(|u(s)|) \tag{2.2.15}$$

and, for any $t, t_0 \in \mathbb{R}_+$ with $t \geq t_0$, we consider the following two cases:
Case(A): *Inequality (2.2.15) holds for all $s \in [t_0, t] \subset \mathbb{R}_+$.*
Case(B): *Inequality (2.2.15) does not holds true for almost all $s \in [t_0, t] \subset \mathbb{R}_+$.*

In Case (A)*, by the Lemma (4) we know that*

$$V(t,x(t)) \leq V(t_0,x(t_0)) \exp\Big(\int_{t_0}^{t}(\phi(\tau)+\gamma(\tau))d\tau\Big) + \int_{t_0}^{t} \exp\Big(\int_{\tau}^{t}(\phi(\lambda)+\gamma(\lambda))d\lambda\Big)\mu(\tau)d\tau \tag{2.2.16}$$

By assumption there exist $\epsilon_i > 0$ and $\delta_1 \geq 0$, $i = 1, 2$ such that

$$\int_{t_0}^{t}(\phi(s)+\gamma(s))ds \leq -(\epsilon_1+\epsilon_2)(t-t_0) + (\delta_1+\delta_2), \quad \forall t, t_0 \in \mathbb{R}_+,\ t \geq t_0$$

Then it follows from (2.2.16) that

$$\begin{aligned}V(t,x(t)) &\leq e^{\delta}e^{-\epsilon(t-t_0)}V(t_0,x(t_0)) + e^{\delta}\int_{t_0}^{t}\mu(\tau)d\tau \\ &\leq e^{\delta}e^{-\epsilon(t-t_0)}\alpha_2(||x_0||) + Me^{\delta}.\end{aligned} \tag{2.2.17}$$

where $\epsilon = \epsilon_1 + \epsilon_2$ and $\delta = \delta_1 + \delta_2$, $M = \int_0^{\infty}\mu(s)ds$.
In case (B)*, Let us consider the set*

$$\Big\{s \in [t_0,t] : V(s,x(s)) \leq \rho(|u(s)|)\Big\}$$

which is non-empty with positive measure. Denote

$$t^* = \sup\Big\{s \in [t_0,t] : V(s,x(s)) \leq \rho(|u(s)|)\Big\}$$

Then we have either $t^* < t$ or $t^* = t$.

- *If $t^* < t$, then for all $s \in [t^*, t]$*
  we have

  $$V(s,x(s)) \geq \rho(|u(s)|)$$

  which implies, by (2.2.13)

  $$\dot{V}(s,x(s))|_{(2.2.1)} \leq (\phi(s)+\gamma(s))V(s,x(s)) + \mu(s),\ \forall s \in [t^*, t]$$

Hence, by lemma (4) we obtain that

$$\begin{aligned} V(t,x(t)) &\leq V(t^*,x(t^*))\exp\Big(\int_{t^*}^{t}(\phi(\tau)+\gamma(\tau))d\tau\Big)+\int_{t^*}^{t}\exp\Big(\int_{\tau}^{t}(\phi(\lambda)+\gamma(\lambda))d\lambda\Big)\mu(\tau)d\tau \\ &= \rho(|u(t^*)|)\exp\Big(\int_{t^*}^{t}(\phi(\tau)+\gamma(\tau))d\tau\Big)+\int_{t^*}^{t}\exp\Big(\int_{\tau}^{t}(\phi(\lambda)+\gamma(\lambda))d\lambda\Big)\mu(\tau)d\tau \\ &\leq \rho(||u(s)||_{[t_0,t]})e^{\delta}+e^{\delta}\int_{t_0}^{t}\mu(s)ds \\ &\leq \rho(||u(s)||_{[t_0,t]})e^{\delta}+Me^{\delta}. \end{aligned} \tag{2.2.18}$$

Where we have noticed that $\mu(t)$ is non negative function.

- *If $t^* = t$, it follows from the definition of t^* that*

$$\begin{aligned} V(t,x(t)) &= V(t^*,x(t^*)) \\ &\leq \rho(|u(t^*)|) \\ &\leq \sup_{s\in[t_0,t]}\rho(|u(s)|) \\ &\leq \rho(||u(s)||_{[t_0,t]}). \end{aligned} \tag{2.2.19}$$

Hence we get from (2.2.17), (2.2.18) and (2.2.19)

$$V(t,x(t)) \leq \max\{e^{\delta}e^{-\epsilon(t-t_0)}\alpha_2(||x_0||),\rho(||u(s)||_{[t_0,t]})e^{\delta}\}+Me^{\delta}. \tag{2.2.20}$$

There from (2.2.11) we can obtain

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}V(t,x(t)) \\ &\leq \alpha_1^{-1}\big(\max\{e^{\delta}e^{-\epsilon(t-t_0)}\alpha_2(||x_0||),\rho(||u(s)||_{[t_0,t]})e^{\delta}\}+Me^{\delta}\big). \end{aligned} \tag{2.2.21}$$

Then from inequality (2.2.21) and by using lemma (6) we get

$$|x(t)| \leq \alpha_1^{-1}\big(2\max\{e^{\delta}e^{-\epsilon(t-t_0)}\alpha_2(||x_0||),\rho(||u(s)||_{[t_0,t]})e^{\delta}\}\big)+\alpha_1^{-1}\big(4Me^{\delta}\big). \tag{2.2.22}$$

Which shows that the time-varying perturbed system (2.2.1) is PUISS.

2.3 Numerical examples

In this section we present several numerical examples that show stability and PUISS for perturbed systems with indefinite Lyapunov functions.

Example 2.3.1. *Consider the following nonlinear perturbed time-varying system*

$$\dot{x}(t) = \Big(\frac{2}{1+t+x^2(t)}-t|\cos t|\Big)x(t)+\frac{2t\cos|t|}{1+x^2(t)}u-\frac{x}{t+\sin x(t)}+\frac{2}{x(t)(1+t^2)} \tag{2.3.1}$$

where

$$\begin{aligned} f(t,x(t),u(t)) &= \Big(\frac{2}{1+t+x^2(t)}-t|\cos t|\Big)+\frac{2t\cos|t|}{1+x^2(t)}u, \\ g(t,x(t)) &= -\frac{x}{t+\sin x(t)}+\frac{2}{x(t)(1+t^2)} \end{aligned}$$

Letting

$$V(t,x(t)) = \frac{1}{2}x^2(t)$$

Then we have

$$\begin{aligned}\dot{V}(t,x(t)) &= 2\Big(\frac{2}{1+t+x^2(t)} - t|\cos t|\Big)V(t,x(t)) + \frac{2tx\cos|t|}{1+x^2}u - \frac{2V(t,x(t))}{t+\sin x} + \frac{2}{(1+t^2)}\\ &\le 2\Big(\frac{2}{1+t} - t|\cos t|\Big)V(t,x(t)) + t|u|\cos|t| - \frac{2V(t,x(t))}{t+1} + \frac{2}{(1+t^2)}\\ &\le 2\Big(\frac{2}{1+t} - t|\cos t|\Big)V(t,x(t)) + t|u|\cos|t| + \frac{2}{(1+t^2)}.\end{aligned} \tag{2.3.2}$$

Which corresponds to (2.2.13) with $\phi(t)+\gamma(t) = \frac{2}{1+t} - 2t|\cos t|$ *and* $\mu(t) = \frac{2}{1+t^2}$ *according to the computation in [25], for all* $t, t_0 \in \mathbb{R}_+$ *with* $t \ge t_0$*, there holds*

$$\int_{t_0}^{t}(\phi(\tau)+\gamma(\tau))d\tau \le -\frac{4}{3\pi}(t-t_0) + 2\ln\Big(1+\frac{3}{2}\pi\Big) + 2$$

Which shows that $\mu(t)$ *is a UES function and*

$$\int_{t_0}^{t}\mu(\tau)d\tau \le \pi, \ \forall t \ge t_0 \in \mathbb{R}_+$$

$$|x(t)| \le e^{\ln(1+3\pi/2)+1}e^{-2(t-t_0)/3\pi}|x(t_0) + \sqrt{\pi}e^{(\ln(1+3\pi/2)+1)} \tag{2.3.3}$$

if

$$|u(t)| \le V(t,x(t))$$

We can obtain from inequality (2.3.2) that

$$\dot{V}(t,x(t)) \le (\phi(t)+\gamma(t))V(t,x(t)) + \mu(t).$$

In which by theorem (2.2.2), implies that the system (2.3.1)is PUISS.

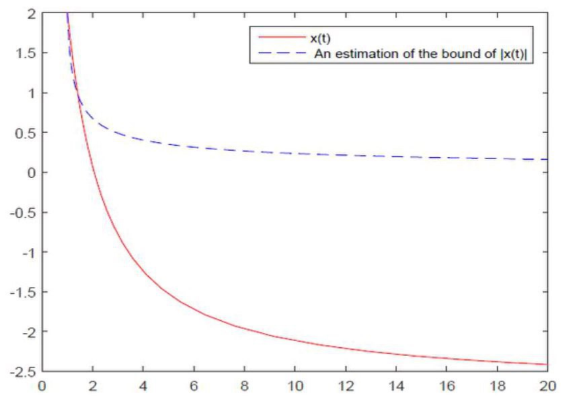

Figure 2.2: Simulation result of the PUISS of system (2.3.1) and the estimation in (2.3.3)

2.4 Conclusion

In this chapter, we study the stability analysis of nonlinear perturbed time-varying systems by using Lyapunov's second method. Differently from the classical Lyapunov approach, the proposed stability theorem does not require that the time-derivative of the Lyapunov function be negative definite, the ability to use an indefinite Lyapunov function makes it easier to find a suitable Lyapunov function. The stability analysis was proposed by using the properties of uniformly stable functions and the improved comparison lemma. The proposed results also improve the related existing results on the same topic by removing some restrictive conditions. Some sufficient conditions verifying globally uniformly asymptotic stability, globally uniformly exponential stability and practical uniform input-to-state stability have been established by employing the indefinite derivative Lyapunov-function.

Two numerical examples demonstrate their effectiveness.

ANNEX

Gronwall's Lemma

Gronwall-type lemmas play an the important role in the area of integral and differential equations, as technical tools used to prove existence and uniqueness of a solution and to obtain various estimates for the solutions. Let $u : [a, a+h] \longrightarrow \mathbb{R}$ be a continuous function satisfying the inequality. where a, b are non negative constants. Then $u(t) \leq ahe^{bh}$, for $t \in [\alpha, \alpha+h]$.
This result was proved by T. H. Gronwall in 1919, and is the prototype for the study of many integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the publications on this subject, the papers of Bellman [13] (1943) and Bihari [23](1956) are well known and have found wide application. Therefore integral inequalities of this type are usually associated with the names of Gronwall, Bellman and Bihari. Other names to be mentioned in connection with the further development of the theory of integral inequalities are: [R.P.Agaraval, N.B.Azbelev, D.D.Bainov, P.R.Beesack, S.G.Deo, U.D.Dhongade, V.Lakshmikantham, S.Leela, B.G.Pacpatte, J.Popenda, P.S.Simenov.]
We begin by giving one of the simple-set and most frequently used integral inequalities.

Theorem 2.4.1. ***(Bellman, 1943)***
Let $u(t)$ and $b(t)$ be non negative continuous functions for $t \geq \alpha$, and let

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)ds, \ t \geq \alpha \tag{2.4.1}$$

where $a \geq 0$ is a constant. Then

$$u(t) \leq a \exp\left(\int_{\alpha}^{t} b(s)ds\right), \ t \geq \alpha. \tag{2.4.2}$$

This lemma is valuable in establishing the existence and uniqueness of solutions to certain integral and differential equations, as well as in deriving estimates and bounds for these solutions. It's widely applied in various branches of mathematics, including analysis, differential equations and mathematical modeling.

Corollary 2.4.1.
Let $u(t)$ and $b(t)$ be non negative continuous functions for $t \geq \alpha$, and let

$$u(t) \leq ae^{-\gamma(t-\alpha)} + \int_{\alpha}^{t} e^{-\gamma(t-s)} b(s)u(s)ds, \ t \geq \alpha \tag{2.4.3}$$

where $a \geq 0$ and γ are constants.
Then

$$u(t) \leq a \exp(-\gamma(t-\alpha)) + \int_{\alpha}^{t} b(s)ds, \ t \geq \alpha. \tag{2.4.4}$$

Lemma 7. *Let $b(t)$ and $f(t)$ be continuous functions for $t \geq \alpha$, let $v(t)$ be a differentiable function for $t \geq \alpha$, and suppose*

$$\dot{v}(t) \leq b(t)v(t) + f(t), v(\alpha) \leq v_0, \ t \geq \alpha \tag{2.4.5}$$

Then, for $t \geq \alpha$

$$v(t) \leq \exp(\int_\alpha^t b(s)ds) + \int_\alpha^t f(s)\exp(\int_s^t b(\tau)d\tau)ds. \tag{2.4.6}$$

Remark 2.3. *Note that the right hand side of (2.4.6) coincides with the unique solution of the equation*

$$\dot{v}(t) = b(t)v(t) + f(t),\ t \geq \alpha \tag{2.4.7}$$

for which

$$v(\alpha) = v_0. \tag{2.4.8}$$

Theorem 2.4.2. *Let $a(t)$, $b(t)$ and $u(t)$ be continuous functions in $\mathbb{J} = [\alpha, \beta]$, and let $b(t)$ be non negative in $\mathbb{J}$. Suppose*

$$u(t) \leq a(t) + \int_\alpha^t b(s)u(s)ds,\ t \in \mathbb{J}$$

Then

$$u(t) \leq a(t) + \int_\alpha^t a(s)b(s)\exp(\int_s^t b(\tau)d\tau)ds,\ t \in \mathbb{J}. \tag{2.4.9}$$

Corollary 2.4.2.

Let, under the conditions of theorem (2.4.2), the functions $a(t)$ also be non decreasing in $\mathbb{J}$. Then

$$u(t) \leq a(t)\exp(\int_\alpha^t a(s)b(s))ds,\ t \in \mathbb{J}.$$

Corollary 2.4.3.

Let $b(t)$ and $u(t)$ be continuous functions in $\mathbb{J} = [\alpha, \beta]$, let $b(t)$ be non negative in $\mathbb{J}$. Suppose

$$u(t) \leq a + \int_\alpha^t b(s)u(s)ds,\ t \in \mathbb{J}$$

where a is a constant.
Then

$$u(t) \leq a\exp(\int_\alpha^t b(s)d(s)),\ t \in \mathbb{J}.$$

Remark 2.4. *The conclusion of corollary (2.4.3) shows that in theorem (2.4.1) and corollary (2.4.1) we may neglect the requirement that $u(t)$ and a be non-negative.*

Corollary 2.4.4.

Let $u(t)$ be a continuous function in $\mathbb{J} = [\alpha, \beta]$, and suppose

$$u(t) \leq a + \int_\alpha^t bu(s)ds,\ t \in \mathbb{J}$$

where $b \geq 0$ and a are constants.
Then

$$u(t) \leq ae^{v(t-\alpha)},\ t \in \mathbb{J}. \tag{2.4.10}$$

Theorem 2.4.3. ***(Chandirov, 1958)***
Let $a(t)$, $b(t)$, $c(t)$ and $u(t)$ be continuous functions in $\mathbb{J} = [\alpha, \beta]$, let $b(t)$ and $c(t)$ be non-negative in $\mathbb{J}$, and suppose

$$u(t) \leq a(t) + \int_{\alpha}^{t} \Big[b(s)u(s) + c(s)) \Big] ds, \ t \in \mathbb{J} \tag{2.4.11}$$

Then

$$u(t) \leq \Big[\sup_{s \in [\alpha, t]} a(s) + \int_{\alpha}^{t} c(s)ds) \Big] \exp(\int_{\alpha}^{t} b(s)ds), \ t \in \mathbb{J}. \tag{2.4.12}$$

Theorem 2.4.4. ***(Chandirov, 1970)***
Let $a(t)$, $b(t)$, $c(t)$ and $u(t)$ be continuous functions in $\mathbb{J} = [\alpha, \beta]$, let $b(t)$ be non negative in $\mathbb{J}$, and suppose

$$u(t) \leq a(t) + \int_{\alpha}^{t} \Big[b(s)u(s) + c(s) \Big] ds, \ t \in \mathbb{J}$$

Then

$$u(t) \leq a(t) + \int_{\alpha}^{t} \Big[a(s)b(s) + c(s) \Big] \exp(\int_{s}^{t} b(\tau)d\tau)ds, \ t \in \mathbb{J}. \tag{2.4.13}$$

Corollary 2.4.5.
If in theorem (2.4.4) the function $a(t)$ is non-decreasing in $\mathbb{J}$, then

$$u(t) \leq a(t) \exp(\int_{\alpha}^{t} b(\tau)d\tau) + \int_{\alpha}^{t} c(s) \exp(\int_{s}^{t} b(\tau)d\tau)ds, \ t \in \mathbb{J}.$$

Corollary 2.4.6.
If in theorem (2.4.4) $a(t) = a$, then

$$u(t) \leq a \exp(\int_{\alpha}^{t} b(\tau)d\tau) + \int_{\alpha}^{t} c(s) \exp(\int_{s}^{t} b(\tau)d\tau)ds, \ t \in \mathbb{J}.$$

Corollary 2.4.7. ***(Quade, 1942, Zadiraka, 1968)***
Let $u(t)$ be a continuous functions for $t \geq \alpha$ and suppose

$$u(t) \leq ae^{-\gamma(t-\alpha)} + \int_{\alpha}^{t} e^{-\gamma(t-s)}[bu(s) + c]ds, \ t \geq \alpha. \tag{2.4.14}$$

Where $b \geq 0$, a, c, and $\gamma \neq b$ are constants. Then

$$u(t) \leq ae^{(b-\gamma)(t-\alpha)} + \frac{c}{\gamma - b}[1 - e^{(b-\gamma)(t-\alpha)}], \ t \geq \alpha.$$

Some generalizations of the gronwall's inequalities

In 1919, Gronwall [19] proved a remarkable inequality which has attracted and continues to attract considerable attention in literature.

Theorem 2.4.5. *Let $u(t)$, $a(t)$ and $b(t)$ be real continuous functions defined in $[\alpha, \beta)$, such that $b(t) \geq 0$ and $a(t)$ is differentiable on $[\alpha, \beta)$. We suppose that on $[\alpha, \beta)$ we have the inequality*

$$u(t) \leq a(t) + \int_{\alpha}^{t} b(s)u(s)ds$$

then

$$u(t) \leq a(\alpha) \exp \Big(\int_{\alpha}^{t} b(s)ds \Big) + \int_{\alpha}^{t} \dot{a}(s) \exp \Big(\int_{\alpha}^{t} b(\sigma)d\sigma \Big) ds, \ \forall t \in [\alpha, \beta).$$

Theorem 2.4.6. ***(Gollwitzer, 1969)***
Let $u(t), f(t), g(t)$ and $h(t)$ be non negative continuous functions defined on $[\alpha, \beta)$ and

$$u(t) \leq f(t) + g(t) \int_{\alpha}^{t} h(s)u(s)ds, \ \forall t \in [\alpha, \beta) \tag{2.4.15}$$

Then

$$u(t) \leq f(t) + g(t) \int_{\alpha}^{t} h(s)f(s) \exp\Big(\int_{s}^{t} h(\tau)d\tau\Big)ds, \ \forall t \in [\alpha, \beta). \tag{2.4.16}$$

Theorem 2.4.7. *Let $u(t), p(t), q(t), f(t)$ and $g(t)$ be non negative continuous functions defined on $[\alpha, \beta]$ and*

$$u(t) \leq p(t) + q(t) \int_{\alpha}^{t} \Big[f(s)u(s) + g(s)\Big]ds, \ \forall t \in [\alpha, \beta) \tag{2.4.17}$$

Then

$$u(t) \leq p(t) + q(t) \int_{\alpha}^{t} \Big[f(s)p(s) + g(s)\Big] \exp\Big(\int_{s}^{t} f(\tau)q(\tau)d\tau\Big)ds, \ \forall t \in [\alpha, \beta). \tag{2.4.18}$$

Theorem 2.4.8. *[3] Let $A, B, C : [\alpha, \beta] \longrightarrow \mathbb{R}_+$, $L, M : \mathbb{J} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be continuous function and*

$$0 \leq L(t, u) - L(t, v) \leq M(t, v)(u - v), t \in \mathbb{J}, 0 \leq v \leq u. \tag{2.4.19}$$

Then for every non-negative continuous function $x : \mathbb{J} \longrightarrow [0, +\infty)$ satisfying the inequality

$$x(t) \leq A(t) + B(t) \int_{\alpha}^{t} C(s)L(s, x(s))ds, \ \forall t \in \mathbb{J} \tag{2.4.20}$$

We have the estimation

$$x(t) \leq A(t) + B(t) \int_{\alpha}^{t} C(u)L(u, A(u)) \exp\Big(\int_{u}^{t} M(s, A(s))B(s)C(s)ds\Big)du. \tag{2.4.21}$$

Theorem 2.4.9. *Let $u(t), v(t), \omega(t)$ be non-negative continuous functions for $t \geq 0$ and suppose*

$$u(t) \leq c + \int_{0}^{t} \big(u(s)v(s) + \omega(s)\big)ds \tag{2.4.22}$$

where c is a positive constant.
Then

$$u(t) \leq e^{\int_0^t v(s)ds}\Big[c + \rho(e^{\int_0^t \frac{\omega(s)}{\rho}ds} - 1)\Big], \forall t \geq 0, \ \forall \rho > 0. \tag{2.4.23}$$

Theorem 2.4.10. *Let $u(t), v(t), \omega(t)$ be non-negative continuous functions for $t \geq 0$, let $c(t)$ be a non-negative differentiable function for $t \geq 0$ and suppose*

$$u(t) \leq c(t) + \int_{0}^{t} \big(u(s)v(s) + \omega(s)\big)ds \tag{2.4.24}$$

Then

$$u(t) \leq e^{\int_0^t v(s)ds}\Big[c(t) + \rho(e^{\int_0^t \frac{\omega(s)}{\rho}ds} - 1)\Big], \ \forall t \geq 0, \ \forall \rho > 0. \tag{2.4.25}$$

Application

The problem of determining the behavior of the solutions of a perturbed differential equation with respect to the solutions of the unperturbed differential equation is studied. The general differential equation considered is

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

and the associated perturbed differential equation is

$$\begin{cases} \dot{y} = f(t, y) + g(t, y) \\ y(t_0) = y_0 \end{cases}$$

The approach used is to examine the difference between the respective solutions $x(t, t_0, x_0)$ and $y(t, t_0, y_0)$ of these two differential equations. The principal mathematical technique employed is Gronwall inequalities.

Example 2.4.1. *Consider the unperturbed differential equation*

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{2.4.26}$$

with (t, x) satisfies a generalized Lipschitz condition

$$||f(t, x_1) - f(t, x_2)|| \leq L(t)||x_1 - x_2|| \tag{2.4.27}$$

where $L(t)$ is integrable on $[t_0, \infty)$.
Solution of (2.4.26) is given by

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Suppose that the trivial solution $x = 0$ of equation (2.4.1) is globally exponentially stable. Further, consider the associated perturbed equation

$$\begin{cases} \dot{y} = f(t, y) + g(t, y) \\ y(t_0) = y_0 \end{cases} \tag{2.4.28}$$

Suppose that the perturbation $g(t, y)$ satisfies

$$||g(t, y)|| \leq \alpha||\psi(t)||, \ t \geq t_0. \tag{2.4.29}$$

Where α is a sufficiently small positive constant and $\psi(t)$ is integrable on $[t_0, \infty)$. Solution of (2.4.28) is given by

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) + g(s, y(s))ds$$

As a consequence

$$\begin{aligned} ||\ y(t) - x(t)\ || &\leq ||y_0 - x_0|| + \int_{t_0}^{t} ||f(s, y) - f(s, x)||ds + \int_{t_0}^{t} ||g(s, y)||ds \\ &\leq ||y_0 - x_0|| + \int_{t_0}^{t} L(s)||(y(s) - x(s))||ds + \alpha \int_{t_0}^{t} ||\psi(s)||ds \\ &\leq ||y_0 - x_0|| + \alpha \int_{t_0}^{\infty} ||\psi(s)||ds + \int_{t_0}^{t} L(s)||(y(s) - x(s))||ds \\ &= C + \int_{t_0}^{t} L(s)||(y(s) - x(s))||ds. \end{aligned}$$

we observe, the quantity C can be made arbitrarily small, by choosing y_0 sufficiently close to x_0.
We now apply Gronwall's Inequality, we obtain

$$\begin{aligned} \| \, y(t) - x(t) \, \| &\leq C + C\int_{t_0}^{t} L(s)\exp\big[\int_{t_0}^{t} L(u)du\big]ds \\ &= C\Big\{1 + \int_{t_0}^{t} L(s)\exp\big[k(t) - k(s)\big]ds\Big\} \\ &= C\Big\{1 + e^{k(t)}\int_{t_0}^{t} L(s)e^{-k(s)}ds\Big\} \\ &= C\Big[1 - e^{k(t)}\big(e^{-k(t)} - e^{-k(t_0)}\big)\Big] \\ &= Ce^{k(t)-k(t_0)} \\ &\leq C\exp\Big[\int_{t_0}^{t} L(s)ds\Big] \\ &= CA. \end{aligned}$$

Where $k(t)$ represents an indefinite integral of $L(t)$.
By choosing y_0 sufficiently close to x_0, we have

$$||y(t) - x(t)|| \leq \lambda, \ t \geq t_0$$

where λ is a positive constant.
Since the trivial solution $x = 0$ of (2.4.26) is globally exponentially stable then, there exist $\lambda_1, \lambda_2 > 0$ such that

$$||x(t)|| \leq \lambda_1 ||x_0|| e^{-\lambda_2(t-t_0)}, \ t \geq t_0 \geq 0 \tag{2.4.30}$$

Which implies that

$$\begin{aligned} ||y(t)|| &= ||y(t) \pm x(t)|| \\ &\leq ||y(t) - x(t)|| + ||x(t)|| \\ &\leq \lambda_1 ||x_0 \pm y_0|| e^{-\lambda_2(t-t_0)} + \lambda \\ &\leq \lambda_1 ||y_0|| e^{-\lambda_2(t-t_0)} + \lambda_1 ||x_0 - y_0|| e^{-\lambda_2(t-t_0)} + \lambda. \end{aligned}$$

$e^{-\lambda_2(t-t_0)} \longrightarrow_{t\to+\infty} 0 \Longrightarrow \exists\alpha$, *a positive constant such that* $|e^{-\lambda_2(t-t_0)}| \leq \alpha$.
So we obtain that

$$||y(t)|| \leq \lambda_1 ||y_0|| e^{-\lambda_2(t-t_0)} + \lambda_1 C\alpha + \lambda.$$

This implies that system (2.4.28) is globally practically uniformly exponentially stable with $r = \lambda_1 C\alpha + \lambda$.

Example 2.4.2.
Suppose that all assumptions of example (2.4.1) are satisfied and the perturbation $g(t, y)$ satisfies

$$||g(t, y)|| \leq \varphi(t), \ t \geq t_0 \tag{2.4.31}$$

Where

$$\int_{t_0}^{\infty} \varphi(t)dt < \infty, \ t \geq t_0$$

As a consequence

$$\| y(t) - x(t) \| \leq \|y_0 - x_0\| + \int_{t_0}^{t} \|f(s,y) - f(s,x)\| ds + \int_{t_0}^{t} \|g(s,y)\| ds.$$

Under the conditions (2.4.27)and(2.4.31) it follows that for any $t \geq t_0$

$$\| y(t) - x(t) \| \leq \|y_0 - x_0\| + \int_{t_0}^{t} L(s)\|y(s) - x(s)\| ds + \int_{t_0}^{t} \|\varphi(s)\| ds$$

By theorem (2.4.10), this implies that

$$\begin{aligned} \| y(t) - x(t) \| &\leq e^{\int_{t_0}^{t} L(s)ds}(\|y_0 - x_0\| + \rho(e^{\int_{t_0}^{t} \frac{\varphi(s)}{\rho} ds} - 1)) \\ &\leq e^{\displaystyle\int_{t_0}^{\infty} L(s)ds}(\|y_0 - x_0\| + \rho(e^{\displaystyle\int_{t_0}^{\infty} \frac{\varphi(s)}{\rho} ds} - 1)) \\ &\leq C(\|y_0 - x_0\| + \rho(e^{\displaystyle\frac{M_\varphi}{\rho}} - 1)). \end{aligned}$$

Where $C = e^{\int_{t_0}^{\infty} L(s)} ds$ *and* $M_\varphi = \int_{t_0}^{\infty} \varphi(s) ds.$

Since the trivial solution $x = 0$ *of (2.4.26) is globally exponentially stable. Therefore, we obtain*

$$\begin{aligned} \|y(t)\| &= \|y(t) \pm x(t)\| \\ &\leq \|y(t) - x(t)\| + \|x(t)\| \\ &\leq C(\|y_0 - x_0\| + \rho(e^{\frac{M_\varphi}{\rho}} - 1)) + \lambda_1 e^{-\lambda_2(t-t_0)}\|x_0 \pm y_0\| \\ &\leq \lambda_1 e^{-\lambda_2(t-t_0)}\|y_0\| + \lambda_1 e^{-\lambda_2(t-t_0)}\|x_0 - y_0\| + C(\|y_0 - x_0\| + \rho(e^{\frac{M_\varphi}{\rho}} - 1)). \end{aligned}$$

In the case where $y_0 = x_0$, *we obtain the following estimation on the trajectories of system (2.4.28)*

$$\|y(t)\| \leq \lambda_1 e^{-\lambda_2(t-t_0)}\|y_0\| + C\rho e^{\frac{M_\varphi}{\rho}}.$$

This implies that system (2.4.2) is globally practically uniformly exponentially stable with $r = C\rho e^{\displaystyle\frac{M_\varphi}{\rho}}$.

Barbalat's Lemma

Barbalat's Lemma is very simple and thus, very attractive. Furthermore, under some conditions, it allowed to finally show that the function $\dot{V}(t)$ ultimately vanishes and in many cases even it allowed reaching the desirable asymptotic stability or asymptotically perfect tracking conclusion. Nevertheless, it also leaves the burdensome impression that any input command, distortion or disturbance that may affect the uniform continuity of Lyapunov derivative may also affect the proof and thus, the very guarantee of stability of nonlinear systems.
However, as we show below, it is only because Barbalat's lemma deals with the functions and not with the systems that it imposes those strict conditions on continuity of functions and even of their derivative. These conditions may happen to hold in various systems, yet if they are not satisfied under less than ideal conditions, it is not necessarily a result of some lack of stability. In any case, most publications work very hard to guarantee uniform continuity of practically all signals involved and end claiming that according to Barbalat's Lemma, the system ends with $\dot{V}(t) = 0$.
However, this claim is not necessarily true, as it seems to ignore that Barbalat's Lemma requires prior knowledge that the Lyapunov function $V(t)$ itself has a finite limit. As a mathematical result, the lemma is perfectly correct, yet as a stability tool, why would the Lyapunov function reach a finite limit unless its derivative tends to zero, but then why would the derivative tend to zero unless the Lyapunov function reaches a finite limit?

Barbalat's Lemma

- ⊛ For autonomous systems, invariant set theorems are power tools to study asymptotic stability when $\dot{V}$ is negative semi-definite.
- ⊛ The invariant set theorem is not valid for non-autonomous systems.
- ⊛ Hence, asymptotic stability of non-autonomous systems is generally more difficult than that of autonomous systems.
- ⊛ An important result that remedy the situation, Barbalat's Lemma.
- ⊛ Asymptotic properties of functions and their derivatives.

For differentiable function f of time t, always keep in mind the following three facts

i) $\dot{f} \longrightarrow 0 \nRightarrow f$ converges

The fact that $\dot{f} \to 0$ does not imply $f(t)$ has a limit as $t \longrightarrow \infty$.

Example 2.4.3.

$$f(t) = \sin(\ln t) \rightsquigarrow \dot{f} = \frac{\cos(\ln t)}{t} \longrightarrow 0 \ \textit{as}\ t \longrightarrow \infty.$$

ii) f converges $\not\Rightarrow \dot{f} \longrightarrow 0$

The fact $f(t)$ has a finite limit at $t \longrightarrow \infty$ does not imply that $\dot{f} \longrightarrow 0$.

Example 2.4.4.

$$f(t) = e^{-t}\sin(e^{2t}) \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

wile its derivative

$$\dot{f} = -e^{-t}\sin(e^{2t}) + 2e^{t}\cos(e^{2t})$$

is unbounded.

iii) If f is lower bounded and decreasing ($\dot{f} \geq 0$), then it converges to a limit. However, it does not say whether the slope of the curve will diminish or not.

Lemma 8. ***(Barbalat's Lemma)***
If the differentiable function has a finite limit as $t \longrightarrow \infty$, and if $\dot{f}$ is uniformly continuity (or $\ddot{f}$ is bounded), then $\dot{f} \longrightarrow 0$ as $t \longrightarrow \infty$.

proved by contradiction

- *A function $g(t)$ is continuous on $[0, \infty)$ if*

$$\forall t_1 \geq 0,\ \forall R > 0,\ \exists \eta(R, t_1) > 0,\ \forall t \geq 0, |t - t_1| < \eta \Longrightarrow |g(t) - g(t_1)| < R.$$

- *A function $g(t)$ is uniformly continuous on $[0, \infty)$ if*

$$\forall R > 0,\ \exists \eta(R) > 0,\ \forall t_1 \geq 0,\ \forall t \geq 0, |t - t_1| < \eta \Longrightarrow |g(t) - g(t_1)| < R.$$

  i.e. *An η can be found independent of specific point t_1.*

- *An immediate and practical corollary of Barbalat's lemma, If the differentiable function $f(t)$ has a finite limit as $t \longrightarrow \infty$ and $\dot{f}$ exists and is bounded, then $\dot{f} \longrightarrow 0$ as $t \longrightarrow \infty$.*

Corollary 2.4.8. ***Lyapunov-Like Lemma***
If a scaler function $V(t, x)$ satisfies the following conditions
1. $V(t, x)$ is lower bounded.
2. $\dot{V}(t, x)$ is negative semi-definite.
3. $\dot{V}(t, x)$ is uniformly continuous in time.
Then, $\dot{V}(t, x) \longrightarrow 0$ as $t \longrightarrow \infty$.

Lemma

Lemma 9. *Let $Y : [0, +\infty[\longrightarrow \mathbb{R}_+$ be a continuously differentiable function such that*

$$\dot{Y}(t) \leq -(a - b\gamma(t))Y(t) + c\delta(t).$$

For all $t \geq 0$, where a, b, c are positive constants, $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\delta : \mathbb{R}_+ \longrightarrow \mathbb{R}$, is continuous non negative functions which satisfied

$$\begin{aligned} \int_0^{+\infty} \gamma(s)ds &\leq M_\gamma &\leq +\infty \\ \int_0^{+\infty} \delta^2(s)ds &\leq M_\delta &\leq +\infty \\ \gamma(t) &< \frac{a}{b}. \end{aligned}$$

Then, $\forall t \geq t_0 \geq 0$

$$Y(t) \leq Y(t_0)e^{bM_\gamma}e^{-a(t-t_0)} + ce^{bM_\gamma}\sqrt{\frac{M_\delta}{2a}}.$$

Proof.
Let

$$\alpha(t) = a - b\gamma(t)$$

and

$$Z(t) = Y(t)e^{\int_{t_0}^{t} \alpha(s)ds}$$

It follows that

$$\begin{aligned} \dot{Z}(t) &= (\dot{Y}(t) + \alpha(t)Y(t))e^{\int_{t_0}^{t} \alpha(s)ds} \\ &\leq c\delta(t)e^{\int_{t_0}^{t} \alpha(s)ds}. \end{aligned}$$

Integrating between t_0 and t, one obtains $\forall t \geq t_0$

$$Z(t) \leq Z(t_0) + c\int_{t_0}^{t} \delta(s)e^{\int_{t_0}^{s} \alpha(\tau)d\tau} ds$$

Then

$$Y(t) \leq Y(t_0)e^{-\int_{t_0}^{t} \alpha(s)ds} + c\Big(\int_{t_0}^{t} \delta(s)e^{\int_{t_0}^{s} \alpha(\tau)d(\tau)} ds\Big)$$

One has

$$\int_{t_0}^{t} \alpha(s)ds = a(t - t_0) - b\int_{t_0}^{t} \gamma(s)ds$$

Which implies that

$$e^{-\int_{t_0}^{t} \alpha(s)ds} \leq e^{bM_\gamma e^{-a(t-t_0)}}.$$

By using the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\int_{t_0}^{t}\delta(s)e^{\int_{t_0}^{s}\alpha(\tau)d(\tau)}ds &= \int_{t_0}^{t}\delta(s)e^{a(s-t_0)}e^{-b\int_{t_0}^{s}\gamma(\tau)d\tau}ds\\ &\leq \left(\int_{t_0}^{t}\delta^2(s)ds\right)^{\frac{1}{2}}\left(\int_{t_0}^{t}e^{2a(s-t_0)}ds\right)^{\frac{1}{2}}\\ &\leq \sqrt{\frac{M_\delta}{2a}}e^{a(t-t_0)}\end{aligned}$$

Thus, we obtain

$$Y(t)\leq Y(t_0)e^{bM_\gamma}e^{-a(t-t_0)+c\sqrt{\frac{M_\delta}{2a}}e^{bM_\gamma}}.$$

Proof lemma

Proof. *(4)*
We write (2.1.14) as

$$\dot{y}(t)-\mu(t)y(t)\leq\pi(t)\ \ t\in\mathbb{R}_+$$

by using which we can obtain

$$\frac{d}{d\lambda}\Big(y(\lambda)\exp\Big(-\int_s^{\lambda}\mu(\omega)d\omega\Big)\Big)$$
$$\leq\pi(\lambda)\exp\Big(-\int_s^{\lambda}\mu(\omega)d\omega\Big),\quad\lambda\geq s\in\mathbb{R}_+.$$

from which it follows that, for all $t\geq s\in\mathbb{R}_+$

$$y(t)\exp\Big(-\int_s^{t}\mu(\omega)d\omega\Big)-y(s)$$
$$\leq\int_s^t\pi(\lambda)\exp\Big(-\int_s^{\lambda}\mu(\omega)d\omega\Big)d\lambda.$$

As $\exp(-\int_s^t\mu(\omega)d\omega)>0$*, the above inequality can be simplified as (2.1.15).*

GENERAL CONCLUSION

The work presented in this memoir concerning the study of the stability analysis of dynamic systems by using the Lyapunov's theory. It is now recognized that this theory in its generalized form is perhaps the most powerful device for the analysis of nonlinear systems, its power being derived from its simplicity, and generality. We use a new Lyapunov function to obtain a global uniform asymptotically stabile of some perturbed systems. we investigate the practical asymptotic and exponential stability of time-varying nonlinear systems. We derive some sufficient conditions that guarante the practical stability of perturbed systems.

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