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DEDICATION

I dedicate this work:

To my mother who helped me a lot to finish my studies, and who encouraged me so much morally and psychologically, and who sacrificed herself for me. To the soul of my father, To my very dear brothers, my sisters, and the whole Zabour family. To all the teachers of the Mathematics department. To all my friends of the mathematics promotion without specifying their names.

∗ **To all of you a big thank you** ∗

⋆ *Z.NASSIRA* ⋆

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⋆ *T.HAYET* ⋆

ABSTRACT

This work investigates the existence and uniqueness of solutions for two distinct classes of fuzzy differential equations using Tarski's fixed point theorem. Firstly, we analyze fuzzy differential equations with boundary conditions, focusing on how Tarski's theorem can establish the existence of solutions that satisfy specified boundary constraints. Secondly, we delve into Caputo-type implicit fractional fuzzy differential equations, applying the theorem to demonstrate the existence of unique solutions within this specialized framework. By leveraging Tarski's fixed point theorem, we establish theoretical foundations that ensure the robustness and reliability of solutions in both problem classes. To illustrate our findings, we present a practical example wherein these methodologies are applied to a realworld scenario, showcasing the effectiveness and applicability of Tarski's theorem in solving complex fuzzy differential equations. This example not only validates our theoretical results but also underscores the practical utility of employing fixed point theorems in mathematical modeling under uncertainty.

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NOTATIONS

Notations

$\mathbb{R}$	:	The set of real numbers.
$\mathbb{R}^+$	:	The set of all non-negative real numbers.
$\lvert . \rvert$	:	Absolute value of a real number.
$\lvert\lvert . \rvert\rvert$	:	Euclidean norm.
$\mathbb{N}$	:	The set of natural numbers.
E	:	The space of all fuzzy numbers.
$K_c(\mathbb{R})$	:	$\{A \subseteq \mathbb{R} : A$ nonempty, compact and convex$\}$.
L^1	:	Lebesgue space (integrable functions).
L^p	:	The Banach space of all measurable functions with $\lvert\lvert f \rvert\rvert_{L_p} < \infty$.
L^∞	:	$\{f, \quad f$ mesurable , $\exists\ c > 0$ tq $\lvert f \rvert \leq c\ p.p\}$.
$\lvert\lvert f \rvert\rvert_{L_p}$	:	$(\int \lvert f \rvert^p dx)^{\frac{1}{p}} \quad \forall \quad p \in [1, +\infty[$.
$\lvert\lvert f \rvert\rvert_{L_\infty}$	:	$\sup\{c\ /\ \lvert f \rvert \leq c,\ p.p\}$.
$\chi(A)$	:	The Hausdorff measure of nonempty bounded set A.
d_H	:	The Hausdorff metric(distance).
$B_\varepsilon(A)$	:	$\{x : d(x, A), < \varepsilon\}$.
d	:	the metric in E.
$[u]^a = [u_{al}, u_{ar}], a \in [0, 1]$	:	the level sets of u.
$d([u]^a)$	:	the diameter of the a-level set of u.

O	:	The fuzzy zero.
$\ominus$ or $(-_H)$	:	Hukuhara difference.
$C(J, E)$	:	The space of all continuous fuzzy functions on J.
$C^1(J, E)$	:	The space of all continuous and continuously differentiabl fuzzy functions on J.
$\ominus_{GH}$	:	Generalized Hukuhara differentiable.
GH	:	Generalized Hukuhara.
$I_{a^+}^{\alpha}$	:	The Riemann-Liouville fractional integral of order α.
$\Gamma(\alpha)$	:	Gamma function.
${}^{RL_f}D_{a^+}^{\alpha}$	:	The Riemann-Liouville -fuzzy-type GH-fractional derivative of order α.
BVP	:	Boundary Value Problem.
${}^{C_f}D_{a^+}^{\alpha}$	:	The Caputo-fuzzy-type fractional derivative of order α.
$BC([a, b], E)$	:	the space of all bounded and continuous functions from $[a, b]$ to E.

GENERAL INTRODUCTION

In many predictive scenarios, uncertainties arise due to imprecise physical measurements or incomplete data. Fuzzy modeling offers an approach to study such models by capturing inherent uncertainties in processes. Models like fuzzy differential equations are particularly suited for analyzing processes under uncertainty, finding applications in economics, engineering, and information sciences. On another note, periodic phenomena play a crucial role in studying various real-world processes. Different viewpoints exist regarding the concept of differentiable fuzzy-valued functions. The Hukuhara differentiability approach is widely adopted but imposes limitations, such as level sets whose diameters must be non-decreasing with respect to time. This limitation complicates the study of periodic phenomena using fuzzy differential models. Some researchers have explored almost periodic functions involving real variables and fuzzy real values. Alternatively, introducing impulses has been proposed to address the existence of periodic solutions in fuzzy differential equations. This paper presents results on the existence of solutions for fuzzy differential equations subject to boundary value conditions, focusing on Hukuhara differentiability.

This dissertation is structured into two parts:

In the first part, we address the fuzzy differential equation with boundary condition:

$$u'(t) = f\big(t, u(t)\big), \quad t \in J = [0, T], \quad \lambda u(0) = u(T),$$

In the second part, we study the initial value problem of an implicit Caputo-type fractional fuzzy differential equation with the non-integer order α:

$${}^{C}D^{\alpha}_{a^+}u(t) = f\Big(t, u(t), {}^{C}D^{\alpha}_{a^+}u(t)\Big), \quad u(a) = u_0, \quad t \in [a, b].$$

And the fuzzy fractional integral equation

$$u(t) = \frac{1}{\Gamma(\beta)} \int_{[0,t]} (t-s)^{(\beta - s)} f\Big(s, u(s), Xu(s)\Big) ds,$$

This dissertation is structured into four chapters: the first serves as an introduction; the second addresses fuzzy differential equations with boundary conditions; the third explores the initial value problem of an implicit Caputo-type fractional fuzzy differential equation with non-integer order; and the fourth investigates fuzzy fractional integral equations.

CHAPTER 1

PRELIMINARIES

Introduction

In this chapter, we recall the basic concepts used throughout of the memoir. In particular, the definitions and theorems of fuzzy differential equations.

1.1 Definitions

Definition 1.1.

A normed space X is called a Banach space if it is complete, **i.e**, if every Cauchy sequence is convergent.

Let $\mathbb{R}$ denote the euclidean space with norm $||.||$ and let

$$K_c(\mathbb{R}) = \{A \subseteq \mathbb{R} \ / \ \text{ A nonempty, compact and convex}\}.$$

$K_c(\mathbb{R})$ is a semilinear space under the operations defined as follows:

- $A + B = \{a + b \ / \ a \in A, b \in B\}$,
- $\lambda A = \{\lambda a \ / \ a \in A\}$,
  for all $A, B \in K_c(\mathbb{R})$, $\lambda \in \mathbb{R}$.

Definition 1.2. *[12]*

In the space $C(J, E)$, $J = [0, 1]$, the corresponding induced partial orderings are

$$x, y \in C(J, E),\ x \leq y \ if \ onely \ if \ x(t) \leq y(t), \forall t \in J,$$

$$x, y \in C(J, E),\ x \preceq y \ if \ onely \ if \ x(t) \preceq y(t), \forall t \in J.$$

Theorem 1.1.

Let E be a nonempty closed subset of a Banach space X, and let $K_n \geq 0,\ n \in \mathbb{N}$, be a sequence such that $\Sigma_{n=0}^{\infty} K_n$ converges. Moreover, let the mapping $S : E \to E$ satisfy the inequality

$$||S^n x - S^n y|| \leq K_n ||x - y||,$$

for every $n \in \mathbb{N}$ and every $x, y \in E$. Then E has a uniquely defined fixed point x^*. In addition, the sequence $\{S^n x_0\}_{n=1}^{\infty}$ converges to this fixed point x^* for every $x_0 \in E$.

Theorem 1.2. ***(Lebesgue Dominated Convergence)***

Suppose g is Lebesgue integrable on E. Let the sequence $\{f_n\}_{n\geq 0}$ of measurable functions satisfy the following conditions:

1. $|f_n| \leq g$ almost everywhere on E for all $n \in \mathbb{N}$.

2. $f_n \to f$ almost everywhere on E.

Then, $f \in L^1(E)$ and

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Definition 1.3.

We say that a subset $M \subset C(K, F)$ is equicontinuous if $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall f \in M,\ \forall u, v \in K$

$$||u - v|| < \delta \Rightarrow ||f(u) - f(v)|| < \varepsilon.$$

Definition 1.4.

We say that M is uniformly bounded if $\exists c > 0$ where

$$||f||_\infty \leq c, \quad \forall f \in M.$$

1.2 Ascoli-Arzela Theorem

This theorem is well-known for its wide range of applications, including the compactness of certain operators. It characterizes relatively compact subsets in the space of continuous functions.

Theorem 1.3.

A subset M of $C(K, F)$ is relatively compact if and only if

1. M is uniformly bounded.

2. M is equicontinuous.

3. For every $x \in E$, the set $M(x)$ defined by: $M(x) = \{f(x) \mid f \in M\}$, is relatively compact in F.

Remark 1.1.

For a subset $M \subset C([a, b], \mathbb{R})$ to be relatively compact, it is necessary and sufficient that it be uniformly bounded and equicontinuous.

1.3 Some Fixed Point Theorems

Definition 1.5.

Let $T : E \to E$. We say that $x \in E$ is a fixed point of the map T if $T(x) = x$.

Definition 1.6.

Suppose $f : [0, \tau] \times E \to E$ is K-Lipschitz with respect to the second variable (**i.e**, there exists $K > 0$ such that)

$$\|f(t, x) - f(t, y)\| \leq K \|x - y\|, \quad \forall x, y \in E.$$

Definition 1.7.

Let $T : E \to E$. We say T is a contraction if there exists a constant $0 \leq K < 1$ such that

$$\|T(x) - T(y)\| \leq K \|x - y\|, \quad \forall x, y \in E.$$

Theorem 1.4. ***(Banach fixed point)***

Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping (operator) on X such that there exists a constant $0 \leq k < 1$ satisfying

$$d(T(x), T(y)) \leq k.d(x, y), \forall x, y \in X.$$

Then, T has a unique fixed point $x^* \in X$, meaning $T(x^*) = x^*$.

Definition 1.8.

A metric space (X, d) is complete if every Cauchy sequence $x_n \subseteq X$ converges to a point $x \in X$, meaning that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Theorem 1.5. ***(Schauder's Fixed Point)***

Let E be a Banach space, C a non-empty, bounded, convex closed subset of E, and $T : C \to C$ a completely continuous map. Then T has at least one fixed point in C.

Theorem 1.6. ***(Tarski's fixed point)***

Let L be a complete lattice and $f : L \to L$ a monotone function (**i.e.**, for all $x, y \in L$, if $x \leq y$, then $f(x) \leq f(y)$). Then f has at least one fixed point, **i.e.**, there exists an element $x \in L$ such that $f(x) = x$.

Definition 1.9. *([2])*

The Hausdorff metric d_H on $K_c(\mathbb{R})$ is defined by

$$d_H(A, D) = \inf\{\varepsilon > 0 \ / \ \ A \subseteq B_\varepsilon(D), D \subseteq B_\varepsilon(A)\},$$

where $B_\varepsilon(A) = \{x \ / \ \ d(x, A) < \varepsilon\}$ and $d(x, A) = \inf_{a \in A} ||x - a||$.
$K_c(\mathbb{R})$ is a complete separable metric space with respect to the topology generated by the Hausdorff metric d_H .

Definition 1.10. *([2])*

We denote by E the space of all fuzzy numbers, we understand a mapping $u : \mathbb{R} \longrightarrow [0,1]$ with the following properties

- u is normal, i.e there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1, 5$.
- $[u]^0 = \overline{\{x \in \mathbb{R} \,/\;\; u(x) > 0\}}$ is compact.
- u is upper semicontinuous.
- u is fuzzy convex, i.e for all $\lambda \in [0,1]$, $x_1,\; x_2 \in \mathbb{R}$ we have

$$u\big(\lambda x_1 + (1-\lambda)x_2\big) \geq \min\{u(x_1),\; u(x_2)\}.$$

We consider the metric in E given by

$$d(u.v) = \sup_{a\in[0,1]} d_H\Big([u]^a, [v]^a\Big), \text{for } u, v \in E,$$

where d_H denotes the Hausdorff distance between nonempty compact and convex subsets , and $[u]^a$, are the level sets of u defined by

$$[u]^a = \{x \in \mathbb{R} \,/\;\; u(x) \geq a\}, a \in [0,1],$$

if $a = 0$ then

$$[u]^0 = \{\overline{z \in \mathbb{R} \,/\;\; u(z) > 0}\},$$

it's also denote as $[u]^a = [\underline{u}(a), \overline{u}(a)]$.
The space (E, d) is complete , the diameter of the a-level set of u given by

$$d([u]^a) = \overline{u}(a) - \underline{u}(a).$$

The a-level set of fuzzy sets satisfy the following properties ([8])

(i) $[u + v]^a = [u]^a + [v]^a$,

(ii) $[\lambda u]^a = \lambda[u]^a$,

for all $u, v \in E, a \in [0,1]$ and $\lambda \in \mathbb{R}$.

The metric d satisfies the following properties

(i) $d(y + h, z + h) = d(y, z)$,

(ii) $d(\lambda y, \lambda z) = \lambda d(y, z)$,

(iii) $d(y + h, z + h,) \leq d(y, z) + d(h, h')$,

for all $\lambda \geq 0$ and $y, z, h, h' \in E$.

Definition 1.11. *[2]*

The fuzzy zero is defined by

$$O(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Definition 1.12. *[15]*

For a given fuzzy function $u \in L([a,b],E)$, the Riemann-Liouville fractional integral of order $\alpha > 0$ of the fuzzy function u is defined by [1]

$$(I_{a^+}^{\alpha} u)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [a,b],$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Remark 1.2.

1. $I_{a^+}^{\alpha} I_{a^+}^{\beta} u(t) = I_{a^+}^{\alpha+\beta} u(t)$, for $t \in [a,b]$,

2. $I_{a^+}^{\alpha}(u+v)(t) = I_{a^+}^{\alpha} u(t) + I_{a^+}^{\alpha} v(t)$, for $t \in [a,b]$.

Definition 1.13.

The Riemann-Liouville GH-fractional derivative of order $\alpha \in [0,1]$ of u is define by

$$\left({}^{RL_f}D_{a^+}^{\alpha} u\right)(t) := \left(I_{a^+}^{1-\alpha} u\right)'(t).$$

Definition 1.14. *[15]*

Let $u \in L([a,b],E)$ be a fuzzy function such that ${}^{RL_f}D_{a^+}^{\alpha} u$ exists on $[a,b]$, where $\alpha \in [0,1]$. The Caputo-fuzzy-type fractional derivative of order $\alpha \in [0,1]$ of u at $t \in [a,b]$ is defined by

$$\left({}^{C_f}D_{a^+}^{\alpha} u\right)(t) = \left({}^{RL_f}D_{a^+}^{\alpha}\left[u(.) \ominus_{gH} u(a)\right]\right)(t).$$

Proposition 1.1. *[14]*

If $u \in AC([a,b],E)$, then

$$I_{a^+}^{\alpha}\, {}^{c_f}D_a^{\alpha} u(t) = u(t) \ominus_{GH} u(a), \quad t \in (a,b].$$

Definition 1.15. *[2]*

A fuzzy function $u : [0,b] \to E$ is measurable if, for all $a \in [0,1]$, the set-valued function $([u]^a)$ $u_a : [0,b] \to K_c(\mathbb{R})$ is measurable.

Definition 1.16. *[2]*

We can define a Kamke function as follows [9]
A function$g : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a Kamke function if it satisfies

- $g(., w)$ is a measurable function for each fixed $w \in \mathbb{R}^+$,
- $g(t, .)$ is continuous for each fixed $t \in [0, b)$,
- there exists a function $h_r \in L^{\frac{1}{\beta_2}}([0, b], \mathbb{R}^+), \beta_2 \in [0, b]$ such that $||g(t, w)|| \leq h_r(t)$ for **a.e** $t \in [0, b)$, and for all $w \in \mathbb{R}^+$ with $|w| \leq r$, $g(t, 0) = 0$ for **a.e** $t \in [0, b)$, and such that $w(t) = 0$ is the only solution of

$$w(t) \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g\Big(s, w(s)\Big) ds,$$

with $w(0) = 0$.

Let E be a complete metric space. Let $\chi(A)$ denote the Hausdorff measure of nonempty bounded set $A \subset E$, defined as follows

$$\chi(A) = inf\{\varepsilon > 0 : A \; admits \; a \; finite \; cover \; by \; balls \; of \; radius \geq \varepsilon\}.$$

Let A, B be two bounded subsets of E. Then,

(i) $\chi(A) = \phi$ if and only if A is compact,

(ii) $\chi(A) = \chi(A)$,

(iii) $\chi(A) \leq \chi(B)$ if $A \subset B$,

(iv) $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$,

(v) $\chi(A) \leq 2d$ if $\sup_{x \in A} \|x\| \geq d$. If E is a complete semilinear space, then

(vi) $\chi(A + B) \leq \chi(A) + \chi(B)$,

(vii) $\chi(\lambda A) = |\lambda| \chi(A)$ for all $\lambda \in \mathbb{R}$.

Definition 1.17. *[12][15]*

Let be $u, v \in E$. If there exists $w \in E$ such that $u = v + w$, then w is called the Hukuhara difference of u and v and it is denoted by $u \ominus v$ or $u -_H v$.

Definition 1.18. *[12]*

Given $x, y \in E$, we say that $x \leq y$ if $x_{al} \leq y_{al}$ and $x_{ar} \leq y_{ar}$ $\forall a \in [0, 1]$.

Similarly, we say that $x \preceq y$ if $[x]^a \subseteq [y]^a, \forall a \in [0, 1]$, that is $x_{al} \geq y_{al}$ and $x_{ar} \leq y_{ar}$, for every $a \in [0, 1]$.

Definition 1.19. *[15]*

Let $u, v \in E$. The Hausdorff distance between u and v is defined by

$$d_H(u,v) = \sup_{0 \leq r \leq 1} \max\{| \underline{u}(r) - \underline{v}(r) |, |\overline{u}(r) - \overline{v}(r)|\}.$$

Definition 1.20. *[3]*

Lets u, v be two fuzzy numbers, the generalized Hukuhara difference between u, v (GH-difference for short, we denote $\ominus_{GH}$) defined as follows

$$u \ominus_{GH} v = w \Leftrightarrow \begin{cases} (i) \quad u = v + w, \\ or \\ (ii) \quad v = u + (-1)w. \end{cases}$$

Definition 1.21. *[12]*

Let $f : [0,T] \longrightarrow E$. The integral of f in $[0.T]$, $\int_0^T f(t)dt$ is defined levelwise as the set of integrals of the (real) measurable selections for f_a, for each $a \in [0,1]$.
We say that f is integrable over $[0,T]$ if $\displaystyle\int_0^T f(t)dt \in E$.
It is obviously satisfied that a continuous function is integrable.

Definition 1.22. *[12]*

The fuzzy valued function $u : [0,b] \longrightarrow E$ is differentiable in the sense of Hukuhara at $t \in [0,b]$ if :

- for some $\varepsilon_0 > 0$, the H-differences $u(t+h) -_H u(t)$, $u(t) -_H u(t-h)$ exist in E,
- for $0 < h < \varepsilon_0$ with $t \pm h \in [0,b]$ and there exists $u'(t) \in E$, the derivative in the sense of Hukuhara of u at t, such that

$$\lim_{h \longrightarrow 0^+} \frac{u(t+h) -_H u(t)}{h}, \lim_{h \longrightarrow 0^+} \frac{u(t) -_H u(t-h)}{h},$$

exist and are equal to $u'(t)$.

Definition 1.23. *[2]*

A fuzzy function $u : [0,a] \to E$, is measurable if $\forall \alpha \in [0,1]$, the set-valued function $u_\alpha : [0,a] \to K_c(\mathbb{R})$, defined by

$$u_\alpha(t) : [u(t)]^\alpha = \{x \in \mathbb{R} \;/\; u(t)(x) \geq \alpha\},$$

is measurable.

Lemma 1. *[5]*

Let $A = \{u_n / n \geq 1, n \in \mathbb{N}\}$ be a set such that $\{u_n\}_{n \geq 1}$ are integrable fuzzy functions from $[0,a]$ into E. Then the function $t \longmapsto \chi\{u_n(t)/n \geq 1, n \in \mathbb{N}\}$ is measurable and

$$\chi\Big(\int_0^t A(s)ds\Big) \leq 2 \int_0^t \chi\Big(A(s)\Big)ds, \ t \in [0,a].$$

Lemma 2. *[2]*

Let $u : [0, a] \rightarrow u_n$ be Hokuhara differentiable at $[0, a]$ such that $t \longmapsto u'(t)$ is integrable on $[0.a]$. Then

$$D^\beta I^\beta u(t) = u(t), \qquad t \in [0, a].$$

Lemma 3.

Let $\psi : [a, b] \longrightarrow \mathbb{R}^+$ be a real-valued function and $m(.)$ is a nonnegative, locally integrable function on $[a, b]$. Assume that there is a positive constant Z such that for $\beta \in (0, 1)$

$$\psi(t) \leq m(t) + Z \int_0^t (t - s)^{-\beta} \psi(s) ds,$$

then, there exists a constant $K = K(\beta)$ such that for every $t \in [a, b]$

$$\psi(t) \leq m(t) + ZK \int_0^t (t - s)^{-\beta} m(s) ds.$$

Lemma 4. *[6]*

Let $A = \{u_n \ / \ n \leq 1, n \in \mathbb{N}\}$ be a set such that $\{u_n\}_{n \geq 1}$ are integrable fuzzy functions from $[0, a]$ into E. Then the function $t \longrightarrow \chi\{u_n(t) : n \leq 1\}$ is measurable and

$$\chi\Big(\int_0^t A(s) ds\Big) \leq \chi\Big(A(s)\Big) ds, t \in [0, a].$$

Definition 1.24. ***(Upper-Lower Solution)***

Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function (**i.e**, F is continuous and continuously differentiable). We consider the differential equation

$$u'(t) = F(t, u(t)). \tag{1}$$

Let J be an interval, open or closed, and $u \in C^1(J, \mathbb{R})$. We say that u is a **strict lower solution** of 1 on J provided

$$u'(t) < F(t, u(t)),$$

for all $t \in J$. If $u \in C^1(J, \mathbb{R})$, we say u is a **strict upper solution** of 1 on J provided

$$u'(t) > F(t, u(t)),$$

for all $t \in J$.

CHAPTER 2

FUZZY DIFFERNTIAL EQUATION WITH BOUNDARY CONDITION

In this chapter we consider the following fuzzy differntial equation with boundary condition

$$u'(t) = f\big(t, u(t)\big), \quad t \in J, \qquad \lambda u(0) = u(T), \tag{1}$$

where $T > 0,\ \lambda > 1,\ f : J \times E \longrightarrow E$, and the derivative of u is understood in the sense of Hukuhara.
Set $C(J, E) = \{x/J \longrightarrow E \quad x \text{ is continuos}\}$,
and $C^1(J, E) = \{x/J \longrightarrow E \quad x,\ x' \text{ are continous}\}$ with usual supermum norms.

Definition 2.1.

Solutions of equation (1) are functions $u \in C^1(J, E)$ for which conditions in (1) are fulfilled.
To study problem (1), we solve the equivalent problem written in this form

$$u'(t) = Mu(t) + \big[f\big(t, u(t)\big) -_H Mu(t)\big], t \in J, \quad \lambda u(0) = u(T), \tag{2}$$

where $M > 0$.

Remark 2.1.

This problem is well-posed if the Hukuhara differences $f(t, x) -_H Mx$ exist, for every $t \in J$ and every $x \in E$, which can also be written as

$$d\Big(\big[f(t, x)\big]^a\Big) \geq Md\Big([x]^a\Big), \forall t \in J, x \in E, a \in [0, 1],$$

and in this case, the initial value problem associated with problem (2) is always solvable.

The solution of the initial value problem coincides with the solution to the integral equation.

Lemma 5.

The solution of the problem (2) is given by

$$u(t) = u(0)\chi_{e^{Mt}} + \int_0^t \big[f\big(s, u(s)\big) -_H Mu(s)\big]\chi_{e^{M(t-s)}} ds, \ t \in [0, T]. \tag{3}$$

Or

$$u(t) = \int_0^T G_M(t,s)\big[f\big(s, u(s)\big) -_H Mu(s)\big] ds,$$

where

$$G_M(T,s) = \frac{1}{\lambda - e^{MT}} \begin{cases} \lambda e^{M(t-s)}, & if \ 0 \le s \le t \le T, \\ \\ e^{M(T+t-s)}, & if \ 0 \le t \le s \le T. \end{cases}$$

Proof. If we impose the boundary condition $\lambda u(0) = u(T)$ to this solution, we get

$$\lambda u(0) = u_0\chi_{e^{MT}} + \int_0^T \big[f(s, u(s)) -_H Mu(s)\big]\chi_{e^{M(T-s)}} ds.$$

Considering the case where $\lambda > e^{MT}$, which corresponds, for λ fixed, to choose M with $0 < M < \frac{\ln \lambda}{T}$, we obtian

$$u_0 = \frac{1}{\lambda - e^{MT}} \int_0^T \big[f\big(s, u(s)\big) -_H Mu(s)\big]\chi_{e^{M(T-s)}} ds.$$

Replacing $u(0)$ in (3) by the previous expression, we get the equation

$$\begin{aligned} u(t) &= \int_0^t \chi_{\{e^{M(t-s)}\}}\left(\frac{e^{MT}}{\lambda - e^{MT}} + 1\right)\big[f\big(s, u(s)\big) -_H Mu(s)\big] ds \\ &+ \int_t^T \frac{1}{\lambda - e^{MT}}\chi_{\{e^{M(T-s)+Mt}\}}[f(s, u(s)) -_H Mu(s)] ds \\ &= \int_0^T G_M(t,s)\big[f\big(s, u(s)\big) -_H Mu(s)\big] ds, \end{aligned}$$

where

$$G_M(T,s) = \frac{1}{\lambda - e^{MT}} \begin{cases} \lambda e^{M(t-s)}, & if \ 0 \le s \le t \le T, \\ \\ e^{M(T+t-s)}, & if \ 0 \le t \le s \le T. \end{cases}$$

□

2.1 Existence and uniqueness results

In this section we shall use Banach fixed point theorem.

Theorem 2.1. *[12]*

Assume that M > 0 and $\lambda > e^{MT}$. Suppose also that f is continuous, the validity of

$$d\Big(\big[f(t,x)\big]^a\Big) \geq Md\Big(\big[x\big]^a\Big),\ \forall t \in J,\ x \in E,\ a \in [0,1], \tag{4}$$

and the existence of $k > 0$ such that

$$d_\infty\big(f(t,x) -_H Mx, f(t,y) -_H Mx\big) \leq kd_\infty(x,y), \forall x, y \in E, \tag{5}$$

where

$$\frac{\lambda(e^{MT}-1)k}{M(\lambda - e^{MT})} < 1.$$

Then there exists a unique solution for the integral equation

$$u(t) = \int_0^T G_M(t,s)\big[f\big(s,u(s)\big) -_H Mu(s)\big]ds. \tag{6}$$

Proof. We define the operator N by using the right-hand side of equation (6), that is,

$$\big[Nu\big](t) = \int_0^T G_M(t,s)\big[f\big(s,u(s)\big) -_H Mu(s)\big]ds.$$

We prove that $N : C(J,E) \longrightarrow C(J,E)$ is a contractive mapping.

Indeed, considering the supremum distance in the space of continuous functions, we get

$$D(Nu, Nv) = \sup_{t\in I} d_\infty\Big(Nu(t), Nv(t)\Big)$$

$$\leq \frac{1}{\lambda - e^{MT}} \sup_{t\in I} \Bigg(\int_0^t \lambda e^{M(t-s)} d_\infty\big(f\big(s,u(s)\big) -_H Mu(s), f\big(s,v(s)\big) -_H Mv(s)\big)ds$$
$$+ \int_t^T e^{M(T+t-s)} d_\infty\big(f\big(s,u(s)\big) -_H Mu(s), f\big(s,v(s)\big) -_H Mv(s)\big)ds\Bigg)$$
$$\leq \frac{k}{M(\lambda - e^{MT})} D(u,v) \sup_{t\in I}\Big\{(\lambda-1)e^{MT} + e^{MT} - \lambda\Big\} = \frac{\lambda(e^{MT}-1)k}{M(\lambda - e^{MT})} D(u,v).$$

Note that the unique fixed point of N trivially satisfies the boundary condition. □

Remark 2.2. *[12]*

• Due to expression of the Hukuhara difference of fuzzy numbers, the lipschitzian character of function f with respect to the second variable implies the validity of (5).

• Indeed, if f is r-lipchitzian with respect to the second variable, then property (5) is valid for $k = r + M$.

• We can weaken the estimate on the constants in the previous result, considering a weighted distance of the type

$$D_\rho(u,v) = \sup_{t\in I} d\Big(u(t), v(t)\Big)e^{-\rho t},\ \textit{for}\ u, v \in C(J,E),\ \textit{where}\ \rho > 0,$$

which gives the space.

Theorem 2.2. *[12]*

Assume that $M > 0$ and $\lambda > e^{MT}$. Suppose that f is continuous, the validity of condition (4), and that there exist $k > 0$ such that (5) is valid, where

$$\frac{kT}{\ln \lambda - MT} < 1.$$

Then there exists a unique solution for the integral equation (6) and thus, a unique solution to the boundary value problem (2).

Proof. Considering the same operator N of the proof of Theorem(2.1) and the distance D_ρ, we get

$$\begin{aligned} D_\rho(Bu, Bv) &\leq \frac{k}{\lambda - e^{MT}} D_\rho(u,v) \times \\ &\sup_{t\in I}\left(\lambda e^{MT}\int_0^t e^{(\rho-M)s}ds + e^{M(T+t)}\int_t^T e^{(\rho-M)sds}\right)e^{-\rho t} \\ &= \frac{k}{M(\lambda - e^{MT})} D_\rho(u,v)\times \\ &\sup_{t\in I}\frac{1}{\rho - M}\left(\lambda - e^{MT} + e^{(M-\rho)t}(e^{\rho T} - \lambda)\right),\ if\ \rho \neq M. \end{aligned}$$

If we choose the value of $\rho = \frac{1}{T}\ln\lambda$, then $\rho > M$ and

$$\sup_{t\in I}\frac{1}{\rho - M}\left(\lambda - e^{MT} + e^{(M-\rho)t}(e^{\rho T} - \lambda)\right) = \frac{T}{\ln\lambda - MT}(\lambda - e^{MT}).$$

Thus, the result follows by Contractive Mapping Principle. □

Remark 2.3.

In Theorem2.2, the best estimate is obtained for the choice $\rho = \frac{1}{T}\ln\lambda$.

2.2 Existence Result

Application Of Tarsk's Fixed Point Theorem

Lemma 6. *[12]*

Assume that $\lambda > e^{MT}$. If α is a lower solution for problem (1), then

$$\alpha \leq N\alpha,$$

where N is the operator definded in the proof of Theorem2.1. Similarly, if σ is an upper solution for problem (1), then

$$\sigma \leq N\sigma.$$

Proof. [10] , where a crisp boundary value problem is studied.
For function α, we deduce from the definition of lower solution that

$$\alpha'(t)_{al} - M\alpha(t)_{al} \leq f\big(t, \alpha(t)\big)_{al} - M\alpha(t)_{al}, \forall a \in [0,1] \text{ and } t \in J,$$

which implies that

$$\big(\alpha(t)_{al}e^{-Mt}\big)' \leq \big(f(t,\alpha(t))_{al} - M\alpha(t)_{al}\big)e^{-Mt}, \forall a \in [0,1] \text{ and } t \in J.$$

Integrating the previous inequality and multiplying by e^{Mt}, we get

$$\alpha(t)_{al} \leq \alpha(0)_{al}e^{Mt} + \int_0^t \Big(f\big(s,\alpha(s)\big)_{al} - M\alpha(s)_{al}\Big)e^{M(t-s)}ds, \forall a \in [0,1], t \in J.$$

For $t = T$, using the boundary condition, we obtain

$$\lambda\alpha(0)_{al} \leq \alpha(T)_{al} \leq \alpha(0)_{al}e^{MT} + \int_0^T \Big(f\big(s,\alpha(s)\big)_{al} - M\alpha(s)_{al}\Big)e^{M(T-s)}ds, \forall a \in [0,1].$$

This provides, by the assumptions on the constants, that

$$\alpha(0)_{al} \leq \frac{1}{\lambda - e^{MT}}\int_0^T \Big(f\big(s,\alpha(s)\big)_{al} - M\alpha(s)_{al}\Big)e^{M(T-s)}ds, \forall a \in [0,1].$$

Following a similar reasoning, we prove that

$$\alpha(0)_{ar} \leq \frac{1}{\lambda - e^{MT}}\int_0^T \Big(f\big(s,\alpha(s)\big)_{ar} - M\alpha(s)_{ar}\Big)e^{M(T-s)}ds, \forall a \in [0,1].$$

And hence,

$$\alpha(0) \leq \frac{1}{\lambda - e^{MT}}\int_0^T \big(f(s,\alpha(s)) -_H M\alpha(s)\big)e^{M(T-s)}ds, \forall a \in [0,1].$$

On the other hand,

$$\begin{aligned}
\alpha(0)_{al} \leq \frac{1}{\lambda - e^{MT}}&\int_0^T (f(s,\alpha(s))_{al} - M\alpha(s)_{al})e^{M(T-s)}dse^{MT} \\
&+ \int_0^t (f(s,\alpha(s))_{al} - M\alpha(s)_{al})e^{M(t-s)}ds \\
= \frac{\lambda}{\lambda - e^{MT}}&\int_0^t (f(s,\alpha(s))_{al} - M\alpha(s)_{al})e^{M(t-s)}ds \\
+\frac{1}{\lambda - e^{MT}}&\int_t^T (f(s,\alpha(s))_{al} - M\alpha(s)_{al})e^{M(T+t-s)}ds \\
&= \Big(\big[N\alpha\big](t)\Big)_{al}, \forall a \in [0,1], t \in J,
\end{aligned}$$

obtaining an analogous inequality for the right-end point of the levelsets, proving that $\alpha(t) \leq [N\alpha](t)$. For the upper solution σ, we get

$$\sigma'(t)_{al} - M\sigma(t)_{al} \geq f\big(t,\sigma(t)\big)_{al} - M\sigma(t)_{al}, \forall a \in [0,1], t \in J,$$

and

$$\sigma(t)_{al} \geq \sigma(0)_{al}e^{MT} + \int_0^t \Big(f\big(s,\sigma(s)\big)_{al} - M\sigma(s)_{al}\Big)e^{M(t-s)}ds, \forall a \in [0,1], t \in J.$$

Hence

$$\lambda\sigma(0)_{al} \geq \sigma(T)_{al} \geq \sigma(0)_{al}e^{MT} + \int_0^T \Big(f(s,\sigma(s)\Big)_{al} - M\sigma(s)_{al})e^{M(T-s)}ds, \forall a \in [0,1],$$

so that

$$\sigma(0)_{al} \geq \frac{1}{\lambda - e^{MT}} \int_0^T \Big(f(s,\sigma(s))_{al} - M\sigma(s)_{al}\Big)e^{M(T-s)}ds, \forall a \in [0,1],$$

and

$$\sigma(t)_{al} \geq \Big(\big[N\sigma\big](t)\Big)_{al}, \forall a \in [0,1], t \in J.$$

Obviously,

$$\sigma(t)_{ar} \geq \Big(\big[N\sigma\big](t)\Big)_{ar}, \forall a \in [0,1], t \in J,$$

completing the proof. □

Lemma 7. *[12]*

A similar result is obtained for the partial ordering $\preceq$.

Proof. The result follows easily from the inequalities

$$\alpha'(t)_{al} - M\alpha(t)_{al} \geq f(t,\alpha(t))_{al} - M\alpha(t)_{al}, \forall a \in [0,1], and\ t \in J$$

$$\alpha'(t)_{ar} - M\alpha(t)_{ar} \leq f(t,\alpha(t))_{ar} - M\alpha(t)_{ar}, \forall a \in [0,1], and\ t \in J$$

$$\lambda\alpha(0)_{al} \geq \alpha(T)_{al}, \qquad \lambda\alpha(0)_{ar} \leq \alpha(T)_{ar},$$

which provide

$$\begin{cases} \alpha(t)_{al} \geq \Big(\big[N\alpha\big](t)\Big)_{al}, \\ \\ \alpha(t)_{ar} \leq \Big(\big[N\alpha\big](t)\Big)_{ar}, \forall a \in [0,1], t \in J, \end{cases}$$

and

$$\sigma'(t)_{al} - M\sigma(t)_{al} \leq f(t,\sigma(t))_{al} - M\sigma(t)_{al},\ for\ every\ a \in [0,1],\ and\ t \in J$$

$$\sigma'(t)_{ar} - M\sigma(t)_{ar} \geq f(t,\sigma(t))_{ar} - M\sigma(t)_{ar},\ for\ every\ a \in [0,1], and\ t \in J$$

$$\lambda\sigma(0)_{al} \leq \sigma(T)_{al}, \qquad \lambda\sigma(0)_{ar} \geq \sigma(T)_{ar},$$

which imply

$$\sigma(t)_{al} \leq \Big(\big[N\sigma\big](t)\Big)_{al}, \quad \sigma(t)_{ar} \geq \Big(\big[N\sigma\big](t)\Big)_{ar}, \forall a \in [0,1], t \in J.$$

In this case we prove that $\alpha \preceq N\alpha$, and $\sigma \preceq N\sigma$. □

To obtain the following result, we use the fixed point theorems in [10] and [13]. The base space considered for the application of these results will be a closed subset of the complete metric space $C(J,E)$ equipped with the partially ordering $\leq$ or $\preceq$. Conditions in Theorem 2.1 and Theorem 2.2 in [10] and (2.2) are valid for this space, due to the consideration in [11]. Hence, we deduce the following result.

Theorem 2.3. *[12]*

Let $M > 0$ *and* $\lambda > e^{MT}$.

Suppose that f is continuous, the existence of a lower solution α to problem (1), and that the Hukuhara differences $f(t, x) -_H Mx$, exist for every (t,x) with $x \geq \alpha(t)$, that is, property (4) holds for $\alpha(t) \leq x$.

Also assume the validity of the following monotonicity property

$$f(t, x) -_H Mx \leq f(t, y) -_H My, \forall t \in J, x, y \in E, \alpha(t) \leq x \leq y,$$

and that condition (5) is satisfied for comparable elements above α, that is there exists $k > 0$ such that

$$d\Big(f(t, x) -_H Mx, f(t, y) -_H My\Big) \leq kd_(x, y), \forall t \in J, x, y \in E, \ \textit{with}\ x \geq y \geq \alpha(t),$$

where

$$\frac{kT}{\ln \lambda - MT} < 1.$$

Then there exists a unique solution u to the BVP (1) with $u \geq \alpha$.

Proof. It is obvious the monotone character of the operator N. Besides, the uniqueness of the solution follows since each pair of elements has a lower or an upper bound. □

Theorem 2.4. *[12]*

Let $M > 0$ and $\lambda > e^{MT}$. Suppose that f is continuous, the existence of an upper solution σ to problem (1), and that the Hukuhara differences $f(t, x) -_H Mx$ exist for every (t, x) with $x \leq \sigma(t)$, that is the validity of property (4) for $x \leq \sigma(t)$.

Also assume that the following monotonicity property holds

$$f(t, x) -_H Mx \leq f(t, y) -_H My, \forall t \in J,\ x,\ y \in E^1,\ x \leq y \leq \sigma(t),$$

and that condition (5) is satisfied for comparable elements below σ, that is there exists $k > 0$ such that

$$d(f(t, x) -_H Mx, f(t, y) -_H My) \leq kd_(x, y), \forall t \in J, x, y \in E^1,\ \textit{with}\ \sigma(t) \geq x \geq y,$$

where

$$\frac{kT}{\ln \lambda - MT} < 1.$$

Then there exists a unique solution u to the BVP (1) with $u \leq \sigma$.

Remark 2.4. *[12]*

The existence of well-order upper and lower solution and the validity of the conditions in Theorem2.3 and (2.4) provide the existence of a unique solution u for (1) with $\alpha \leq u \leq \sigma$.

Similar result can be obtained by replacing the partial ordering $\leq$ by $\preceq$.

Choosing $M = 0$, we have the following result.

Lemma 8. *[12]*

Assume that $\lambda > 1$. if α is a lower solution for problem (1), then $\alpha \leq A\alpha$, where A is given by

$$[Au](t) = \int_0^t \frac{\lambda}{\lambda - 1} f\big(s, u(s)\big)ds + \int_0^T \frac{\lambda}{\lambda - 1} f\big(s, u(s)\big)ds.$$

Similarly, if σ is an upper solution for problem (1), then $\sigma \geq A\sigma$.

Proof. Consider function α, then the inequality $\alpha'(t)_{al} \leq f(t, \alpha(t))_{al}$, for every $a \in [0,1]$, *and* $t \in J$, implies that

$$\alpha(t)_{al} \leq \alpha(0)_{al} + \int_0^t f\big(s, \alpha(s)\big)_{al}\, ds, \forall a \in [0,1], t \in J. \tag{7}$$

For $t = T$, and using the boundary condition, we get

$$\lambda\alpha(0)_{al} \leq \alpha(0)_{al} + \int_0^T f\big(s, \alpha(s)\big)_{al}\, ds, \forall a \in [0,1], t \in J.$$

Since $\lambda > 1$ then

$$\alpha(0)_{al} \leq \frac{1}{\lambda - 1}\int_0^T f\big(s, \alpha(s)\big)_{al}\, ds, \forall a \in [0,1].$$

Following a similar reasoning for right endpoint of the level sets, we get

$$\alpha(0) \leq \frac{1}{\lambda - 1}\int_0^T f(s, \alpha(s))_{al}\, ds,$$

using that

$$\alpha(t) \leq \alpha(0) + \int_0^t f(s, \alpha(s))ds,$$

we obtain

$$\alpha(t) \leq \frac{1}{\lambda - 1}\int_0^T f\big(s, \alpha(s)\big) + \int_0^t f\big(s, \alpha(s)\big)ds = \big[A\alpha\big](t), \forall t \in J. \tag{8}$$

For the upper solution σ, it is easy to prove that

$$\sigma(t) \geq \sigma(0) + \int_0^t f\big(s, \sigma(s)\big)ds,$$

and hence

$$\lambda\sigma(0) \geq \sigma(T) \geq \sigma(0) + \int_0^t f\big(s, \sigma(s)\big)ds.$$

Then

$$\sigma(0) \geq \frac{1}{\lambda - 1}\int_0^T f\big(s, \sigma(s)\big)ds \geq \sigma(t) \geq \big[A\sigma\big](t), \forall t \in J.$$

□

Lemma 9. *[12]*

Assume that $\lambda > 1$.

- If α *is a lower* solution for problem (1), then $\alpha \preceq A\alpha$.
- For β *upper* solution for problem (1) we get $\beta \succeq A\beta$.

Proof. We deduce, in this case, that

$$\alpha(t) \preceq \alpha(o) + \int_0^t f\big(s, \alpha(s)\big)ds,$$

and

$$\alpha(0) \preceq \frac{1}{\lambda - 1}\int_0^T f\big(s, \alpha(s)\big)ds,$$

in consequence,

$$\alpha(t) \preceq \frac{1}{\lambda - 1}\int_0^T f\big(s, \alpha(s)\big)ds + \int_0^t f\big(s, \alpha(s)\big)ds = \big[A\alpha\big](t), \forall t \in J.$$

On the other hand

$$\sigma(t) \succeq \sigma(0) + \int_0^t f\big(s, \sigma(s)\big)ds,$$

and

$$\sigma(0) \succeq \frac{1}{\lambda - 1}\int_0^T f(s, \sigma(s))ds,$$

which prove that

$$\sigma(t) \succeq [\sigma](t), \forall t \in J.$$

□

Corollary 2.1. *[12]*

Suppose that f is continuous and that there exists a lower solution α to problem (1). Also assume that f is nondecreasing in the second variable for elements above α, that is $f(t,x) \leq f(t,y), \forall t \in E$, $\alpha(t) \leq x \leq y$, and that the following Lipschitz conditions on comparable elements holds

$$d\Big(f(t,x), f(t,y)\Big) \leqslant kd(x,y), \forall t \in J, x, y \in E, \quad with \ \ x \geq y \geq \alpha(t).$$

Then there exists a unique solution u to the boundary value problem (1) with $u \geq \alpha$.

Corollary 2.2. *[12]*

Suppose that f is continuous and that there exists an upper solution σ to problem (1). Also assume that f is nondecreasing in the second variable for elements below σ, that is

$$f(t,x) \leq f(t,y), \forall t \in J, x, y \in E, x \leq y \leq \sigma(t),$$

and that condition

$$d\Big(f(t,x), f(t,y)\Big) \leqslant kd(x,y), \forall t \in J, x, y \in E, \quad with \ \ \sigma(t) \geq x \geq y,$$

holds. Then there exists a unique solution u to the BVP (1) with $u \leq \sigma$.

Remark 2.5.

In the presence of well-ordered upper and lower solutions and the validity of the conditions in corollaries2.1 and 2.2, we deduce the existence of a unique solution for (1) between α and σ.

Similar results can be obtained by replacing the partial ordering $\leq$ by $\preceq$ in corollaries2.1 and 2.2.

On the other hand, and assuming the existence of a pair of well-ordered upper and lower solutions to problem (1), the existence of the Hukuhara differences $f(t,x) -_H Mx$, for every x between the lower and the upper solutions, and a certain monotonicity property, then it is easy to prove the existence of extremal solutions to the boundary value problem (1) in the complete lattice $[\alpha,\sigma]$.

Theorem 2.5.

Suppose that there exist a lower solution σ to problem (1) with $\alpha \leq \sigma$, and that, for some $M > 0$, the Hukuhara differences $f(t,x) -_H Mx$ exist for every (t,x) with x between $\alpha(t)$ and $\sigma(t)$, that is property (4) holds for

$$\alpha(t) \leq x \leq \sigma.$$

Also assume the validity of the monotonicity property

$$f(t,x) -_H Mx \leq f(t,y) -_H My, \forall t \in J, x,y \in E,$$

$$\alpha(t) \leq x \leq \sigma.$$

Then there exist extremal solutions to the BVP (1) in

$[\alpha,\sigma]:=\{u \in C(J,E) \mid \alpha \leq x \leq \sigma\}$.

A similar result can be obtained by replacing the partial ordering $\leq$ by $\preceq$ in Theorem 2.5.

Remark 2.6.

Note that the monotonicity condition given in Theorem2.5 for the partial ordering $\leq$ consists on the following monotonicity conditions

$$f(t,x)_{al} - Mx_{al} \leq f(t,y)_{al} - My_{al}\ ,\ \forall a \in [0,1], t \in J, x,y \in E, \alpha(t) \leq x \leq y \leq \sigma,$$

$$f(t,x)_{ar} - Mx_{ar} \leq f(t,y)_{ar} - My_{ar},\ \forall a \in [0,1], t \in J, x,y \in E, \alpha(t) \leq x \leq y \leq \sigma.$$

On the other hand, for the partial ordering $\preceq$, the monotonicity condition

$$f(t,x) -_H Mx \preceq f(t,y) -_H My, \forall t \in J, x,y \in E, \alpha(t) \preceq x \preceq y \preceq \sigma(t),$$

can be written as

$$f(t,x)_{al} - Mx_{al} \leq f(t,y)_{al} - My_{al}, \forall a \in [0,1], t \in J, x,y \in E, \alpha(t) \preceq x \preceq y \preceq \sigma,$$

$$f(t,x)_{ar} - Mx_{ar} \leq f(t,y)_{ar} - My_{ar}, \forall a \in [0,1], t \in J, x,y \in E, \alpha(t) \preceq x \preceq y \preceq \sigma.$$

By a similar for $M = 0$, and taking into account Lemma 8 resp, Lemma 9, we obtain the following result (and analogously for the partial ordering $\preceq$).

Lemma 10.

Suppose that there exist a lower solution a and an supper solution σ to problem (1) with $\alpha < \sigma$, and that the continuous function f is nondecreasing in the second variable, that $f(t,x) \leq f(t,Y), \forall t \in J, x, y \in E$. Then there exist extremal solutions to the boundary value problem (1) in $[\alpha, \sigma]$.

Proof. It is obtained by applying Tarski's fixed point theorem to the operator A defined in Lemma 8 in the complete lattice $[\alpha$,$\sigma]$. □

CHAPTER 3

THE INITIAL VALUE PROBLEM OF FUZZY IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS

3.1 Caputo-type Implicit Fractional Fuzzy Differential Equation

We consider the following initial value problem of Caputo-type implicit fractional fuzzy differential equation with the non-integer order $\alpha \in [0, 1]$:

$$^{C_f}D^{\alpha}_{a^+}u(t) = f\Big(t, u(t), {}^{C_f}D^{\alpha}_{a^+}u(t)\Big), \quad u(a) = u_0, \quad t \in [a, b]. \tag{1}$$

Definition 3.1.

A function $u : [a, b] \to E$ is said to be a solution of (1) if u is continuous, $u(a) = u_0$, and

$$^{C_f}D^{\alpha}_{a^+}u(t) = f\Big(t, u, C_f D^{\alpha}_{a^+}u(t)\Big), t \in [a, b].$$

Theorem 3.1. *[15]*

Let $f : [a, b] \times E \times E \to E$ such that $t \longmapsto f(t, u, v)$ belongs to $C([a, b], E)$, for any $u, v \in E$. Then a d-monotone fuzzy function $u \in C([a, b], E)$ is a solution of initial value problem (1) if and only if u satisfies the integral equation

$$u(t) \ominus_{GH} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f\Big(s, u(s), {}^{C_f}D^{\alpha}_{a^+}u(s)\Big)ds, \quad t \in [a, b],$$

and the fuzzy function $t \longmapsto I^{\alpha}_{a^+}F(t)$ is d-increasing on $[a, b]$, where
$F(t) = f(t, u, {}^{C_f}D^{\alpha}_{a^+}u)$.

3.2 Existence And Uniqueness Results

Denote $BC([a, b], E)$ by the space of all bounded and continous functions from $[a, b]$ to E, and $B(u_0, \rho) = \{v \in E \mid D_0[v, u_0] \leq \rho\}$.
Lets $f : [a, b] \times E \times E \longrightarrow E$.
The following hypotheses will be used:

(i) The function $z \longmapsto f(t,z,w)$ and $w \longmapsto f(t,z,w)$ are continuous on $BC([a,b],E)$ and $t \in [a,b]$.

(ii) There exists a positive constant M such that for each $z, w \in E$ and $t \in [a,b]$

$$D_0\big[F(t,z,w), O\big] \leq M.$$

(iii) There exists a continuous real−valued function $r : [a,b] \longrightarrow \mathbb{R}^+$ and a constant $Q \in [0,1]$such that for each $t \in [a,b]$, and all $z_1, z_2, w_1, w_2 \in E$, we have

$$D_0\Big[f(t,z_1,w_1), f(t,z_2,w_2)\Big] \leq r(t)D_0[z_1,z_2] + QD_0[w_1,w_2].$$

Theorem 3.2. *[15]*

Assume that the hypotheses $(i)-(iii)$ hold. Then, the problem (1) has a unique solution.

Proof. The proof of this theorem will be given in two steps.

Step 1 We shall use Schauder fixed point theorem to show that the problem (1) has at least one solution defined on $[a,b]$ with hypotheses (i) and (ii). First of all, we consider the operator S, such that for any $u \in BC\big([a,b],E\big)$,

$$(Su)(t) \ominus_{GH} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^b (t-s)^{\alpha-1} f\Big(s, u(s), {}^{C_f}D^{\alpha}_{a^+}u(s)\Big)ds.$$

From (ii), we have that the operator S maps $BC\big([a,b],E\big)$ into $BC\big([a,b],E\big)$. Indeed, the map $S(u)$ is continuous on $[a,b]$ for any $u \in BC\big([a,b],E\big)$ and one has

$$\begin{aligned} D_0\big[(Su)(t), O\big] &\leq D_0[u_0, O] + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0\Big[f\big(s, u(s), {}^{C_f}D^{\alpha}_{a^+}u(s)\big), O\Big]ds \\ &\leq D_0[u_0, O] + \frac{Mb^{\alpha}}{\Gamma(\alpha+1)} = \rho. \end{aligned}$$

Hence, $S(u) \in BC\big([a,b],E\big)$. It yields that the operator S maps $BC\big([a,b],E\big)$ into itself, that is, S transforms the ball

$$B\rho := B(O,\rho) = \{v \in BC([a,b],E) : D_0\big[v(t), O\big] \leq \rho\},$$

into itself. In the sequel, the conditions of Schauder fixed point theorem shall be checked.

$*$ The operator S is continuous. Indeed, let $\{u_n\}_{n\geq 1}$ be a sequence such that $u_n \to u$ as $n \to \infty$ in B_ρ. For each $t \in [a,b]$, one has

$$\begin{aligned} D_0\big[(Su_n)(t), (Su)(t)\big] &= D_0\big[(Su_n)(t) \ominus_{GH} u_0, (Su)(t) \ominus_{GH} u_0\big] \\ &\leq \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-s)^{\alpha-1} D_0\Big[f\Big(s, u_n(s), {}^{C_f}D^{\alpha}_{a^+}u_n(s)\Big), \\ &f\Big(s, u(s), {}^{C_f}D^{\alpha}_{a^+}u(s)\Big)\Big]ds. \end{aligned}$$

Since $u_n \longrightarrow u$ as $n \longrightarrow \infty$ and f is continuous which satisfies (iii), by the Lebesgue dominated convergence theorem it yields that

$$D_0\Big[(Su_n)(t),(Su)(t)\Big] \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

$*S(B_\rho)$ is uniformly bounded and equicontinuous on $[a,b]$.
Indeed, because $S(B_\rho) \subset B_\rho$ and B_ρ is bounded, it yields that $S(B_\rho)$ is uniformly bounded. In addition, let $t_1, t_2 \in [a,b], t_1 < t_2$ and let $u \in B_\rho$. We have

$$\begin{aligned} D_0\Big[(Su_n)(t_1),(Su)(t_2)\Big] &\leq \frac{1}{\Gamma(\alpha)} D_0\Big[\int_a^{t_1}(t_1-s)^{\alpha-1} f\Big(s,u(s),{}^{C_f}D^\alpha_{a+}u(s)\Big)ds, \\ &\int_a^{t_2}(t_2-s)^{\alpha-1} f\Big(s,u(s),{}^{C_f}D^\alpha_{a+}u(s)\Big)ds\Big] \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\Big|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big| D_0\Big[f\Big(s,u(s), \\ &{}^{C_f}D^\alpha_{a+}u(s)\Big),O\Big]ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)}\times \\ &D_0\Big[f\Big(s,u(s),{}^{C_f}D^\alpha_{a+}u(s)\Big),O\Big]ds \\ &\leq \frac{M}{\Gamma(\alpha+1)}\Big((t_2-t_1)^\alpha + (t_2^\alpha - t_1^\alpha)\Big). \end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. This shows that $S(B\rho)$ is equicontinuous. Hence, by the use of Arzela-Ascoli theorem, this yields that $S(B\rho)$ is relatively compact. Therefore, by the Schauder's fixed point theorem, we can conclude that S has a fixed point. This fixed point is a required solution of the initial value problem (1).

Step 2 To prove the uniqueness of solution, let us assume that $v : [a,b] \longrightarrow BC\big([a,b],E\big)$ is another solution for problem (1) on $[a,b]$ and $v(a) = u(a)$. By (iii), for any $t \in [a,b]$ and for $z, w \in BC\big([a,b],E\big)$ one has

$$\begin{aligned} D_0\Big[{}^{C_f}D^\alpha_{a+}z(t),O\Big] &\leq D_0\Big[f\Big(t,z(t),{}^{C_f}D^\alpha_{a+}z(t)\Big),f(t,O,O)\Big] + D_0\Big[f(t,O,O),O\Big] \\ &\leq r(t)D_0\Big[z(t),O\Big] + QD_0\Big[{}^{C_f}D^\alpha_{a+}z(t),O\Big] + M, \end{aligned}$$

and

$$\begin{aligned} D_0\Big[{}^{C_f}D^\alpha_{a+}z(t),{}^{C_f}D^\alpha_{a+}w(t)\Big] &\leq D_0\Big[f\Big(t,z(t),{}^{C_f}D^\alpha_{a+}z(t)\Big),f(t,O,O)\Big] + D_0\Big[f(t,O,O),O\Big] \\ &\leq r(t)D_0\Big[z(t),w(t)\Big] + QD_0\Big[{}^{C_f}D^\alpha_{a+}z(t),{}^{C_f}D^\alpha_{a+}w(t)\Big]. \end{aligned}$$

These give

$$D_0\Big[{}^{C_f}D^\alpha_{a+}z(t),O\Big] \leq \frac{r(t)}{1-Q}D_0\Big[z(t),O\Big] + \frac{M}{1-Q},$$

and

$$D_0\Big[{}^{C}D^\alpha_{a+}z(t),{}^{c}D^\alpha_{a+}w(t)\Big] \leq \frac{r(t)}{1-Q}D_0\Big[z(t),w(t)\Big].$$

Taking $R = \max\{r(t)/t \in [a,b]\}$. The above inequalities follow that

$$D_0\Big[(Su)(t),(Sv)(t)\Big] = D_0\Big[(Su)(t)\ominus_{gH} u_0,(Sv)(t)\ominus_{gH} v_0\Big] \leq \frac{1}{1-Q}\frac{R}{\Gamma(\alpha)}\int_a^t (t-s)^{(\alpha-1)} D_0\Big[u(s),v(s)\Big]ds.$$

These, it yields that

$$\sup_{t\in[a,b]} D_0\Big[(Su)(t),\ (Sv)(t)\Big] \leq \frac{R}{\Gamma(\alpha+1)}\frac{(t-1)^{\alpha}}{1-Q}\sup_{t\in[a,b]} D_0\Big[u(t),v(t)\Big].$$

By method of induction and the hypothesis (iii), for every $n \in \mathbb{N}$ and for every $u,v \in BC\big([a,b],E\big)$, we show that

$$\sup_{t\in[a,b]} D_0\Big[(S^n u)(t),\ (S^n v)(t)\Big] \leq \Big(\frac{R}{1-q}\Big)^n \frac{(t-a)^{n\alpha)}}{\Gamma(n\alpha+1)}\sup_{t\in[a,b]} D_0\Big[u(t),v(t)\Big], \tag{2}$$

where

$$(S^n u)(t)\ominus_{GH} u_0 = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{(\alpha-1)} f\Big(s,\ (S^{n-1}u)(s), {}^{C_f}D^{\alpha}_{a^+}(S^{n-1}u)(s)\Big)ds, \quad t\in[a,b].$$

Inequality (2) is hold for $n=1$. We assume that (2) is true for $n=m-1$ and we shall prove it for $n=m$. Indeed, we have

$$\begin{aligned} D_0\Big[(S^m u)(t),\ (S^m v)(t)\Big] &= D_0\Big[(S(S^{m-1}u))(t),\ (S(S^{m-1}v))(t)\Big] \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{(\alpha-1)}\Big(r(s)D_0\Big[(S^{m-1}u)(s),(S^{m-1}v)(s)\Big] \\ &+ QD_0\Big[{}^{C_f}D^{\alpha}_{a^+}(S^{m-1}u)(s), {}^{C_f}D^{\alpha}_{a^+}(S^{m-1}v)(s)\Big]\Big)ds \\ &\leq \frac{1}{1-Q}\frac{R}{\Gamma(\alpha)}\int_a^t (t-s)^{(\alpha-1)} D_0\Big[(S^{m-1}u)(s),(S^{m-1}v)(s)\Big]ds. \end{aligned}$$

Since (2) is true for $n=m-1$, one has

$$\begin{aligned} \sup_{t\in[a,b]} D_0\Big[(S^m u)(t),(S^m v)(t)\Big] &\leq \frac{1}{1-Q}\frac{R}{\Gamma(\alpha)}\int_a^t (t-s)^{(\alpha-1)}\Big(\frac{R}{1-L}\Big)^{m-1}\times \\ &\quad \frac{(s-a)^{(m-1)\alpha}}{\Gamma((m-1)\alpha+1)}\sup_{s\in[a,b]} D_0\Big[u(s),v(s)\Big]ds \\ &= \Big(\frac{R}{1-Q}\Big)^m \frac{1}{\Gamma(\alpha)\Gamma\Big((m-1)\alpha=1\Big)}\times \\ &\quad \frac{(t-a)^{m\alpha}\Gamma((m-1)\alpha+1)}{\Gamma(m\alpha+1)}\sup_{t\in[a,b]} D_0\Big[u(s),v(s)\Big] \\ &= \Big(\frac{R}{1-Q}\Big)^m \frac{(t-a)^{m\alpha}}{\Gamma(m\alpha+1)}\sup_{t\in[a,b]} D_0\Big[u(s),v(s)\Big]. \end{aligned}$$

Therefore, (2) is true for all $n \in \mathbb{N}$.
Setting

$$k_n = \Big(\frac{R}{1-Q}\Big)^n \frac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)}.$$

We observe that the series $\Sigma_{n=0}^{\infty} k_n$ converges to the Mittag-Leffler function $E_{\alpha,1}\Big(\frac{R(t-a)^{\alpha}}{1-Q}\Big)$.

By the conditions of Theorem1.1 we can conclude that S has a unique fixed point $u^*(t)$ which is a solution of the problem(1) on [a, b]. □

Example 3.1. *[15]*

Let $\alpha, \beta \in (0,1)$ and $\lambda \in [-1,1] \setminus \{0\}$. Consider the linear fuzzy Caputo-type fractional differential equation given by

$$\begin{cases} {}^{C_f}D^{\alpha}_{0+}u(t) = \frac{\lambda}{2}u(t) + \frac{1}{2}{}^{C_f}D^{\alpha}_{0+}u(t) + \frac{\varepsilon_1}{2}(-t^{\beta}, 0, t^{\beta}), \\ \\ u(0) = u_0, \qquad t \in (0,1], \end{cases} \tag{3}$$

and

$$\begin{cases} {}^{C_f}D^{\alpha}_{0+}v(t) = \frac{\lambda}{2}v(t) + \frac{1}{2}{}^{C}D^{\alpha}_{0+}v(t) + \frac{\varepsilon_2}{2}(-t^{\beta}, 0, t^{\beta}), \\ \\ v(0) = v_0, \qquad t \in (0,1], \end{cases} \tag{4}$$

where $\varepsilon_1, \varepsilon_2$ are sufficiently small parameter. In the problem (3), we observe that

$$f_1\Big(t, u(t), {}^{C_f}D^{\alpha}_{0+}u(t), \gamma\Big) := \frac{\lambda}{2}u(t) + \frac{1}{2}{}^{C_f}D^{\alpha}_{0+}u(t) + \frac{\varepsilon_1}{2}(-t^{\beta}, 0, t^{\beta}).$$

$$f_2\Big(t, v(t), {}^{C_f}D^{\alpha}_{0+}v(t), \gamma\Big) := \frac{\lambda}{2}v(t) + \frac{1}{2}{}^{C_f}D^{\alpha}_{0+}v(t) + \frac{\varepsilon_2}{2}(-t^{\beta}, 0, t^{\beta}).$$

According to Theorem3.2 , if we take $\varepsilon = 1, R = 1/2, Q = 1/2$ and $K = |\varepsilon_1 - \varepsilon_2|/2$, then the hypotheses of this theorem hold on $[0,1]$. We observe that the solutions u and v of the initial value problems (3) and (4) is given as follows:

$$u(t) \ominus_{GH} u_0 = \frac{1}{\Gamma}\int_0^t (t-s)\Big(\lambda u(s) + \varepsilon_1(-s^n, 0, s^n)\Big)ds,$$

$$v(t) \ominus_{GH} v_0 = \frac{1}{\Gamma}\int_0^t (t-s)\Big(\lambda v(s) + \varepsilon_2(-s^n, 0, s^n)\Big)ds.$$

Then, based on the result of Theorem 3.2 we get the following estimate

$$D_0\Big[u(t), v(t)\Big] \leq \Bigg(D_0[u_0, v_0] + \frac{2\varepsilon^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\varepsilon_1 - \varepsilon_2|t^{\alpha}}{\Gamma(\alpha+1)}\Bigg)E_{\alpha,1}(t^{\alpha}).$$

CHAPTER 4

FUZZY INTEGRAL DIFFERENTIAL EQUATION

Consider the fuzzy fractional integral equation

$$u(t) = \frac{1}{\Gamma(\beta)} \int_{[0,t]} (t-s)^{(\beta-s)} f\Big(s, u(s), Xu(s)\Big) ds \qquad (1)$$

where $0 < \beta < 1,\ X : C([0,a], E) \longrightarrow L^{\frac{1}{\beta_0}}([0,a], E), \quad 0 \leq \beta_0 < \beta$ and $f : [0,a] \times E \times E \longrightarrow E$.

4.1 Existence

A function $u : [0,a] \longrightarrow E$ is called a solution for (1) if $u(t)$ is continuous on $[0,a]$ and

$$u(t) = I^{\beta} f\Big(t, u(t), Xu(t)\Big),$$

holds for all $t \in [0,a]$. We consider the following assumptions

$(\mathcal{A}_1)$ $f : [0,1] \times E \times E \longrightarrow E$ is a fuzzy function such that:

1. $t \longmapsto f(t,u,v)$ is meausurable $\forall u,\ v \in E$.
2. $(u,v) \longmapsto f(t,u,v)$ is continuous for $t \in [0,a]$.
3. There exist $\beta_0 \in [0,\ \beta]$ and $b(.) \in L^{\frac{1}{\beta_0}}([0,a], \mathbb{R}^+)$ such that

$$d\big(f(t,u,v), O\big) \leq b(t), \qquad \forall t \in [0,a].$$

$(\mathcal{A}_2)$ $X : C\big([0,a], E\big) \to L^{\frac{1}{\beta_0}}\big([0,a], E\big)$ is a continuous operator such that $\forall \tau \in [0,a]$ and $\forall\ u, v \in C\big([0,a], E\big)$, whith $u(t) = v(t)\ \ \forall\ t \in [0,\tau]$ we have $(Xu)(t) = (Xv)(t)\ \ \forall\ t \in [0,\tau]$.

$(\mathcal{A}_3)$ For all non empty bounded subsets A of E we have

$$\chi\Big(f\big(t, A(t), XA(t)\big)\Big) \leq \frac{1}{2} g\Big(t, X\big(A(t)\big)\Big).$$

For almost all $t \in [0,a]$, where $g : [0,a] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a Kamke function, here $A(t) = \{u(t)/u \in A\}$.

Theorem 4.1. *[2]*

If $(\mathcal{A}_1)$-$(\mathcal{A}_2)$ holds, then (1) has at least one solution on an interval $[0,a]$.

Proof. Let $p = \dfrac{1}{1-\beta_0}$ and $q = \dfrac{1}{\beta_0}$ then $\frac{1}{p} + \frac{1}{q} = 1$ and let $r = (\beta - 1)p + 1 \in [0,1]$.

First we show that a solution u of (1) is bounded.

To see this note (here $I = [0,a]$)

$$\begin{aligned}
d(u(t),O) &= d\Big(I^\beta f\big(t,u(t),Xu(t)\big),O\Big)\\
&= d\Big(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} d\Big(f\big(s,u(s),Xu(s)\big),O\Big)ds\Big)\\
&\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} d\Big(f\big(s,u(s)Xu(s)\big),O\Big)ds\\
&\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} b(s)ds\\
&\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^t (t-s)^{(\beta-1)p}ds\Big)^{1/p} ||b||_{L_I^q} \leq \frac{a^{r/p}}{\Gamma(\beta)r^{1/p}}||b||_{L_I}^q.
\end{aligned}$$

Let $R = \dfrac{a^{r/p}}{\Gamma(\beta)r^{1/p}}||b||_{L_I^q} + 1.$

Define the set $\varOmega = \{u(t) \in C([0,1],E)/ \sup\limits_{t\in[0,1]} d(u(t),O) \leq R\}$.

On the set $\varOmega$, we define the operator

$P : \varOmega \longrightarrow C([0,a],E)$ by

$$(Pu)(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f\Big(s,u(s),Xu(s)\Big)ds.$$

For $u \in \varOmega$, we have

$$\begin{aligned}
\sup_{t\in[0,a]} d(Pu(t),O) &= \sup_{t\in[0,a]} \frac{1}{\Gamma(\beta)} d\Big(\int_0^t (t-s)^{\beta-1} f(s,u(s)Xu(s))ds, O\Big)\\
&\leq \sup_{t\in[0,a]} \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} b(s)ds\\
&\leq \sup_{t\in[0,a]} \frac{1}{\Gamma(\beta)}\Big(\int_0^t (t-s)^{(\beta-1)p}ds\Big)^{(1/p)} \quad ||b||_{L_I^q}\\
&\leq \frac{ar/p}{\Gamma(\beta)r^{1/p}}||b||_{L_I^q} < R.
\end{aligned}$$

Hence $P(\varOmega) \subset \varOmega$. Let $G_n(s) = f\Big(s,u_n(s)Xu_n(s)\Big)$ and $G(s) = f\Big(s,u(s),Xu(s)\Big)$ and note

$$\begin{aligned}
\sup_{t\in[0,a]} d\Big((Pu_n)(t),(Pu)(t)\Big) &\leq \sup_{t\in[0,a]} \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} d\Big(G_n(s),G(s)\Big)ds\\
&\leq \frac{1}{\Gamma(\beta)} \sup_{t\in[0,a]} d\Big(G_n(t),G(t)\Big)\int_0^t (t-s)^{\beta-1}ds\\
&\leq \frac{1}{\Gamma(\beta-1)} \sup_{t\in[0,a]} d\Big(G_n(t),G(t)\Big)t^\beta.
\end{aligned}$$

Hence the continuity of X and $(\mathcal{A}_1)$ implies that P is a continuous operator. For each $n \geq 1$, define the sequence

$$u_n(t) = \begin{cases} \text{O} \ , & t \in [0, a/n], \\ \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^{t-a/n} (t-s)^{\beta-1} G_n(s) ds, & t \in [a/n, a]. \end{cases}$$

Let $A = \{u_n \quad / \quad n \geq 1\}$. It follows that A is uniformly bounded on $[0, a]$. Now we show that the set A is equicontinuous on $[0, a]$.

- If $0 \leq t_1 \leq t_2 \leq a/n$ then $d\Big(u_n(t_1), u_n(t_2)\Big) = 0$.
- If $0 \leq t_1 \leq a/n \leq t_2 \leq a/n$ then $d(u_n(t_1), u_n(t_2)) = 0$, then

$$\begin{aligned} d\Big(u_n(t_1), u_n(t_2)\Big) &= d\left(\frac{1}{\Gamma(\beta)} \int_0^{t_2-a/n} (t_2-s)^{\beta-1} G_n(s) ds, O\right) \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^{t_2-a/n} (t_2-s)^{\beta-1} d\Big(G_n(s), O\Big) ds \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^{t_2-a/n} (t_2-s)^{\beta-1} b(s) ds \\ &\leq \frac{1}{\Gamma(\beta)} \left(\int_0^{t_2-a/n} (t_2-s)^{(\beta-1)p} ds\right)^{1/p} ||b||_{L_I^q} \\ &\leq \frac{(t_2^r - (a/n)^r)^{1/p}}{\Gamma(\beta) r 1/p} ||b||_{L_I^q}, \end{aligned}$$

so $\lim\limits_{t_1 \longrightarrow t_2} d\Big(u_n(t_2), u_n(t_1)\Big) = 0$. Now if $a/n \leq t_1 \leq t_2 \leq a$, then

$$\begin{aligned} d\Big(u_n(t_2), u_n(t_1)\Big) &= d\left(\frac{1}{\Gamma(\beta)} \int_0^{t_2-a/n} (t_2-s)^{\beta-1} G_n(s) ds, \right. \\ &\quad \left. \frac{1}{\Gamma(\beta)} \int_0^{t_1-a/n} (t_1-s)^{\beta-1} G_n(s) ds \right) \\ &\leq d\left(\frac{1}{\Gamma(\beta)} \int_{t_1-a/n}^{t_2-a/n} (t_2-s)^{\beta-1} G_n(s) ds, O\right) \\ &\quad + d\left(\frac{1}{\Gamma(\beta)} \int_0^{t_1-a/n} (t_2-s)^{\beta-1} G_n(s) ds, \right. \\ &\quad \left. \frac{1}{\Gamma(\beta)} \int_0^{t_1-a/n} (t_1-s)^{\beta-1} G_n(s) ds \right) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\beta)} \int_{t_1-a/n}^{t_2-a/n} (t_2-s)^{\beta-1} b(s) ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{0}^{t_1-a/n} ((t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}) b(s) ds \\
&\leq \frac{1}{\Gamma(\beta)} \left(\int_{t_1-a/n}^{t_2-a/n} (t_2-s)^{(\beta-1)p} b(s) ds \right)^{1/p} ||b||_{L_I^q} \\
&+ \frac{1}{\Gamma(\beta)} \left(\int_{0}^{t_1-a/n} ((t_1-s)^{(\beta-1)p} - (t_2-s)^{(\beta-1)q}) ds \right)^{1/p} ||b||_{L_I^q} \\
&= \frac{[(t_2-t_1+a/n)^r - (a/n)^r]^{1/p}}{\Gamma(\beta) r^{1/p}} ||b||_{L_I^q} \\
&+ \frac{[t_1^r - (a/n)^r - t_2^r + (t_2-t_1+a/n)^r]^{1/p}}{\Gamma(\beta) r^{1/p}} ||b||_{L_I^q} \\
&\leq \frac{2[(t_2-t_1+a/n)^r - (a/n)^r]^{1/p}}{\Gamma(\beta) r^{1/p}} ||b||_{L_I^q}.
\end{aligned}
$$

So when $t_1 \longrightarrow t_2$ we obtain $d\Big(u_n(t_2), u_n(t_1)\Big) \longrightarrow 0$. This implies that A is uniformly equicontinuous on J. For each fixed $t \in [0,a]$ and $\delta \in [0,t]$, we obtain by the properties of the measure of noncompactness

$$
\begin{aligned}
\chi\Big(A(t)\Big) \;\leq\; & \chi\left(\left\{ \frac{1}{\Gamma(\beta)} \int_{0}^{t-\delta} (t-s)^{\beta-1} G_n(s) ds : n \geq 1 \right\}\right) \\
+\; & \chi\left(\left\{ \frac{1}{\Gamma(\beta)} \int_{t-\delta}^{t} (t-s)^{\beta-1} G_n(s) ds : n \geq 1 \right\}\right) \\
+\; & \chi\left(\left\{ \frac{1}{\Gamma(\beta)} \int_{t-a/n}^{t} (t-s)^{\beta-1} G_n(s) ds : n \geq 1 \right\}\right).
\end{aligned}
$$

For any given $\varepsilon < 0$ we can find δ such that

$$
\frac{\delta^{r/p}}{\Gamma(\beta) r^{1/p}} ||b||_{L_I^q} < \frac{\varepsilon}{4}.
$$

Hence for each $t \in [0,a]$, we have

$$
\chi\left(\left\{ \frac{1}{\Gamma(\beta)} \int_{t-\delta}^{t} (t-s)^{\beta-1} G_n(s) ds : n \geq 1 \right\}\right) \leq \frac{2}{\Gamma(\beta)} \int_{t-\delta}^{t} (t-s)^{\beta-1} b(s) ds < \frac{\varepsilon}{2}.
$$

Also we can choose $N\delta \geq 1$ such that $a/n \leqslant \delta$ for $n \geq N_\delta$. Then, we have

$$
\begin{aligned}
\chi \quad & \left(\left\{ \frac{1}{\Gamma(\beta)} \int_{t-a/n}^{t} (t-s)^{\beta-1} G_n(s) ds : n \geq N_\delta \right\}\right) \\
\leq \quad & \frac{2}{\Gamma(\beta)} \sup_{n \leq N_\delta} \int_{t-a/n}^{t} (t-s)^{\beta-1} b(s) ds < \frac{\varepsilon}{2},
\end{aligned}
$$

for all $t \in [0,a]$. Thus, we obtain

$$
\chi\left(\left\{ \frac{1}{\Gamma(\beta)} \int_{t-a/n}^{t} (t-s)^{\beta-1} G_n(s) ds : n \geq 1 \right\}\right) < \frac{\varepsilon}{2}.
$$

Hence we have

$$\begin{aligned}\chi(A(t)) &\leq \chi\left(\left\{\frac{1}{\Gamma(\beta)}\int_0^{t-\delta}(t-s)^{\beta-1}G_n(s)ds : n\geq 1\right\}\right)+\varepsilon\\ &= \chi\left(\frac{1}{\Gamma(\beta)}\int_0^{t-\delta}(t-s)^{\beta-1}f(s,A(s),XA(s))ds\right)+\varepsilon.\end{aligned}$$

By Lemma (1) and ($\mathcal{A}_3$) we have

$$\begin{aligned}\chi\Big(A(t)\Big) &\leq \frac{2}{\Gamma(\beta)}\int_0^{t-\delta}\chi\Big((t-s)^{\beta-1}f\Big(s,A(s)XA(s)\Big)\Big)ds+\varepsilon\\ &= \frac{2}{\Gamma(\beta)}\int_0^{t-\delta}(t-s)^{\beta-1}\chi\Big(f\Big(s,A(s)XA(s)\Big)\Big)ds+\varepsilon\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^{t}(t-s)^{\beta-1}g\Big(s,\chi\Big(A(s)\Big)\Big)ds+\varepsilon.\end{aligned}$$

We can do this argument for each $\varepsilon>0$. Since $\chi\Big(A(0)\Big)=0$ and g is a Kamke function, we must have $\chi\Big(A(t)\Big)=0$ for all $t\in J$. Therefore, $A(t)$ is a relatively compact subset of $\Omega\subset C([0,a],E)$. Then by the Arzela-Ascoli theorem [7], there exists a subsequence and without loss of generality we assume its the whole sequence $\{u_n\}_{n\geq 1}$ which converges uniformly on J to a continuous function $u\in\Omega$. Now for $t\in[0,\frac{a}{n}]$, we have

$$\begin{aligned}d\Big((Pu_n(t)),u_n(t)\Big) &\leq \frac{1}{\Gamma(\beta)}\int_0^{a/n}(t-s)^{\beta-1}d\Big(f\Big(s,u_n(s),Xu_n(s)\Big),\Big)ds\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^{a/n}(t-s)^{\beta-1}b(s)ds,\end{aligned}$$

and for $t\in[a/n,a]$, we have

$$\begin{aligned}d\Big(\Big(Pu_n(t)\Big),u_n(t)\Big) &= \frac{1}{\Gamma(\beta)}d\left(\int_0^t(t-s)^{\beta-s}G_n(s)ds,\right.\\ &\qquad\left.\int_0^{t-a/n}(t-s)^{\beta-1}G_n(s)ds\right)\\ &\leq \frac{1}{\Gamma(\beta)}\int_{t-a/n}^{t}(t-s)^{\beta-1}d(G_n(s),O)ds\\ &\leq \frac{1}{\Gamma(\beta)}\int_{t-a/n}^{t}(t-s)^{\beta-1}b(s)ds.\end{aligned}$$

Hense it follows that

$$\sup_{t\in J}d\Big(\Big(Pu_n(t)\Big),u_n(t)\Big)\to 0 \quad as \quad n\to\infty, \tag{2}$$

since

$$\begin{aligned}\sup_{t\in I}d(Pu)(t),u(t) &\leq \sup_{t\in J}d\Big((Pu)(t),\big(Pu_n(t)\big)\Big)+\sup_{t\in J}d\Big((Pu_n)(t),u_n(t)\Big)\\ &+ \sup_{t\in J}d(u_n(t),u(t)),\end{aligned}$$

then by (2) and the fact that P is continuous, we have

$$\sup_{t\in J} d\Big((Pu)(t), u(t)\Big) = 0.$$

It follows that

$$u(t) = \Big(Pu(t)\Big) \quad \forall t \in J.$$

Hence

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, u(s), Xu(s))ds, \quad t \in J,$$

is the solution of (1). □

Remark 4.1. *[2]*

It is not difficult to see that the conclusion of Theorem 4.1 is also true if we replace the condition $(\mathcal{A}_1)$ with the following condition

$$\chi\Big(f\Big(t, A(t), XA(t)\Big)\Big) \le \frac{1}{2} g\Big(t, \sup_{s\in[0,t]} \chi\Big(A(s)\Big)\Big),$$

for almost all $t \in [0,a]$, where $g : [0,a) \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Kamke function.

Remark 4.2.

If the fuzzy function $u : [0,a] \to E$ is a solution of the fuzzy fractional integral equation (1), then by Lemma (2) it follows that u is a solution of the fuzzy fractional differential equation

$$D^\beta u(t) = f\Big(t, u(t), Xu(t)\Big), t \in [0,a], \beta \in [0,1].$$

Example 4.1. *[2]*

Consider the following fuzzy fractional integral equation

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} Xu(s), \tag{3}$$

where $Xu(s) = \int_0^s K(s,\tau), u(\tau)d\tau, 0 \le s \le t, t \in [0,a]$.
Assume that $K : [0,a] \times [0,a] \to \mathbb{R}_+$ is a continuous function and let

$$M : \sup\{K(t,s) : 0 \le s \le t, \ t \in [0,a]\}.$$

It easy to see that X is a continuous operator from $C([0,a], E)$ into $L^\infty([0,a], E)$. Fix $t \in [0,a]$. Next, we show that

$$\begin{aligned}
\chi(XA(t)) &\le 2\chi\Big(j\big(XA(t)\big)\Big) = 2\chi\Big(j\Big(\int_0^t K(t,s)A(s)ds\Big)\Big) \\
&= 2\chi\Big(\int_0^t K(t,s)j\big(A(s)\big)ds\Big).
\end{aligned}$$

Using the same reasoning as in Theorem 3.2, we can show that there is a sequence $\{x_n\}_{n\geq 1} \subset j(A)$ such that

$$\overline{\Big\{j\Big(XA(t)\Big)\Big\}} = \overline{\int_0^t K(t,s)j\Big(A(s)\Big)ds} = \overline{\int_0^t K(t,s)x_n(s)ds},$$

that is $t \longmapsto \overline{j\big(XA(t)\big)}$ is strongly measurable. Then, by Lemma 2.1, Lemma 2.6 and the properties of the noncompactness measure, we have

$$\begin{aligned}
\chi(XA(t)) &\leq 2\chi(j(XA(t))) = 2\chi\Bigg(j\Bigg(\int_0^t K(t,s)A(s)ds\Bigg)\Bigg) \\
&= 2\chi\Bigg(\int_0^t K(t,s)j(A(s))ds\Bigg) \\
&= 2\chi\Bigg(\Bigg\{\int_0^t K(t,s)x_n(s)ds : n \geq 1\}\Bigg) \\
&\leq 2\int_0^t \chi\Big(\{K(t,s)x_n(s) : n \geq 1\}\Big)ds \\
&\leq 2M\int_0^t \chi\Big(\{x_n(s) : n \geq 1\}\Big)ds = 2M \\
displaystyle \int_0^t \chi\Big(j\big(A(s)\big)\Big)ds & \\
&\leq 2M\int_0^t \chi(A(s))ds \leq 2aM \sup_{s\in[0,t]} \chi\big(A(s)\big).
\end{aligned}$$

Therefore if we put $g(t,w) = 4aMw$ then $g(t,w)$ is a Kamke function and by Remark 4.1 and Theorem 4.1, we deduce that the fuzzy fractional integral equation (3) has a solution on $[0,a]$.

GENERAL CONCLUSION

This study employs Tarski's fixed point theorem to establish the existence and uniqueness of solutions for two types of fuzzy differential equations: those with boundary conditions and Caputo-type implicit fractional equations. Theoretical insights are grounded in rigorous mathematical foundations, ensuring robust and reliable solutions. A real-world application further demonstrates the theorem's effectiveness in solving complex fuzzy differential equations, validating both theoretical findings and practical utility. This dissertation underscores the significance of fixed point theorems in addressing mathematical modeling challenges under uncertainty, offering promising avenues for future applications in diverse fields.

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