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Monotone iterative method for systems of nonlinear conformable fractional differential equations

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Dedication

I dedicate this work:

To my dear parents, ***Mohamed and BELARBI Fatma*** *who have never stopped supporting me and for all their blessings, their love, their tenderness and their prayers throughout my studies.*

To my dear brothers, for their support and encouragement.

To all my friends.

To you dear readers.

I dedicate this work to you with all my best wishes for happiness, health and success.

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Abstract

In this work, we present the existence of extremal solutions for nonlinear conformable fractional differential equation involving integral boundary condition, and for a coupled system of nonlinear first order ordinary differential equations with initial conditions. Also, we present existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations with initial conditions.

Existence results for these problems are obtained by using the monotone iterative technique combined with the method of upper and lower solutions.

Key words and phrases: Conformable fractional derivative, conformable fractional calculus, systems of conformable fractional differential equations, upper and lower solutions, monotone iterative technique.

Résumé

Dans ce mémoire, nous présentons l'existence de solutions extrêmes pour une équation différentielle fractionnaire conforme non linéaire avec condition intégrale, et pour un système couplé d'équations différentielles ordinaires non linéaires du premier ordre avec des conditions initiales. Aussi, nous présentons l'existence de solutions extrêmes pour un système couplé d'équations différentielles fractionnaires conformes non linéaires avec conditions initiales. Ces résultats sont obtenus grâce à la technique itérative monotone combinée à la méthode des sous et sur solutions.

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Mots Clés: Dérivée fractionnaire conforme, calculs fractionnaire conforme, systèmes d'équations différentielles fractionnaires conformes, sous et sur solutions, technique des itérations monotones.

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Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, aerodynamics and electrodynamics, etc. For examples and details, the reader can see the references [18, 21, 23, 26, 27].

Recently, a new fractional derivative, called the conformable fractional derivative, was introduced by Khalil et al. [17]. For recent results on conformable fractional derivatives we refer the reader to [1, 2, 4, 5, 10, 12, 15, 16]. We point out that the method of lower and upper solutions has been applied by several authors to obtain the existence of solutions of initial value problems and boundary value problems for fractional differential equations, see [6, 28, 29, 30].

In this work, we present existence of extremal solutions for nonlinear conformable fractional differential equation involving integral boundary condition, and for a coupled system of nonlinear first order ordinary differential equations with initial conditions. Also, we present existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations with initial conditions. Existence results for these problems are obtained by using the monotone iterative technique combined with the method of upper and lower solutions. The purpose of this method is to:

(i) constructing two monotone iterative sequences, by using γ, δ the lower and upper solutions with $\gamma \leq \delta$,

(ii) showing the convergence of the constructed sequences,

(iii) proving these two sequences approximate the extremal solutions of the given problem.

A solution x^* in $[\gamma, \delta]$ is a maximal solution if $x^* \geq x$ for any other solution x in $[\gamma, \delta]$. A solution y^* in $[\gamma, \delta]$ is a minimal solution if $y^* \leq x$ for any other solution x in $[\gamma, \delta]$, when both a minimal and a maximal solution in $[\gamma, \delta]$ exist, we call them the extremal solutions in $[\gamma, \delta]$.

For applications of monotone iterative technique combined with the method of upper and lower solutions to differential equations and differential systems, one can refer to literatures [3, 8, 9, 19, 20, 22].

We have organized this work as follows:

In Chapter 1, we present some definitions and results which are used throughout this thesis.

In Chapter 2, we prove the existence of extremal solutions for the following nonlinear conformable fractional differential equation involving integral boundary condition, using the method of upper and lower solutions and its associated monotone iterative technique:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), & t \in I = [0, 1], \\ x(0) = \displaystyle\int_0^1 x(t)dt, \end{cases} \tag{1}$$

where $0 < \alpha \leq 1$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α and $\sigma \in \mathbb{R}$.

In Chapter 3, we investigate the existence of extremal solutions for a coupled system of nonlinear first order ordinary differential equations with initial conditions, by using the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions:

$$\begin{cases} x^{'}(t) = f(t, x(t), y(t)), & t \in I = [a, b], \\ y^{'}(t) = g(t, y(t), x(t)), & t \in I = [a, b], \\ x(a) = \lambda_0,\ y(a) = \beta_0. \end{cases} \tag{2}$$

where $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\lambda_0, \beta_0 \in \mathbb{R}$ with $\lambda_0 \leq \beta_0$.

In Chapter 4, we investigate the existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations with initial conditions, by using the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t), y(t)), & t \in I = [a, b], \\ y^{(\alpha)}(t) = g(t, y(t), x(t)), & t \in I = [a, b], \\ x(a) = \lambda_0,\ y(a) = \beta_0. \end{cases} \tag{3}$$

where $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\lambda_0, \beta_0 \in \mathbb{R}$, $\lambda_0 \leq \beta_0$, $x^{(\alpha)}$, $y^{(\alpha)}$ are the conformable fractional derivatives with $0 < \alpha \leq 1$.

Chapter 1

Preliminaries

In this chapter, we present some definitions and results which we will use in this work.

1.1 Elements of Functional Analysis

Let $C(J, \mathbb{R})$ be the Banach space of continuous functions from $J = [a, b]$ into $\mathbb{R}$ with the norm

$$\|u\| = \sup\{|u(t)| : t \in J\}.$$

Definition 1.1.1. *[25]. Let E, F be Banach spaces and $T : E \to F$.*

(i) The operator T is said to be bounded if it maps any bounded subset of E into a bounded subset of F.

(ii) The operator T is called compact if $T(E)$ is relatively compact (i.e., $\overline{T(E)}$ is compact).

(iii) The operator T is said to be completely continuous if it is continuous and maps any bounded subset of E into a relatively compact subset of F.

Theorem 1.1.2. (*Arzela-Ascoli theorem [24]*). *A subset $\mathcal{F}$ of $C([a, b], \mathbb{R}^n)$ is relatively compact (i.e. $\overline{\mathcal{F}}$ is compact) if and only if the following conditions hold:*

1. $\mathcal{F}$ is uniformly bounded i.e, there exists $M > 0$ such that

$$\|f(t)\| < M \; for\, each\, t \in [a, b] \text{ and each } f \in \mathcal{F}.$$

2. $\mathcal{F}$ is equicontinuous i.e, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t_1, t_2 \in [a, b]$, $|t_2 - t_1| \leqslant \delta$ implies $\|f(t_2) - f(t_1)\| \leqslant \varepsilon$, for every $f \in \mathcal{F}$.

Theorem 1.1.3. *(Schauder's fixed point theorem [11]). Let C be a convex (not necessarily closed) subset of a normed linear space E. Then each compact map $N : C \to C$ has at least one fixed point.*

1.2 Conformable Fractional Calculus

In this section, we introduce some necessary definitions and properties of the conformable fractional calculus which are used in this thesis and can be found in [1, 13, 17].

Definition 1.2.1. *[17] Given a function $f : [0, \infty) \to \mathbb{R}$ and a real constant $\alpha \in (0, 1]$. The conformable fractional derivative of f of order α is defined by,*

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1.1}$$

for all $t > 0$. If $f^{(\alpha)}(t)$ exists and is finite, we say that f is α-differentiable at t.

If f is α-differentiable in some interval $(0, a),\ a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then the conformable fractional derivative of f of order α at $t = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Theorem 1.2.2. *[17] Let $\alpha \in (0, 1]$ and $f : [0, \infty) \to \mathbb{R}$ a α-differentiable function at $t_0 > 0$, then f is continuous at t_0.*

Theorem 1.2.3. *[17] Let $\alpha \in (0, 1]$ and assume f, g to be α-differentiable at a point $t > 0$. Then,*

(i) $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$, *for all* $a, b \in \mathbb{R}$;

(ii) $(fg)^{(\alpha)} = fg^{(\alpha)} + gf^{(\alpha)}$;

(iii) $(f/g)^{(\alpha)} = \dfrac{gf^{(\alpha)} - fg^{(\alpha)}}{g^2}$.

(iv) If, in addition, f is differentiable at a point $t > 0$, then

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t).$$

Additionaly, conformable fractional derivatives of certain functions as follow:

1. $(t^p)^{(\alpha)} = p\, t^{p-\alpha}$, for all $p \in \mathbb{R}$.
2. $(\lambda)^{(\alpha)} = 0$, for all $\lambda \in \mathbb{R}$.
3. $(e^{ct})^{(\alpha)} = c\, t^{1-\alpha} e^{ct}$, for all $c \in \mathbb{R}$.
4. $(e^{\frac{p}{\alpha} t^\alpha})^{(\alpha)} = p\, e^{\frac{p}{\alpha} t^\alpha}$, for all $p \in \mathbb{R}$.

Remark 1.2.4. *It is not difficult to verify the following assertions:*

(i) The function $x : t \mapsto e^{\frac{p}{\alpha}t^{\alpha}}$, $p \in \mathbb{R}$, *is the unique solution to the conformable fractional differential equation*

$$x^{(\alpha)}(t) = p\,x(t),\ t \in [0, \infty),\ x(0) = 1.$$

(ii) If f *is differentiable at* t, *then* f *is* α*-differentiable at* t.

Theorem 1.2.5. *[13] Let* $a > 0$ *and* $f : [a, b] \to \mathbb{R}$ *be a given function that satisfies*

(i) f *is continuous on* $[a, b]$,

(ii) f *is* α*-differentiable for some* $\alpha \in (0, 1)$.

Then we have the following:

(1) If $f^{\alpha}(x) > 0$ *for all* $x \in (a, b)$, *then* f *is increasing on* $[a, b]$.

(2) If $f^{\alpha}(x) < 0$ *for all* $x \in (a, b)$, *then* f *is decreasing on* $[a, b]$.

Definition 1.2.6. *(Conformable fractional integral [17]). Let* $\alpha \in (0, 1]$ *and* $f : [a, \infty) \to \mathbb{R}$. *The conformable fractional integral of* f *of order* α *from* a *to* t, *denoted by* $I_{\alpha}^{a}(f)(t)$, *is defined by*

$$I_{\alpha}^{a}(f)(t) := \int_a^t f(s) d_{\alpha}s := \int_a^t f(s)s^{\alpha-1}ds.$$

The considered integral is the usual improper Riemann one.
For $a = 0$ *we put* $I_{\alpha}^{0}(f)(t) = I_{\alpha}(f)(t)$.

Theorem 1.2.7. *[17] If* f *is a continuous function in the domain of* I_{α}^{a} *then, for all* $t \geq a$ *we have*

$$\left(I_{\alpha}^{a}(f)\right)^{(\alpha)}(t) = f(t).$$

Lemma 1.2.8. *[17] Let* $f : (a, b) \to \mathbb{R}$ *be differentiable and* $0 < \alpha \leq 1$. *Then, for all* $t > a$ *we have*

$$I_{\alpha}^{a}(f^{(\alpha)})(t) = f(t) - f(a). \tag{1.2}$$

Proposition 1.2.9. *[13] Let* $0 < a < b$, $f : [a, b] \to \mathbb{R}$ *be continuous function and* $0 < \alpha < 1$. *Then for all* $t \in [a, b]$ *we have,*

$$|I_{\alpha}^{a}(f)(t)| \leq I_{\alpha}^{a}|f|(t).$$

Theorem 1.2.10. (*Rolle's theorem [13]*) *Let* $a > 0$ *and* $f : [a, b] \to \mathbb{R}$ *be a function with the properties that*

(1) f *is continuous on* $[a, b]$,

(2) f is $\alpha-$differentiable on (a, b) for some $\alpha \in (0, 1)$,

(3) $f(a) = f(b)$.

Then, there exists $c \in (a, b)$, such that $f^{\alpha}(c) = 0$.

Theorem 1.2.11. (*Mean Value Theorem [13]*) *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a function with the properties that*

(1) f is continuous on $[a, b]$,

(2) f is $\alpha-$differentiable on (a, b) for some $\alpha \in (0, 1)$,

Then, there exists $\theta \in]a, b[$, such that $f^{(\alpha)}(\theta) = \dfrac{f(b) - f(a)}{\frac{1}{\alpha}(b^{\alpha} - a^{\alpha})}$.

Remark 1.2.12. *We introduce the following space:*

$$C^{\alpha}(J, \mathbb{R}) \;=\; \{f : J \to \mathbb{R}, \;\; \textit{is} \;\; \alpha\textit{-differentiable on } J \;\; \textit{and } f^{(\alpha)} \in C(J, \mathbb{R})\}.$$

If $\alpha = 1$, we have

$$C^{1}(J, \mathbb{R}) \;=\; \{f : J \to \mathbb{R}, \;\; \textit{is} \;\; \textit{differentiable on } J \;\; \textit{and } f^{'} \in C(J, \mathbb{R})\}.$$

Chapter 2

Extremal solutions to conformable fractional differential equations

2.1 Introduction

In this chapter, we consider the existence of solutions for the following nonlinear conformable fractional differential equation involving integral boundary condition:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), & t \in I = [0,1], \\ x(0) = \displaystyle\int_0^1 x(t)dt, \end{cases} \tag{2.1}$$

where $0 < \alpha \leq 1$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α.

S. Meng et al. in [22], studied the existence of extremal iteration solution to the following nonlinear conformable fractional differential equation involving integral boundary condition:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), & t \in I = [0,1], \\ x(0) = \displaystyle\int_0^1 x(t)d\mu(t), \end{cases} \tag{2.2}$$

where $0 < \alpha \leq 1$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α and $\displaystyle\int_0^1 x(t)d\mu(t)$ denotes the Riemann-Stieltjes integral with positive Stieltjes measure of μ.

In [14] T. Jankowski studied the existence of extremal solutions to the following nonlinear ordinary differential equations with integral boundary conditions:

$$\begin{cases} x^{'}(t) = f(t, x(t)), \qquad t \in J = [0, T],\ T > 0 \\ x(0) = \lambda \displaystyle\int_0^T x(t)dt + d, \end{cases} \tag{2.3}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $d \in \mathbb{R}$ and $\lambda = -1$ or $\lambda = 1$.

The existence result of (2.1) is obtained by means of the method of upper and lower solutions and its associated monotone iterative technique. Based on a comparison result, two monotone iterative sequences are obtained using the upper and lower solutions, and these two sequences approximate the extremal solutions of the (2.1).

2.2 Linear fractional differential equations

In this section, we study the expression of the solutions of a linear conformable fractional differential equation involving integral boundary problem:

$$\begin{cases} x^{(\alpha)}(t) = -p(t)x(t) + g(t), \qquad t \in I = [0, 1], \\ x(0) = \displaystyle\int_0^1 x(t)dt + \sigma, \end{cases} \tag{2.4}$$

with $0 < \alpha \leq 1$, $\sigma \in \mathbb{R}$ and $p,\ g \in C([0, 1], \mathbb{R})$.

Once we have such expression, we derive comparison results for the considered problems.

Theorem 2.2.1. *If* $1 - \displaystyle\int_0^1 e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}dt \neq 0$, *then problem (2.4) has a unique solution* $x \in C([0, 1], \mathbb{R})$, *and it is given by the following expression:*

$$\begin{aligned} x(t) = &\frac{e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}}{1 - \displaystyle\int_0^1 e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}dt}\left[\int_0^1 \int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}dsdt + \sigma\right] \\ &+ \int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}ds. \end{aligned} \tag{2.5}$$

Proof. Let x be a solution of problem (2.4), we have $x^{(\alpha)}(t) + p(t)x(t) = g(t)$. By Theorem 1.2.3, we have that the following property holds:

$$\begin{aligned} [e^{\int_0^t \tau^{\alpha-1}p(\tau)d\tau}x(t)]^{(\alpha)} &= e^{\int_0^t \tau^{\alpha-1}p(\tau)d\tau}x^{\alpha}(t) + p(t)x(t)e^{\int_0^t \tau^{\alpha-1}p(\tau)d\tau} \\ &= \left(x^{\alpha}(t) + p(t)x(t)\right)e^{\int_0^t \tau^{\alpha-1}p(\tau)d\tau} \\ &= g(t)e^{\int_0^t \tau^{\alpha-1}p(\tau)d\tau}. \end{aligned}$$

Applying I_α the conformable fractional integral of order α to both sides of, we have

$$\begin{aligned} e^{\int_0^t \tau^{\alpha-1}p(\tau)d\tau}x(t) - x(0) &= I_\alpha\left[g(t)e^{\int_0^t \tau^{\alpha-1}p(\tau)d\tau}\right] \\ &= \int_0^t s^{\alpha-1}g(s)e^{\int_0^s \tau^{\alpha-1}p(\tau)d\tau}ds. \end{aligned}$$

Then

$$\begin{aligned} x(t) &= e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}\left(x(0) + \int_0^t s^{\alpha-1}g(s)e^{\int_0^s \tau^{\alpha-1}p(\tau)d\tau}ds\right) \\ &= x(0)e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau} + \int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}ds. \end{aligned} \tag{2.6}$$

From the boundary condition of (2.4), we have

$$\int_0^1 x(t)dt + \sigma = x(0)\int_0^1 e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}dt + \int_0^1\int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}dsdt + \sigma.$$

So,

$$\left(1 - \int_0^1 e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}dt\right)x(0) = \int_0^1\int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}dsdt + \sigma.$$

On account of condition $1 - \displaystyle\int_0^1 e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}dt \neq 0$, then

$$x(0) = \frac{1}{1 - \int_0^1 e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}dt}\left[\int_0^1\int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}dsdt + \sigma\right]. \tag{2.7}$$

Now, by substituting (2.7) into (2.6), we get

$$\begin{aligned} x(t) = &\frac{e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}}{1 - \int_0^1 e^{-\int_0^t \tau^{\alpha-1}p(\tau)d\tau}dt}\left[\int_0^1\int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}dsdt + \sigma\right] \\ &+ \int_0^t s^{\alpha-1}g(s)e^{-\int_s^t \tau^{\alpha-1}p(\tau)d\tau}ds. \end{aligned}$$

Thus problem (2.4) has a unique solution. The proof is finished. □

As a direct consequence of the previous result, we deduce the following expression for the following particular case where p is a constant function i.e., $\forall x \in I:\ p(x) = M \in \mathbb{R}$.

Corollary 2.2.2. *Let $M \in \mathbb{R}$ and $1 - \displaystyle\int_0^1 e^{-\frac{M}{\alpha}t^\alpha} dt \neq 0$. If $g \in C([0,1],\mathbb{R})$, then problem*

$$\begin{cases} x^{(\alpha)}(t) = -Mx(t) + g(t), & t \in I = [0,1], \\ x(0) = \displaystyle\int_0^1 x(t)dt + \sigma, \end{cases} \tag{2.8}$$

has a unique solution $x \in C([0,1],\mathbb{R})$, and it is given by the following expression:

$$\begin{aligned} x(t) = & \frac{e^{-\frac{M}{\alpha}t^\alpha}}{1 - \displaystyle\int_0^1 e^{-\frac{M}{\alpha}t^\alpha} dt} \left[\int_0^1 \int_0^t s^{\alpha-1} g(s) e^{-\frac{M}{\alpha}(t^\alpha - s^\alpha)} ds dt + \sigma \right] \\ & + \int_0^t s^{\alpha-1} g(s) e^{-\frac{M}{\alpha}(t^\alpha - s^\alpha)} ds. \end{aligned} \tag{2.9}$$

In the next Lemmas, we discuss comparison results for the linear problem (2.8).

Lemma 2.2.3. *Let $0 < \alpha \leq 1$, $M \in \mathbb{R}$, $1 - \displaystyle\int_0^1 e^{-\frac{M}{\alpha}t^\alpha} dt > 0$ and $x \in C([0,1],\mathbb{R})$, such that:*

$$\begin{cases} x^\alpha(t) \leq -Mx(t), & t \in [0,1], \\ x(0) \leq \displaystyle\int_0^1 x(t)dt. \end{cases} \tag{2.10}$$

Then $x(t) \leq 0$ for every $t \in I = [0,1]$.

Proof. we put $x^\alpha(t) + Mx(t) = g(t)$ and $x(0) - \displaystyle\int_0^1 x(t)dt = \sigma$. We are know that $g(t) \leq 0$, for every $t \in I = [0,1]$, $a \leq 0$, and

$$\begin{cases} x^{(\alpha)}(t) = -Mx(t) + g(t), & t \in [0,1], \\ x(0) = \displaystyle\int_0^1 x(t)dt + \sigma. \end{cases} \tag{2.11}$$

By Corollary 2.2.2, the expression of $x(t)$ is given by (2.9):

$$\begin{aligned} x(t) = & \frac{e^{-\frac{M}{\alpha}t^\alpha}}{1 - \displaystyle\int_0^1 e^{-\frac{M}{\alpha}t^\alpha} dt} \left[\int_0^1 \int_0^t s^{\alpha-1} g(s) e^{-\frac{M}{\alpha}(t^\alpha - s^\alpha)} ds dt + \sigma \right] \\ & + \int_0^t s^{\alpha-1} g(s) e^{-\frac{M}{\alpha}(t^\alpha - s^\alpha)} ds, \end{aligned}$$

we can conclude that, $x(t) \leq 0$ for every $t \in I = [0,1]$. □

Lemma 2.2.4. *Let* $0 < \alpha \leq 1$, $M \in \mathbb{R}$, $1 - \int_0^1 e^{-\frac{M}{\alpha}t^\alpha} dt > 0$ *and* $x \in C([0,1], \mathbb{R})$, *such that:*

$$\begin{cases} x^\alpha(t) \geq -Mx(t), & t \in [0,1], \\ x(0) \geq \displaystyle\int_0^1 x(t)dt. \end{cases} \tag{2.12}$$

Then $x(t) \geq 0$ *for every* $t \in I = [0,1]$.

Proof. we put $x^\alpha(t) + Mx(t) = g(t)$ and $x(0) - \int_0^1 x(t)dt = \sigma$. We are know that $g(t) \geq 0$, for every $t \in I = [0,1]$, $\sigma \geq 0$, and

$$\begin{cases} x^{(\alpha)}(t) = -Mx(t) + g(t), & t \in [0,1], \\ x(0) = \displaystyle\int_0^1 x(t)dt + \sigma. \end{cases} \tag{2.13}$$

By Corollary 2.2.2, the expression of $x(t)$ is given by (2.9):

$$x(t) = \frac{e^{-\frac{M}{\alpha}t^\alpha}}{1 - \displaystyle\int_0^1 e^{-\frac{M}{\alpha}t^\alpha} dt} \left[\int_0^1 \int_0^t s^{\alpha-1} g(s) e^{-\frac{M}{\alpha}(t^\alpha - s^\alpha)} ds dt + \sigma \right] + \int_0^t s^{\alpha-1} g(s) e^{-\frac{M}{\alpha}(t^\alpha - s^\alpha)} ds,$$

we can conclude that, $x(t) \geq 0$ for every $t \in I = [0,1]$. □

2.3 Main Results

In this section, we prove the existence of extremal solutions for conformable fractional differential equation involving integral boundary condition (2.1). Let us define what we mean by a solution of this problem.

Definition 2.3.1. *A solution of problem (2.1) will be a function* $x \in C^1(I, \mathbb{R})$ *for which (2.1) is satisfied.*

Next, we introduce the notion of lower and upper solutions for the problem (2.1).

Definition 2.3.2. *Let* $\gamma \in C^\alpha(I, \mathbb{R})$. *We say that* γ *is a lower solution of problem* (2.1), *if it satisfies:*

$$\begin{cases} \gamma^{(\alpha)}(t) \leq f(t, \gamma(t)), & \text{for all } t \in I = [0,1], \\ \gamma(0) \leq \displaystyle\int_0^1 \gamma(t)dt. \end{cases} \tag{2.14}$$

Let $\delta \in C^{\alpha}(I,\mathbb{R})$. We say that δ is an upper solution of problem (2.1)*, if it satisfies:*

$$\begin{cases} \delta^{(\alpha)}(t) \geq f(t,\delta(t)), & \text{for all } t \in I = [0,1], \\ \delta(0) \geq \displaystyle\int_0^1 \delta(t)dt. \end{cases} \tag{2.15}$$

We define the sector $[\gamma,\delta] = \{x \in C^{\alpha}(I,\mathbb{R}) : \gamma(t) \leq x(t) \leq \delta(t),\ t \in I\}$.
Now we give the main result on the existence of solutions for the nonlinear problem (2.1).

Theorem 2.3.3. *Assume that:*

(H_1) *$f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function.*

(H_2) *There exists $\gamma, \delta \in C(I,\mathbb{R})$, lower and upper solutions to problem* (2.1)*, respectively, such that $\gamma(t) \leq \delta(t)$, for all $t \in I$.*

(H_3) *There exists constant $M \in \mathbb{R}$ With $1 - \int_0^1 e^{-\frac{M}{\alpha}t^{\alpha}} dt > 0$ which satisfies*

$$f(t,x) - f(t,\overline{x}) \leq M(\overline{x} - x),$$

for

$$\gamma(t) \leq x \leq \overline{x} \leq \delta(t).$$

Then there exist monotone iterative sequences $\{v_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset C(I,\mathbb{R})$ converging uniformly to v, w, respectively, (i.e., $\lim_{n\to\infty} v_n = v, \quad \lim_{n\to\infty} w_n = w$), and v, w are the extremal solutions of problem (2.1) *in the sector $[\gamma,\delta]$, such that*

$$\gamma = v_0 \leq ... \leq v_n \leq ... \leq w_n \leq ... \leq w_0 := \delta, \quad \text{on } I \text{ for all } n \in \mathbb{N}.$$

Proof. For all $v_n, w_n \in C(I,\mathbb{R})$, let

$$\begin{cases} v_{n+1}^{(\alpha)}(t) & = f(t,v_n(t)) - M\left(v_{n+1}(t) - v_n(t)\right), \quad t \in [0,1], \\ w_{n+1}^{(\alpha)}(t) & = f(t,w_n(t)) - M\left(w_{n+1}(t) - w_n(t)\right), \quad t \in [0,1], \\ v_{n+1}(0) & = \displaystyle\int_0^1 v_{n+1}(t)dt, \quad w_{n+1}(0) = \displaystyle\int_0^1 w_{n+1}(t)dt. \end{cases} \tag{2.16}$$

Thus, the iterative sequences $\{v_n\}$ and $\{w_n\}$ can be constructed by Corollary 2.2.2. Firstly, we shall prove that

$$v_n \leq v_{n+1} \leq w_{n+1} \leq w_n, \quad n = 0,1,2,\ldots$$

□

Let $p = v_0 - v_1$. According to (2.16) and Definition 2.3.2, we have

$$\begin{cases} p^{(\alpha)}(t) = v_0^{(\alpha)}(t) - v_1^{(\alpha)}(t) \leq f(t, v_0(t)) - f(t, v_0(t)) + M\left(v_1(t) - v_0(t)\right), & t \in [0,1], \\ p(0) \leq \int_0^1 v_0(t)dt - \int_0^1 v_1(t)dt = \int_0^1 (v_0(t) - v_1(t))dt, \end{cases}$$

i.e.,

$$\begin{cases} p^{(\alpha)}(t) \leq -Mp(t), & t \in [0,1], \\ p(0) \leq \int_0^1 p(t)dt. \end{cases}$$

Therefore, by Lemma 2.2.3, we have $p(t) \leq 0$, $t \in I$, then $v_0(t) \leq v_1(t)$, $t \in I$.

Similarly, Let $q = w_0 - w_1$. According to (2.16) and Definition 2.3.2, we have

$$\begin{cases} q^{(\alpha)}(t) = w_0^{(\alpha)}(t) - w_1^{(\alpha)}(t) \geq f(t, w_0(t)) - f(t, w_0(t)) + M\left(w_1(t) - w_0(t)\right), & t \in [0,1], \\ q(0) \geq \int_0^1 w_0(t)dt - \int_0^1 w_1(t)dt = \int_0^1 (w_0(t) - w_1(t))dt, \end{cases}$$

i.e.,

$$\begin{cases} q^{(\alpha)}(t) \geq -Mq(t), & t \in [0,1], \\ q(0) \geq \int_0^1 q(t)dt. \end{cases}$$

Therefore, by Lemma 2.2.4, we have $q(t) \geq 0$, $t \in I$, then $w_0(t) \geq w_1(t)$, $t \in I$.

Now, let $r = v_1 - w_1$. According to (2.16) and (H_3), we have

$$\begin{cases} r^{(\alpha)}(t) & = f(t, v_0(t)) - f(t, w_0(t)) - M\left(v_1(t) - v_0(t) - w_1(t) - w_0(t)\right) \\ & \leq M\left(w_0(t) - v_0(t)\right) - M\left(v_1(t) - v_0(t) - w_1(t) - w_0(t)\right) \\ & = -Mr(t), \\ r(0) & = \int_0^1 r(t)dt. \end{cases}$$

By Lemma 2.2.3, we have $r(t) \leq 0$ for every $t \in I$, then $v_1(t) \leq w_1(t)$ for every $t \in I$.

Secondly, we show that v_1, w_1 are lower and upper solutions of (2.1), respectively. We have

$$\begin{cases} v_1^{(\alpha)}(t) &= f(t, v_0(t)) - M\left(v_1(t) - v_0(t)\right) - f(t, v_1(t)) + f(t, v_1(t)) \\ &\leq M\left(v_1(t) - v_0(t)\right) - M\left(v_1(t) - v_0(t)\right) + f(t, v_1(t)) \\ &\leq f(t, v_1(t)), \\ v_1(0) &= \displaystyle\int_0^1 v_1(t)dt, \end{cases}$$

and

$$\begin{cases} w_1^{(\alpha)}(t) &= f(t, w_0(t)) - M\left(w_1(t) - w_0(t)\right) - f(t, w_1(t)) + f(t, w_1(t)) \\ &\geq M\left(w_1(t) - w_0(t)\right) - M\left(w_1(t) - w_0(t)\right) + f(t, w_1(t)) \\ &\geq f(t, w_1(t)), \\ w_1(0) &= \displaystyle\int_0^1 w_1(t)dt. \end{cases}$$

According to (H_3) and Definition 2.3.2, we deduce that v_1, w_1 are lower and upper solutions of (2.1), respectively. By the above arguments and mathematical induction, it is clear that:

$$v_0 \leq ... \leq v_n \leq v_{n+1} \leq w_{n+1} \leq w_n \leq .... \leq w_0, \quad n = 0, 1, 2, ... \tag{2.17}$$

Thirdly, we show that $\lim_{n\to\infty} v_n(t) = v$ and $\lim_{n\to\infty} w_n = w$. Hence, we need to conclude that v_n, w_n are uniformly bounded and equicontinuous on $[0, 1]$. Obviously, the uniform boundedness of sequences v_n, w_n follows from (2.17). Thus, there exists $K > 0$ such that for all $n \in \mathbb{N}$.

$$|f(t, v_n(t)) - M(v_{n+1}(t) - v_n(t))| \leq K$$

and

$$|f(t, w_n(t)) - M(w_{n+1}(t) - w_n(t))| \leq K$$

Using Theorem 1.2.11, we get

$$\begin{aligned} |v_n(t_1) - v_n(t_2)| &= \frac{1}{\alpha}|t_1^\alpha - t_2^\alpha||v_n^{(\alpha)}(\theta)| \\ &= \frac{1}{\alpha}|t_1^\alpha - t_2^\alpha||f(\theta, v_{n-1}(\theta)) - M\left(v_n(\theta) - v_{n-1}(\theta)\right)| \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Therefore, $\{v_n\}$ are equicontinuous. Similarly, we obtain that $\{w_n\}$ are equicontinuous too. By the Arzelà-Ascoli Theorems, we conclude that the sequences $\{v_n\}$ and $\{w_n\}$ have subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$, respectively, such that $v_{n_k} \to v$ and $w_{n_k} \to w$ as $k \to \infty$. This, together with the monotonicity of the sequences $\{v_n\}$ and $\{w_n\}$, implies

$$\lim_{n\to\infty} v_n(t) = v(t), \quad \lim_{n\to\infty} w_n(t) = w(t)$$

uniformly on $[0, 1]$. Please note that the sequence $\{v_n\}$ satisfies

$$\begin{cases} v_n(t) &= e^{-\frac{M}{\alpha}t^\alpha}\left[v_{n-1}(0) + \displaystyle\int_0^t s^{\alpha-1}\left(f(t, v_{n-1}(s)) + Mv_{n-1}(s)\right)e^{\frac{M}{\alpha}s^\alpha}ds\right], \quad t \in [0, 1], \\ v_n(0) &= \displaystyle\int_0^1 v_n(t)dt, \quad n = 1, 2, \cdots \end{cases} \tag{2.18}$$

Let $n \to \infty$ in (2.18). We have

$$\begin{cases} v(t) &= e^{-\frac{M}{\alpha}t^\alpha}\left(v(0) + \displaystyle\int_0^t s^{\alpha-1}\left(f(t, v(s)) + Mv(s)\right)e^{\frac{M}{\alpha}s^\alpha}ds\right), \quad t \in [0, 1], \\ v(0) &= \displaystyle\int_0^1 v(t)dt. \end{cases}$$

This shows that v is a solution of the nonlinear problem (2.1). Similarly, we obtain w is a solution of the nonlinear problem (2.1) too. And

$$v_0(t) \le v(t) \le w(t) \le w_0(t), \quad t \in [0, 1].$$

Finally, we are going to prove that v, w are minimal and maximal solutions of (2.1) in the sector $[v_0, w_0]$. In the following, we show this using induction arguments. Suppose that $h(t)$ is any solution of (2.1) in the $[v_0, w_0]$ that is

$$v_0(t) \le h(t) \le w_0(t), \quad t \in [0, 1].$$

Assume that $v_n(t) \le h(t) \le w_n(t)$ hold. Let $p(t) = v_{n+1}(t) - h(t)$, we have

$$\begin{cases} p^{(\alpha)}(t) &= v_{n+1}^{(\alpha)}(t) - h^{(\alpha)}(t) \\ &= f(t, v_n(t)) - M\left(v_{n+1}(t) - v_n(t)\right) - f(t, g(t)) \\ &\le M\left(g(t) - v_n(t)\right) - M\left(v_{n+1}(t) - v_n(t)\right) \\ &= -Mp(t), \\ p(0) &= \displaystyle\int_0^1 p(t)dt. \end{cases} \tag{2.19}$$

Then, by Lemma 2.2.3, we have $v_{n+1}(t) \leq g(t),\ t \in [0,1]$. By similar method, we can show that $h(t) \leq w_{n+1}(t),\ t \in [0,1]$. Therefore,

$$v_n \leq h \leq w_n, \quad n = 1, 2, ...$$

By taking $n \to \infty$ in the above inequalities, we get that $v \leq h \leq w$. That is v, w are extremal solutions of problem (2.1) in $[v_0, w_0]$. Thus, the proof is finished.

2.4 Examples

In this section, we present two examples where we apply Theorem 2.3.3.

Example 2.4.1. *Consider the conformable fractional boundary value problem:*

$$\begin{cases} x^{(\frac{1}{2})}(t) & = -\frac{2}{9}(1+x(t))^3 + \frac{1}{9}\sin t, \quad t \in [0,1], \\ x(0) & = \displaystyle\int_0^1 x(t)dt, \end{cases} \tag{2.20}$$

where, $\alpha = \frac{1}{2}$ and $f(t,x) = -\frac{2}{9}(1+x)^3 + \frac{1}{9}\sin t\ t \in [0,1]$. It is clear that f is continuous function. Take $v_0(t) = -1 \leq w_0(t) = 0$ for $t \in [0,1]$, then

$$\begin{cases} v_0^{(\frac{1}{2})}(t) = 0 \leq \frac{1}{9}\sin t = f(t, v_0(t)), \\ v_0(0) = -1 \leq \displaystyle\int_0^1 v_0(t)dt = -1, \end{cases} \tag{2.21}$$

and

$$\begin{cases} w_0^{(\frac{1}{2})}(t) = 0 > -\frac{1}{9}(2-\sin t) = f(t, w_0(t)), \\ w_0(0) = 0 = \displaystyle\int_0^1 w_0(t)dt. \end{cases} \tag{2.22}$$

Then v_0, w_0 are lower and upper solutions of (2.20) respectively, then assumptions (H_1) and (H_2) holds. Let $x, \overline{x} \in \mathbb{R}$, with $v_0(t) \leq x \leq \overline{x} \leq w_0(t),\ t \in I$, then we have $0 \leq 1 + x \leq 1 + \overline{x} \leq 1$ and

$$\begin{aligned} f(t,x) - f(t,\overline{x}) &= -\frac{2}{9}(1+x)^3 + \frac{2}{9}(1+\overline{x})^3 = \frac{2}{9}\left((1+\overline{x})^3 - (1+x)^3\right) \\ &= \frac{2}{9}(\overline{x} - x)\left((1+\overline{x})^2 + (1+x)(1+\overline{x}) + (1+\overline{x})^2\right) \\ &\leq \frac{2}{9}(\overline{x} - x)(3) = \frac{2}{3}(\overline{x} - x) \\ &\leq 1.(\overline{x} - x). \end{aligned}$$

Hence the assumption (H_3) *holds with* $M = 1$. *In addition,*

$$\int_0^1 e^{-\frac{t^{(\frac{1}{2})}}{\frac{1}{2}}} dt = \int_0^1 e^{-2t^{\frac{1}{2}}} dt < \int_0^1 dt = 1.$$

By Theorem 2.3.3, problem (2.20) has an extremal iterative solutions $(v, w) \in [v_0, w_0] \times [v_0, w_0]$ *on* $[0, 1]$, *which can be obtained by taking limits from the iterative sequences:*

$$v_n(t) = e^{-2t^{1/2}} \left[v_{n-1}(0) + \int_0^t s^{-1/2} \left(-\frac{2}{9}(1 + v_{n-1}(s))^3 + \frac{1}{9} \sin s + v_{n-1}(s) \right) e^{2s^{1/2}} ds \right],$$
$$w_n(t) = e^{-2t^{1/2}} \left[w_{n-1}(0) + \int_0^t s^{-1/2} \left(-\frac{2}{9}(1 + w_{n-1}(s))^3 + \frac{1}{9} \sin s + w_{n-1}(s) \right) e^{2s^{1/2}} ds \right].$$

Example 2.4.2. *Consider the following problem:*

$$\begin{cases} x'(t) &= -x^2(t) + tx(t), \quad t \in [0, 1], \\ x(0) &= \int_0^1 x(t) dt, \end{cases} \tag{2.23}$$

where, $\alpha = 1$ *and* $f(t, x) = -x^2 + tx$. *It is clear that* f *is continuous function. Take* $v_0(t) = 0 \le w_0(t) = 1$ *for* $t \in [0, 1]$, *then*

$$\begin{cases} v_0'(t) = 0 \le f(t, v_0(t)) = 0, \\ v_0(0) = 0 \le \int_0^1 v_0(t) dt = 0, \end{cases} \tag{2.24}$$

and

$$\begin{cases} w_0'(t) = 0 \ge f(t, w_0(t)) = t - 1, \\ w_0(0) = 1 \ge \int_0^1 w_0(t) dt = 1. \end{cases} \tag{2.25}$$

Then v_0, w_0 *are lower and upper solutions of (2.23) respectively, then assumptions* (H_1) *and* (H_2) *holds. Let* $x, \overline{x} \in \mathbb{R}$, *with* $v_0(t) \le x \le \overline{x} \le w_0(t)$, $t \in I$, *then*

$$\begin{aligned} f(t, x) - f(t, \overline{x}) &= -x^2 + tx + \overline{x}^2 - t\overline{x} = \left(\overline{x}^2 - x^2\right) + t\,(x - \overline{x}) \\ &\le \left(\overline{x}^2 - x^2\right) = (\overline{x} - x)\,(\overline{x} + x) \\ &\le 2(\overline{x} - x). \end{aligned}$$

Hence the assumption (H_3) *holds with* $M = 2$. *In addition,*

$$\int_0^1 e^{-2t} dt = \frac{1}{2}(1 - e^{-2}) \le \frac{1}{2} < 1.$$

By Theorem 2.3.3, problem (2.23) has an extremal iterative solutions $(v, w) \in [v_0, w_0] \times [v_0, w_0]$ *on* $[0, 1]$*, which can be obtained by taking limits from the iterative sequences:*

$$v_n(t) = e^{-2t} \left[v_{n-1}(0) + \int_0^t \left(-v_{n-1}^2(s) + s v_{n-1}(s) + 2 v_{n-1}(s) \right) e^{2s} ds \right],$$

$$w_n(t) = e^{-2t} \left[w_{n-1}(0) + \int_0^t \left(-w_{n-1}^2(s) + s w_{n-1}(s) + 2 w_{n-1}(s) \right) e^{2s} ds \right].$$

Chapter 3

Extremal solutions to a coupled system of first order ordinary differential equations

In this chapter, we investigate the existence of extremal solutions for a coupled system of nonlinear first order ordinary differential equations by using the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions:

$$\begin{cases} x^{'}(t) = f(t, x(t), y(t)), & t \in I = [a, b], \\ y^{'}(t) = g(t, y(t), x(t)), & t \in I = [a, b], \\ x(a) = \lambda_0, \; y(a) = \beta_0. \end{cases} \tag{3.1}$$

where $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\lambda_0, \beta_0 \in \mathbb{R}$ with $\lambda_0 \leq \beta_0$.

Now we enunciate the following existence and uniqueness results for initial linear equations and initial linear system.

Lemma 3.0.3. *Let $M \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. If $g \in C([a, b], \mathbb{R})$, then the linear initial value problem:*

$$\begin{cases} x^{'}(t) + Mx(t) = g(t), & t \in I = [a, b], \\ x(a) = x_0, \end{cases} \tag{3.2}$$

has a unique solution $x \in C([a, b], \mathbb{R})$, and it is given by the following expression:

$$x(t) = e^{-Mt} \left(x_0 e^{Ma} + \int_a^t e^{Ms} g(s) ds \right). \tag{3.3}$$

Proof. We first consider the associated linear homogeneous equation:

$$\begin{cases} x^{'}(t) + Mx(t) = 0, & t \in I = [a, b], \\ x(a) = x_0, \end{cases} \tag{3.4}$$

We can easily get that the solution to (3.4) is $x(t) = Ce^{-Mt}$, where C is constant. By $x(a) = x_0$, we have $C = x_0 e^{Ma}$.
Hence, the solution to (3.4) is $x(t) = x_0 e^{Ma} e^{-Mt}$.
Let $x(t) = C(t)e^{-Mt}$ is the general solution to problem (3.2), and then we can get

$$C(t) = x_0 e^{Ma} + \int_a^t e^{Ms} g(s) ds$$

Hence, the solution to (3.2) is

$$x(t) = e^{-Mt} \left(x_0 e^{Ma} + \int_a^t e^{Ms} g(s) ds \right).$$

□

Lemma 3.0.4. *Let $M, N \in \mathbb{R}$, $N \geq 0$ and $h_1, h_2 \in C(I, \mathbb{R})$. The linear system*

$$\begin{cases} x^{'}(t) = h_1(t) - M\ x(t) - N\ y(t), & \textit{for } t \in I = [a, b], \\ y^{'}(t) = h_2(t) - M\ y(t) - N\ x(t), & \textit{for } t \in I = [a, b], \\ x(a) = \lambda_0,\ y(a) = \beta_0, \end{cases} \tag{3.5}$$

has a unique system of solutions $(x, y) \in C^1([a, b], \mathbb{R}) \times C^1([a, b], \mathbb{R})$, with

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, \quad t \in I = [a, b],$$

where

$$z(t) = e^{-(M+N)t} \left((\lambda_0 + \beta_0) e^{(M+N)a} + \int_a^t e^{(M+N)s} (h_1 + h_2)(s) ds \right),$$
$$w(t) = e^{-(M-N)t} \left((\lambda_0 - \beta_0) e^{(M-N)a} + \int_a^t e^{(M-N)s} (h_1 - h_2)(s) ds \right).$$

Proof. The pair $(x, y) \in C^1([a, b], \mathbb{R}) \times C^1([a, b], \mathbb{R})$ is a solution to system (3.5) if and only if

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, t \in [a, b]$$

where $z(t)$ and $w(t)$ are the solutions to the following problems:

$$\begin{cases} z^{'}(t) = (h_1 + h_2)(t) - (M+N)z(t), \quad t \in [a,b], \\ z(a) = \lambda_0 + \beta_0, \end{cases}$$
$$\begin{cases} w^{'}(t) = (h_1 - h_2)(t) - (M-N)w(t), \quad t \in [a,b], \\ w(a) = \lambda_0 - \beta_0, \end{cases}$$

By Lemma 3.0.3, we have

$$z(t) = e^{-(M+N)t}\left((\lambda_0 + \beta_0)e^{(M+N)a} + \int_a^t e^{(M+N)s}(h_1 + h_2)(s)ds\right),$$
$$w(t) = e^{-(M-N)t}\left((\lambda_0 - \beta_0)e^{(M-N)a} + \int_a^t e^{(M-N)s}(h_1 - h_2)(s)ds\right).$$

□

In the next Lemma, we prove a comparison result for the initial linear problem (3.2).

Lemma 3.0.5. *Let $x \in C^1([a,b],\mathbb{R})$ satisfy*

$$\begin{cases} x^{'}(t) + Mx(t) \geqslant 0, \quad t \in [a,b] \\ x(a) \geqslant 0, \end{cases}$$

where $M \in \mathbb{R}$, then $x(t) \geqslant 0$ for all $t \in [a,b]$.

Proof. we put $x^{'}(t) + Mx(t) = g(t)$ and $x(a) = x_0 \geq 0$. We know that $g(t) \geq 0$, for every $t \in I = [a,b]$ and

$$\begin{cases} x^{'}(t) + Mx(t) = g(t), \qquad t \in [a,b], \\ x(a) = x_0 \geq 0. \end{cases} \tag{3.6}$$

By Lemma 3.0.3, the expression of $x(t)$ is:

$$x(t) = e^{-Mt}\left(x_0 e^{Ma} + \int_a^t e^{Ms}g(s)ds\right),$$

we can conclude that, $x(t) \geq 0$ for every $t \in I = [a,b]$. □

Now we are in a position to prove the following comparison result for system (3.5).

Lemma 3.0.6. *(Comparison principle).*
Let $(x, y) \in C^1([a, b], \mathbb{R}) \times C^1([a, b], \mathbb{R})$ *satisfy*

$$\begin{cases} x^{'}(t) \geq -M\ x(t) + N\ y(t), & \textit{for } t \in I = [a, b], \\ y^{'}(t) \geq -M\ y(t) + N\ x(t), & \textit{for } t \in I = [a, b], \\ x(a) \geq 0,\ y(a) \geq 0, \end{cases} \tag{3.7}$$

where $M, N \in \mathbb{R}$ *with* $N \geq 0$. *Then* $x(t) \geq 0, y(t) \geq 0$ *for all* $t \in [a, b]$.

Proof. Let $u(t) = x(t) + y(t)$, then (3.7) is equivalent to the following:

$$\begin{cases} u^{'}(t) \geq -(M - N)u(t), & t \in [a, b] \\ u(a) \geq 0, \end{cases}$$

i.e.,

$$\begin{cases} u^{'}(t) + \overline{M}u(t) \geqslant 0, & t \in [a, b], \\ u(a) \geq 0. \end{cases}$$

By Lemma 3.0.5, we know that

$$u(t) \geq 0, \text{for all} t \in [a, b], \quad \text{i.e.,} \quad x(t) + y(t) \geq 0, \text{for all} t \in [a, b].$$

By (3), we have

$$\begin{cases} x^{'}(t) + (M + N)x(t) \geq 0, & \text{for } t \in I = [a, b], \\ y^{'}(t) + (M + N)y(t) \geq 0, & \text{for } t \in I = [a, b], \\ x(a) \geq 0,\ y(a) \geq 0. \end{cases}$$

By Lemma 3.0.5, we have $x(t) \geq 0, y(t) \geq 0$ for all $t \in [a, b]$. The proof is completed. □

3.1 Main Result

In this section, we prove the existence of extremal solutions for problem (3.1). Let us defining what we mean by a solution of this problem.

Definition 3.1.1. *A solution of problem* (3.1) *will be a pair* $(x, y) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$ *for which* (3.1) *is satisfied.*

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 3.1.2. *We say that* $\gamma,\ \delta \in C^1(I,\mathbb{R})$ *is a pair of coupled lower and upper solutions of the problem* (3.1)*, if* $\gamma(t) \leq \delta(t)$ *for all* $t \in I$ *and the following inequalities hold:*

$$\begin{cases} \gamma'(t) \leq f(t,\gamma(t),\delta(t)), & \textit{for} \ \ t \in I,\ \gamma(a) \leq \lambda_0, \\ \delta'(t) \geq g(t,\delta(t),\gamma(t)), & \textit{for} \ \ t \in I,\ \delta(a) \geq \beta_0. \end{cases} \tag{3.8}$$

We assume the following hypothesis:

(F_1) $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

(F_2) There exists $\gamma, \delta \in C^1(I,\mathbb{R})$, a pair of coupled lower and upper solutions of the problem (3.1).

(F_3) There exist constants $M \in \mathbb{R}$ and $N \geq 0$ such that

$$\begin{cases} f(t,x,y) - f(t,\overline{x},\overline{y}) \geq -M(x-\overline{x}) - N(y-\overline{y}), \\ g(t,\overline{y},\overline{x}) - g(t,y,x) \geq -M(\overline{y}-y) - N(\overline{x}-x), \end{cases}$$

where $\gamma(t) \leq \overline{x} \leq x \leq \delta(t)$, $\gamma(t) \leq y \leq \overline{y} \leq \delta(t)$ for all $t \in I$, and

$$g(t,y,x) - f(t,x,y) \geq -M(y-x) - N(x-y),$$

where $\gamma(t) \leq x \leq y \leq \delta(t)$ for all $t \in I$.

Now, we can obtain our main theorem.

Theorem 3.1.3. *Assume that* (F_1), (F_2) *and* (F_3) *hold. Then* (3.1) *has an extremal system of solutions* $(x^*, y^*) \in [\gamma,\delta] \times [\gamma,\delta]$*, and there exist two monotone iterative sequences* $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ *converging uniformly to* x^*, y^**, respectively, where* $x_n, y_n \in [\gamma,\delta]$*, such that*

$$\gamma = x_0 \leq x_1 \leq \cdots \leqslant x_n \leqslant \cdots \leqslant x^* \leqslant y^* \leqslant \cdots \leqslant y_n \leqslant \cdots \leqslant y_1 \leqslant y_0 = \delta,\ on I,\ \forall n \in \mathbb{N}.$$

Proof. Firstly, for all $x_n, y_n \in C^1([a,b],\mathbb{R})$, we consider the linear system:

$$\begin{cases} x_{n+1}'(t) = f(t, x_n(t), y_n(t)) + M\left(x_n(t) - x_{n+1}(t)\right) + N\left(y_n(t) - y_{n+1}(t)\right), t \in [a,b], \\ y_{n+1}'(t) = g(t, y_n(t), x_n(t)) + M\left(y_n(t) - y_{n+1}(t)\right) + N\left(x_n(t) - x_{n+1}(t)\right), t \in [a,b], \\ x_{n+1}(a) = \lambda_0, \quad y_{n+1}(a) = \beta_0. \end{cases} \tag{3.9}$$

By Lemma 3.0.4, the linear system (3.9) has a unique system of solutions in $C^1([a,b],\mathbb{R}) \times C^1([a,b],\mathbb{R})$, which is defined by

$$x_{n+1}(t) = \frac{p_{n+1}(t) + q_{n+1}(t)}{2}, \quad y_{n+1}(t) = \frac{p_{n+1}(t) - q_{n+1}(t)}{2}, \text{for all } t \in [a,b] \tag{3.10}$$

where

$$\begin{aligned} p_{n+1}(t) = & e^{-(M+N)t}\Big((\lambda_0 + \beta_0)e^{(M+N)a} + \int_a^t e^{(M+N)s}\left[f(s, x_n(s), y_n(s))\right. \\ & \left. + g(s, y_n(s), x_n(s)) + (M+N)\left(x_n(s) + y_n(s)\right)\right] ds\Big), \end{aligned} \tag{3.11}$$

$$\begin{aligned} q_{n+1}(t) = & e^{-(M-N)t}\Big((\lambda_0 - \beta_0)e^{(M-N)a} + \int_a^t e^{(M-N)s}\left[f(s, x_n(s), y_n(s))\right. \\ & \left. - g(s, y_n(s), x_n(s)) + (M-N)\left(x_n(s) - y_n(s)\right)\right] ds\Big). \end{aligned} \tag{3.12}$$

Secondly, we shall prove that

$$x_n \leqslant x_{n+1} \leqslant y_{n+1} \leqslant y_n, \quad on\ I\ for\ all\ n \in \mathbb{N}.$$

Let $p = x_1 - x_0, q = y_0 - y_1$. According to (3.9) and $(F_1) - (F_2)$, we have

$$\begin{cases} p'(t) \geqslant M\left(x_0(t) - x_1(t)\right) + N\left(y_0(t) - y_1(t)\right), \quad \text{for } t \in I \\ p(a) \geqslant \lambda_0 - \lambda_0 = 0, \\ q'(t) \geqslant -M\left(y_0(t) - y_1(t)\right) - N\left(x_0(t) - x_1(t)\right), \quad \text{for } t \in I \\ q(a) \geqslant \beta_0 - \beta_0 = 0, \end{cases}$$

i.e.,

$$\begin{cases} p'(t) \geqslant -Mp(t) + Nq(t), \text{ for } t \in I \quad p(a) \geqslant 0, \\ q'(t) \geqslant -Mq(t) + Np(t), \text{ for } t \in I \quad q(a) \geqslant 0. \end{cases}$$

Then, by Lemma 3.0.6, we have $p(t) \geqslant 0,\ q(t) \geqslant 0$, i.e., $x_1 \geqslant x_0,\ y_1 \leqslant y_0$.

Let $w = y_1 - x_1$. According to (3.9) and (F_3), we have

$$\begin{cases} w^{'}(t) & = y_1^{'}(t) - x_1^{'}(t) \\ & = g\left(t, y_0(t), x_0(t)\right) + M\left(y_0(t) - y_1(t)\right) + N\left(x_0(t) - x_1(t)\right) \\ & \quad -f\left(t, x_0(t), y_0(t)\right) - M\left(x_0(t) - x_1(t)\right) - N\left(y_0(t) - y_1(t)\right) \\ & \geqslant -M\left(y_1(t) - x_1(t)\right) + N\left(y_1(t) - x_1(t)\right) = -(M-N)w(t), \\ w(a) & = \beta_0 - \lambda_0 \geqslant 0. \end{cases}$$

By Lemma 3.0.5, we have $w(t) \geqslant 0$, i.e., $y_1(t) \geqslant x_1(t)$ for all $t \in I = [a, b]$. By mathematical induction, we can prove that

$$x_n \leqslant x_{n+1} \leqslant y_{n+1} \leqslant y_n, \quad on\ I\ for\ all\ n \in \mathbb{N}.$$

Thirdly, the sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ are monotone and bounded, hence

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*$$

(x^*, y^*) is an extremal system of solutions to (3.1).

Finally, we prove that (3.1) has at most one extremal system of solutions. Assume that $(x, y) \in [\gamma = x_0, \delta = y_0] \times [x_0, y_0]$ is the system of solutions to (3.1), then

$$x_0 = \gamma \leqslant x, \quad y \leqslant y_0 = \delta$$

. For some $k \in \mathbb{N}$, assume that the following relation holds

$$x_k(t) \leqslant x(t), \quad y(t) \leqslant y_k(t), \quad t \in [a, b].$$

Let $u(\mathrm{t}) = \mathrm{x(t)} - \mathrm{x_{k+1}(t)}, \quad v(\mathrm{t}) = \mathrm{y_{k+1}(t)} - \mathrm{y(t)}$. According to (3.9) and (F_3), we have

$$\begin{cases} u^{'}(t) & = x^{'}(t) - x^{'}_{k+1}(t) \\ & = f\left(t, x(t), y(t)\right) - f(t, x_k(t), y_k(t)) - M\left(x_k(t) - x_{k+1}(t)\right) \\ & \qquad -N\left(y_k(t) - y_{k+1}(t)\right), \\ & \geq -M\left(x(t) - x_k(t)\right) - N\left(y(t) - y_k(t)\right) - M\left(x_k(t) - x_{k+1}(t)\right) \\ & \qquad -N\left(y_k(t) - y_{k+1}(t)\right) \\ & = -M\left(x(t) - x_{k+1}(t)\right) + N\left(y_{k+1}(t) - y(t)\right). \end{cases}$$

and

$$\begin{cases} v^{'}(t) & = y^{'}_{k+1}(t) - y^{'}(t) \\ & = g\left(t, y_k(t), x_k(t)\right) + M\left(y_k(t) - y_{k+1}(t)\right) + N\left(x_k(t) - x_{k+1}(t)\right) \\ & \qquad -g(t, y(t), x(t)) \\ & \geq -M\left(y_k(t) - y(t)\right) - N\left(x_k(t) - x(t)\right) + M\left(y_k(t) - y_{k+1}(t)\right) \\ & \qquad +N\left(x_k(t) - x_{k+1}(t)\right. \\ & = -M\left(y_{k+1}(t) - y(t)\right) + N\left(x(t) - x_{k+1}(t)\right), \end{cases}$$

we can get

$$\begin{cases} u^{'}(t) \geq -Mu(t) + Nv(t), \text{ for } t \in I \quad u(a) \geq 0, \\ v^{'}(t) \geq -Mv(t) + Nu(t), \text{ for } t \in I \quad v(a) \geq 0. \end{cases}$$

Then, by Lemma 3.0.6, we have $u(t) \geqslant 0,\ v(t) \geqslant 0$, i.e.,

$$x_{k+1}(t) \leqslant x(t),\ y(t) \leqslant y_{k+1}(t),\ t \in [a, b].$$

By the induction arguments, the following relation holds

$$x_n(t) \leqslant x(t), \quad y(t) \leqslant y_n(t), \quad on\ I \quad for\ all\ n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$, we get that $x^* \leqslant x,\ y \leqslant y^*$.
Hence, $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ is the extremal system of solutions to (3.1).
So the proof is finished. □

3.2 An example

Consider the following system of nonlinear first order ordinary differential equations:

$$\begin{cases} x^{'}(t) = 2t^3(t-x(t))^3 - t^4y^2(t), & t \in I = [0,1], \\ y^{'}(t) = 2t^3(t-y(t))^3 - t^4x^2(t), & t \in I = [0,1], \\ x(0)=0, \quad y(0)=0, \end{cases} \tag{3.13}$$

where $f(t,x,y) = 2t^3(t-x)^3 - t^4y^2$ and $g(t,y,x) = 2t^3(t-y)^3 - t^4x^2$.
It is clear that f,g are continuous functions. Take $\gamma(t) = 0$ and $\delta(t) = t$ for $t \in [0,1]$, then

$$\gamma^{'}(t) = 0 \leq f(t,\gamma(t),\delta(t)) = 2t^6 \; for \; t \in [0,1], \quad \gamma(0) = 0 \leq 0,$$

and

$$\delta^{'}(t) = 1 \geq g(t,\delta(t),\gamma(t)) = 0 \; for \; t \in [0,1], \quad \delta(0) = 0 \geq 0.$$

So, γ and δ, are lower and upper solutions of problem (3.13), respectively with $\gamma(t) = 0 \leq \delta(t) = t \; for \; t \in [0,1]$, then assumptions (F_1) and (F_2) holds.
Let $x, \overline{x}, y, \overline{y} \in \mathbb{R}$, then we have:

$$\begin{aligned} f(t,x,y) - f(t,\overline{x},\overline{y}) &= 2t^3[(t-x)^3 - (t-\overline{x})^3] - t^4(y^2 - \overline{y}^2) \\ &\geq -2t^3(x-\overline{x})[(t-x)^2 + (t-x)(t-\overline{x}) + (t-\overline{x})^2] \\ &\geq -6t^3(x-\overline{x}) \\ &\geq -6(x-\overline{x}) - 0(y-\overline{y}), \end{aligned}$$

$$\begin{aligned} g(t,\overline{y},\overline{x}) - g(t,y,x) &= 2t^3[(t-\overline{y})^3 - (t-y)^3] - t^4(\overline{x}^2 - x^2) \\ &\geq -2t^3(\overline{y}-y)[(t-\overline{y})^2 + (t-\overline{y})(t-y) + (t-y)^2] \\ &\geq -6t^3(\overline{y}-y) \\ &\geq -6(\overline{y}-y) - 0(\overline{x}-x), \end{aligned}$$

with $\gamma(t) \leq \overline{x} \leq x \leq \delta(t)$, $\gamma(t) \leq y \leq \overline{y} \leq \delta(t)$ for all $t \in I$, and we have

$$\begin{aligned} g(t,y,x) - f(t,x,y) &= 2t^3[(t-y)^3 - (t-x)^3] - t^4(x^2 - y^2) \\ &\geq -2t^3(y-x)[(t-y)^2 + (t-y)(t-x) + (t-x)^2] \\ &\geq -6t^3(y-x) \\ &\geq -6(y-x) + 0.(x-y). \end{aligned}$$

with $\gamma(t) \leq x \leq y \leq \delta(t)$, for all $t \in I$.
Hence the assumption (F_3) holds with $M = 6$ and $N = 0$. By Theorem 3.1.3, the

nonlinear system (3.13) has the extremal solution $(x^*, y^*) \in C^1([0,1]) \times C^1([0,1])$, such that $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ on $[0,1]$, which can be obtained by taking limits from the iterative sequences:

$$x_{n+1}(t) = e^{-6t} \int_0^t e^{6s} \left[2s^3(s - x_n(s))^3 - s^4 y_n^2(s) + 6x_n(s)\right] ds, \quad t \in I = [0,1],$$

$$y_{n+1}(t) = e^{-6t} \int_0^t e^{6s} \left[2s^3(s - y_n(s))^3 - s^4 x_n^2(s) + 6y_n(s)\right] ds, \quad t \in I = [0,1].$$

Chapter 4

Extremal solutions to a coupled system of conformable fractional differential equations

In this chapter, we investigate the existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations by using the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t), y(t)), & t \in I = [a, b], \\ y^{(\alpha)}(t) = g(t, y(t), x(t)), & t \in I = [a, b], \\ x(a) = \lambda_0, \ y(a) = \beta_0. \end{cases} \tag{4.1}$$

where $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\lambda_0, \beta_0 \in \mathbb{R}$, $\lambda_0 \leq \beta_0$, $x^{(\alpha)}$, $y^{(\alpha)}$ are the conformable fractional derivatives with $0 < \alpha \leq 1$.

B. Bendouma. in [7], studied the existence of extremal iteration solution to the following coupled system of nonlinear conformable fractional dynamic equations on time scales:

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t), y^{\sigma}(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ y_{\Delta}^{(\alpha)}(t) = g(t, y^{\sigma}(t), x^{\sigma}(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ x(a) = \lambda_0, \ y(a) = \beta_0. \end{cases} \tag{4.2}$$

where, $\mathbb{T}$ is an arbitrary bounded time scale, $J = [a, \sigma(b)]_{\mathbb{T}}$ with $a, b \in \mathbb{T}$, $0 < a < b$, $\lambda_0, \ \beta_0 \in \mathbb{R}$, $\lambda_0 \leq \ \beta_0$, $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $x_{\Delta}^{(\alpha)}$, $y_{\Delta}^{(\alpha)}$ are the conformable fractional derivatives (on time scales) with $\alpha \in (0, 1]$.

4.1 Linear fractional differential equations

In this section, we study the expression of the solutions of a linear conformable fractional differential equation with initial value conditions.

Lemma 4.1.1. *Let $0 < \alpha \leq 1$, $M \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. If $g \in C([a,b],\mathbb{R})$, then the linear problem:*

$$\begin{cases} x^{(\alpha)}(t) + Mx(t) = g(t), \qquad t \in I = [a,b], \\ x(a) = x_0, \end{cases} \tag{4.3}$$

has a unique solution $x \in C([a,b],\mathbb{R})$, and it is given by the following expression:

$$x(t) = e^{-\frac{M}{\alpha}t^\alpha}\left(x_0 e^{\frac{M}{\alpha}a^\alpha} + \int_a^t s^{\alpha-1}e^{\frac{M}{\alpha}s^\alpha}g(s)ds\right). \tag{4.4}$$

Proof. Let x be a solution of problem (4.3). By Theorem 1.2.3, we have that the following property holds:

$$\begin{aligned} [e^{\frac{M}{\alpha}(t^\alpha - a^\alpha)}x(t)]^{(\alpha)} &= e^{\frac{M}{\alpha}(t^\alpha - a^\alpha)}x^\alpha(t) + Mx(t)e^{\frac{M}{\alpha}(t^\alpha - a^\alpha)} \\ &= e^{\frac{M}{\alpha}(t^\alpha - a^\alpha)}\left(x^\alpha(t) + Mx(t)\right) \\ &= e^{\frac{M}{\alpha}(t^\alpha - a^\alpha)}g(t). \end{aligned}$$

Applying I_α^a the conformable fractional integral of order α to both sides of, we have

$$\begin{aligned} e^{\frac{M}{\alpha}(t^\alpha - a^\alpha)}x(t) - x(a) &= I_\alpha^a\left[e^{\frac{M}{\alpha}(t^\alpha - a^\alpha)}g(t)\right] \\ &= \int_a^t s^{\alpha-1}e^{\frac{M}{\alpha}(s^\alpha - a^\alpha)}g(s)ds. \end{aligned}$$

Then

$$\begin{aligned} x(t) &= e^{-\frac{M}{\alpha}(t^\alpha - a^\alpha)}\left(x(a) + \int_a^t s^{\alpha-1}e^{\frac{M}{\alpha}(s^\alpha - a^\alpha)}g(s)ds\right) \\ &= e^{-\frac{M}{\alpha}t^\alpha}\left(e^{\frac{M}{\alpha}a^\alpha}x_0 + \int_a^t s^{\alpha-1}e^{\frac{M}{\alpha}s^\alpha}g(s)ds\right). \end{aligned} \tag{4.5}$$

Thus problem (4.3) has a unique solution. The proof is finished. □

In the next Lemmas, we discuss comparison results for the linear problem (4.3)

Lemma 4.1.2. *Let $x \in C^\alpha([a,b],\mathbb{R})$ satisfy*

$$\begin{cases} x^{(\alpha)}(t) + Mx(t) \geqslant 0, & t \in [a,b] \\ x(a) \geqslant 0, \end{cases}$$

where $0 < \alpha \leqslant 1, M \in \mathbb{R}$, then $x(t) \geqslant 0$ for all $t \in [a,b]$.

Proof. we put $x^\alpha(t)+Mx(t) = g(t)$ and $x(a) = x_0 \geq 0$. We are know that $g(t) \geq 0$, for every $t \in I = [a,b]$ and

$$\begin{cases} x^{(\alpha)}(t) + Mx(t) = g(t), & t \in [a,b], \\ x(a) = x_0 \geq 0. \end{cases} \tag{4.6}$$

By Lemma 4.1.1, the expression of $x(t)$ is:

$$x(t) = e^{-\frac{M}{\alpha}t^\alpha}\left(x_0 e^{\frac{M}{\alpha}a^\alpha} + \int_a^t s^{\alpha-1}e^{\frac{M}{\alpha}s^\alpha}g(s)ds\right)$$

we can conclude that, $x(t) \geq 0$ for every $t \in I = [a,b]$. $\square$

4.2 Main Result

In this section, we prove the existence of extremal solutions for problem (4.1). Let us defining what we mean by a solution of this problem.

Definition 4.2.1. *A solution of problem* (4.1) *will be a pair $(x,y) \in C^\alpha(I,\mathbb{R}) \times C^\alpha(I,\mathbb{R})$ for which* (4.1) *is satisfied.*

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 4.2.2. *We say that γ, $\delta \in C^\alpha(I,\mathbb{R})$ is a pair of coupled lower and upper solutions of the problem* (4.1)*, if $\gamma(t) \leq \delta(t)$ for all $t \in I$ and the following inequalities hold:*

$$\begin{cases} \gamma^{(\alpha)}(t) \leq f(t,\gamma(t),\delta(t)), \quad for \;\; t \in I, \;\; \gamma(a) \leq \lambda_0, \\ \delta^{(\alpha)}(t) \geq g(t,\delta(t),\gamma(t)), \quad for \;\; t \in I, \;\; \delta(a) \geq \beta_0. \end{cases} \tag{4.7}$$

We assume the following hypothesis:

(H_1) *$f,g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.*

(H_2) *There exists $\gamma, \delta \in C^{\alpha}(I, \mathbb{R})$, a pair of coupled lower and upper solutions of the problem* (4.1).

(H_3) *There exist constants $M \in \mathbb{R}$ and $N \geq 0$ such that*

$$\begin{cases} f(t, x, y) - f(t, \overline{x}, \overline{y}) \geq -M(x - \overline{x}) - N(y - \overline{y}), \\ g(t, \overline{y}, \overline{x}) - g(t, y, x) \geq -M(\overline{y} - y) - N(\overline{x} - x), \end{cases}$$

where $\gamma(t) \leq \overline{x} \leq x \leq \delta(t)$, $\gamma(t) \leq y \leq \overline{y} \leq \delta(t)$ for all $t \in I$, and

$$g(t, y, x) - f(t, x, y) \geq -M(y - x) - N(x - y),$$

where $\gamma(t) \leq x \leq y \leq \delta(t)$ for all $t \in I$.

To study the nonlinear system (4.1), *we first consider the associated linear system:*

$$\begin{cases} x^{(\alpha)}(t) = h_1(t) - M\ x(t) - N\ y(t), & \text{for } t \in I = [a, b], \\ y^{(\alpha)}(t) = h_2(t) - M\ y(t) - N\ x(t), & \text{for } t \in I = [a, b], \\ x(a) = \lambda_0,\ y(a) = \beta_0, \end{cases} \tag{4.8}$$

where $\alpha \in (0, 1]$, $(\lambda_0,\ \beta_0) \in \mathbb{R}^2$, $\lambda_0 \leq\ \beta_0$, $M, N \in \mathbb{R}$, $N \geq 0$ and $h_1, h_2 \in C(I, \mathbb{R})$.

Lemma 4.2.3. *The linear system (4.8) has a unique system of solutions $(x, y) \in C^{\alpha}([a, b], \mathbb{R}) \times C^{\alpha}([a, b], \mathbb{R})$, with*

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, \quad t \in I = [a, b],$$

where

$$z(t) = e^{-\frac{M+N}{\alpha} t^{\alpha}} \left((\lambda_0 + \beta_0)\, e^{\frac{M+N}{\alpha} a^{\alpha}} + \int_a^t s^{\alpha - 1} e^{\frac{M+N}{\alpha} s^{\alpha}} \left(h_1 + h_2\right)(s) ds \right),$$
$$w(t) = e^{-\frac{M-N}{\alpha} t^{\alpha}} \left((\lambda_0 - \beta_0)\, e^{\frac{M-N}{\alpha} a^{\alpha}} + \int_a^t s^{\alpha - 1} e^{\frac{M-N}{\alpha} s^{\alpha}} \left(h_1 - h_2\right)(s) ds \right).$$

Proof. The pair $(x, y) \in C^{\alpha}([a, b], \mathbb{R}) \times C^{\alpha}([a, b], \mathbb{R})$ is a solution to system (4.8) if and only if

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, t \in [a, b]$$

where $z(t)$ and $w(t)$ are the solutions to the following problems:

$$\begin{cases} z^{(\alpha)}(t) = (h_1 + h_2)(t) - (M+N)z(t), \quad t \in [a,b], \\ z(a) = \lambda_0 + \beta_0, \end{cases}$$
$$\begin{cases} w^{(\alpha)}(t) = (h_1 - h_2)(t) - (M-N)w(t), \quad t \in [a,b], \\ w(a) = \lambda_0 - \beta_0, \end{cases}$$

By Lemma 4.1.1, we have

$$z(t) = e^{-\frac{M+N}{\alpha}t^\alpha}\left((\lambda_0 + \beta_0)\, e^{\frac{M+N}{\alpha}a^\alpha} + \int_a^t s^{\alpha-1} e^{\frac{M+N}{\alpha}s^\alpha}(h_1 + h_2)(s)ds\right),$$
$$w(t) = e^{-\frac{M-N}{\alpha}t^\alpha}\left((\lambda_0 - \beta_0)\, e^{\frac{M-N}{\alpha}a^\alpha} + \int_a^t s^{\alpha-1} e^{\frac{M-N}{\alpha}s^\alpha}(h_1 - h_2)(s)ds\right).$$

The proof is finished. □

Lemma 4.2.4. *(Comparison principle).*
Let $(x,y) \in C^\alpha([a,b],\mathbb{R}) \times C^\alpha([a,b],\mathbb{R})$ *satisfy*

$$\begin{cases} x^{(\alpha)}(t) \geq -M\ x(t) + N\ y(t), \quad \text{for } t \in I = [a,b], \\ y^{(\alpha)}(t) \geq -M\ y(t) + N\ x(t), \quad \text{for } t \in I = [a,b], \\ x(a) \geq 0,\ y(a) \geq 0, \end{cases} \tag{4.9}$$

where $0 < \alpha \leqslant 1, M, N \in \mathbb{R}$ *with* $N \geq 0$. *Then* $x(t) \geq 0, y(t) \geq 0$ *for all* $t \in [a,b]$.

Proof. Let $u(t) = x(t) + y(t)$, then (4.9) is equivalent to the following:

$$\begin{cases} u^{(\alpha)}(t) \geq -(M-N)u(t), \quad t \in [a,b] \\ u(a) \geq 0, \end{cases}$$

i.e.,

$$\begin{cases} u^{(\alpha)}(t) + \bar{M}u(t) \geqslant 0, \quad t \in [a,b], \\ u(a) \geq 0. \end{cases}$$

By Lemma 4.1.2, we know that

$$u(t) \geq 0, \text{for all} t \in [a,b], \quad \text{i.e.,} \quad x(t) + y(t) \geq 0, \text{for all} t \in [a,b].$$

By (4.2), we have

$$\begin{cases} x^{(\alpha)}(t) + (M+N)x(t) \geq 0, & \text{for } t \in I = [a,b], \\ y^{(\alpha)}(t) + (M+N)y(t) \geq 0, & \text{for } t \in I = [a,b], \\ x(a) \geq 0, \ \ y(a) \geq 0. \end{cases}$$

By Lemma 4.1.2, we have $x(t) \geq 0, y(t) \geq 0$ for all $t \in [a,b]$. The proof is completed. □

Now, we can obtain our main theorem.

Theorem 4.2.5. *Assume that* (H_1), (H_2) *and* (H_3) *hold. Then* (4.1) *has an extremal system of solutions* $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$, *and there exist two monotone iterative sequences* $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ *converging uniformly to* x^*, y^*, *respectively, where* $x_n, y_n \in [\gamma, \delta]$, *such that*

$$\gamma = x_0 \leq x_1 \leq \cdots \leqslant x_n \leqslant \cdots \leqslant x^* \leqslant y^* \leqslant \cdots \leqslant y_n \leqslant \cdots \leqslant y_1 \leqslant y_0 = \delta, \ \textit{on } I, \ \forall n \in \mathbb{N}.$$

Proof. Firstly, for all $x_n, y_n \in C^\alpha([a,b],\mathbb{R})$, we consider the linear system:

$$\begin{cases} x_{n+1}^{(\alpha)}(t) = f(t, x_n(t), y_n(t)) + M\left(x_n(t) - x_{n+1}(t)\right) + N\left(y_n(t) - y_{n+1}(t)\right), t \in [a,b], \\ y_{n+1}^{(\alpha)}(t) = g(t, y_n(t), x_n(t)) + M\left(y_n(t) - y_{n+1}(t)\right) + N\left(x_n(t) - x_{n+1}(t)\right), t \in [a,b], \\ x_{n+1}(a) = \lambda_0, \quad y_{n+1}(a) = \beta_0. \end{cases} \tag{4.10}$$

By Lemma 4.2.3, the linear system (4.10) has a unique system of solutions in $C^\alpha([a,b],\mathbb{R}) \times C^\alpha([a,b],\mathbb{R})$, which is defined by

$$x_{n+1}(t) = \frac{p_{n+1}(t) + q_{n+1}(t)}{2}, \quad y_{n+1}\gimel = \frac{p_{n+1}(t) - q_{n+1}(t)}{2}, \text{for all } t \in [a,b] \tag{4.11}$$

where

$$\begin{aligned} p_{n+1}(t) =& e^{-\frac{M+N}{\alpha}t^\alpha}\left((\lambda_0 + \beta_0)e^{\frac{M+N}{\alpha}a^\alpha} + \int_a^t s^{\alpha-1}e^{\frac{M+N}{\alpha}s^\alpha}\left[f(s, x_n(s), y_n(s))\right.\right. \\ &\left.\left. + g(s, y_n(s), x_n(s)) + (M+N)\left(x_n(s) + y_n(s)\right)\right] ds\right), \end{aligned} \tag{4.12}$$

$$\begin{aligned} q_{n+1}(t) =& e^{-\frac{M-N}{\alpha}t^\alpha}\left((\lambda_0 - \beta_0)e^{\frac{M-N}{\alpha}a^\alpha} + \int_a^t s^{\alpha-1}e^{\frac{M-N}{\alpha}s^\alpha}\left[f(s, x_n(s), y_n(s))\right.\right. \\ &\left.\left. - g(s, y_n(s), x_n(s)) + (M-N)\left(x_n(s) - y_n(s)\right)\right] ds\right). \end{aligned} \tag{4.13}$$

Secondly, we shall prove that

$$x_n \leqslant x_{n+1} \leqslant y_{n+1} \leqslant y_n, \quad on\ I\ for\ all\ n \in \mathbb{N}.$$

Let $p = x_1 - x_0, q = y_0 - y_1$. According to (4.10) and $(H_1) - (H_2)$, we have

$$\begin{cases} p^{(\alpha)}(t) \geqslant M\left(x_0(t) - x_1(t)\right) + N\left(y_0(t) - y_1(t)\right), & \text{for } t \in I \\ p(a) \geqslant \lambda_0 - \lambda_0 = 0, \\ q^{(\alpha)}(t) \geqslant -M\left(y_0(t) - y_1(t)\right) - N\left(x_0(t) - x_1(t)\right), & \text{for } t \in I \\ q(a) \geqslant \beta_0 - \beta_0 = 0, \end{cases}$$

i.e.,

$$\begin{cases} p^{(\alpha)}(t) \geqslant -Mp(t) + Nq(t), \ \text{for } t \in I \quad p(a) \geqslant 0, \\ q^{(\alpha)}(t) \geqslant -Mq(t) + Np(t), \ \text{for } t \in I \quad q(a) \geqslant 0. \end{cases}$$

Then, by Lemma 4.2.4, we have $p(t) \geqslant 0,\ q(t) \geqslant 0$, i.e., $x_1 \geqslant x_0,\ y_1 \leqslant y_0$.

Let $w = y_1 - x_1$. According to (4.10) and (H$_3$), we have

$$\begin{cases} w^{(\alpha)}(t) & = y_1^{(\alpha)}(t) - x_1^{(\alpha)}(t) \\ & = g\left(t, y_0(t), x_0(t)\right) + M\left(y_0(t) - y_1(t)\right) + N\left(x_0(t) - x_1(t)\right) \\ & \quad -f\left(t, x_0(t), y_0(t)\right) - M\left(x_0(t) - x_1(t)\right) - N\left(y_0(t) - y_1(t)\right) \\ & \geqslant -M\left(y_1(t) - x_1(t)\right) + N\left(y_1(t) - x_1(t)\right) = -(M-N)w(t), \\ w(a) & = \beta_0 - \lambda_0 \geqslant 0. \end{cases}$$

By Lemma 4.1.2, we have $w(t) \geqslant 0$, i.e., $y_1(t) \geqslant x_1(t)$ for all $t \in I = [a, b]$. By mathematical induction, we can prove that

$$x_n \leqslant x_{n+1} \leqslant y_{n+1} \leqslant y_n, \quad on\ I\ for\ all\ n \in \mathbb{N}.$$

Thirdly, the sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ are monotone and bounded, hence

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*$$

(x^*, y^*) is an extremal system of solutions to (4.1).

Finally, we prove that (4.1) has at most one extremal system of solutions. Assume that $(x, y) \in [\gamma = x_0, \delta = y_0] \times [x_0, y_0]$ is the system of solutions to (4.1), then

$$x_0 = \gamma \leqslant x, \quad y \leqslant y_0 = \delta$$

. For some $k \in \mathbb{N}$, assume that the following relation holds

$$x_k(t) \leqslant x(t), \quad y(t) \leqslant y_k(t), \quad t \in [a, b].$$

Let $u(\mathrm{t}) = \mathrm{x(t)} - \mathrm{x_{k+1}(t)}, \quad v(\mathrm{t}) = \mathrm{y_{k+1}(t)} - \mathrm{y(t)}$. According to (4.10) and ($\mathrm{H_3}$), we have

$$\begin{cases} u^{(\alpha)}(t) & = x^{(\alpha)}(t) - x_{k+1}^{(\alpha)}(t) \\ & = f\left(t, x(t), y(t)\right) - f(t, x_k(t), y_k(t)) - M\left(x_k(t) - x_{k+1}(t)\right) \\ & \quad -N\left(y_k(t) - y_{k+1}(t)\right), \\ & \geq -M\left(x(t) - x_k(t)\right) - N\left(y(t) - y_k(t)\right) - M\left(x_k(t) - x_{k+1}(t)\right) \\ & \quad -N\left(y_k(t) - y_{k+1}(t)\right) \\ & = -M\left(x(t) - x_{k+1}(t)\right) + N\left(y_{k+1}(t) - y(t)\right). \end{cases}$$

and

$$\begin{cases} v^{(\alpha)}(t) & = y_{k+1}^{(\alpha)}(t) - y^{(\alpha)}(t) \\ & = g\left(t, y_k(t), x_k(t)\right) + M\left(y_k(t) - y_{k+1}(t)\right) + N\left(x_k(t) - x_{k+1}(t)\right) \\ & \quad -g(t, y(t), x(t)) \\ & \geq -M\left(y_k(t) - y(t)\right) - N\left(x_k(t) - x(t)\right) + M\left(y_k(t) - y_{k+1}(t)\right) \\ & \quad +N\left(x_k(t) - x_{k+1}(t)\right. \\ & = -M\left(y_{k+1}(t) - y(t)\right) + N\left(x(t) - x_{k+1}(t)\right), \end{cases}$$

we can get

$$\begin{cases} u^{(\alpha)}(t) \geq -Mu(t) + Nv(t), \text{ for } t \in I \quad u(a) \geq 0, \\ v^{(\alpha)}(t) \geq -Mv(t) + Nu(t), \text{ for } t \in I \quad v(a) \geq 0. \end{cases}$$

Then, by Lemma 4.2.4, we have $u(t) \geqslant 0, \; v(t) \geqslant 0$, i.e.,

$$x_{k+1}(t) \leqslant x(t), \; y(t) \leqslant y_{k+1}(t), \; t \in [a, b].$$

By the induction arguments, the following relation holds

$$x_n(t) \leqslant x(t), \quad y(t) \leqslant y_n(t), \quad on\ I \quad for\ all\ n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$, we get that $x^* \leqslant x,\ y \leqslant y^*$.
Hence, $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ is the extremal system of solutions to (4.1).
So the proof is finished. □

Remark 4.2.6. *The result (Theorem 4.2.5) in this Section 4.2 generalize the previous one (Theorem 3.1.3) given in Section 3.1(Chapter* 3*) for the coupled system of nonlinear first order ordinary differential equations (3.1).*

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4.3 An example

We present an example where we apply Theorem 4.2.5.

Example 4.3.1. *Consider the system of nonlinear conformable fractional differential equations:*

$$\begin{cases} x^{(\frac{1}{3})}(t) = \dfrac{t(2-x(t))^2 - y^2(t)}{\sqrt[3]{t}}, & t \in I = [1,2], \\ y^{(\frac{1}{3})}(t) = t^{\frac{2}{3}}(2-y(t))^3 - t^{-\frac{1}{3}}x^2(t), & t \in I = [1,2], \\ x(1) = 0, \quad y(1) = 0.5, \end{cases} \tag{4.14}$$

where $\alpha = \frac{1}{3}$, $f(t,x,y) = \dfrac{t(2-x)^2 - y^2}{\sqrt[3]{t}}$ *and* $g(t,y,x) = t^{\frac{2}{3}}(2-y)^3 - t^{-\frac{1}{3}}x^2$.
It is clear that f, g *are continuous functions. Take* $\gamma(t) = 0$ *and* $\delta(t) = 2$ *for* $t \in [1,2]$*, then*

$$\gamma^{(\frac{1}{3})}(t) = 0 \leq f(t, \gamma(t), \delta(t)) = \frac{4(t-1)}{\sqrt[3]{t}}\ for\ t \in [1,2], \quad \gamma(1) = 0 \leq 0,$$

and

$$\delta^{(\frac{1}{3})}(t) = 0 \geq g(t, \delta(t), \gamma(t)) = 0\ for\ t \in [1,2], \quad \delta(1) = 2 \geq 0.5.$$

So, γ *and* δ*, are lower and upper solutions of problem* (4.14)*, respectively with* $\gamma(t) = 0 \leq \delta(t) = 2$ *for* $t \in [1,2]$*, then assumptions* (H_1) *and* (H_2) *holds.*

Let $x, \overline{x}, y, \overline{y} \in \mathbb{R}$, then we have:

$$\begin{aligned}
f(t,x,y) - f(t,\overline{x},\overline{y}) &= t^{1-\frac{1}{3}}\left((2-x)^2 - (2-\overline{x})^2\right) - \frac{1}{\sqrt[3]{t}}(y^2 - \overline{y}^2) \\
&\geq t^{1-\frac{1}{3}}\left(-4(x-\overline{x}) + x^2 - \overline{x}^2\right) - (y^2 - \overline{y}^2) \\
&\geq -4t^{\frac{2}{3}}(x-\overline{x}) \\
&\geq -24(x-\overline{x}) - 0(y-\overline{y}),
\end{aligned}$$

$$\begin{aligned}
g(t,\overline{y},\overline{x}) - g(t,y,x) &= t^{\frac{2}{3}}\left((2-\overline{y})^3 - (2-y)^3\right) - t^{-\frac{1}{3}}(\overline{x}^2 - x^2) \\
&\geq -t^{\frac{2}{3}}(\overline{y}-y)\left((2-\overline{y})^2 + (2-\overline{y})(2-y) + (2-y)^2\right) \\
&\geq -12t^{\frac{2}{3}}(\overline{y}-y), \\
&\geq -24(\overline{y}-y) - 0.(\overline{x}-x),
\end{aligned}$$

with $\gamma(t) \leq \overline{x} \leq x \leq \delta(t)$, $\gamma(t) \leq y \leq \overline{y} \leq \delta(t)$ for all $t \in I$, and we have

$$\begin{aligned}
g(t,y,x) - f(t,x,y) &= t^{\frac{2}{3}}\left((2-y)^3 - (2-x)^2\right) + t^{-\frac{1}{3}}(y^2 - x^2) \\
&\geq -t^{\frac{2}{3}}(y-x)\left((2-y)^2 + (2-y)(2-x) + (2-x)^2\right) \\
&\geq -12t^{1-\frac{1}{3}}(y-x) \\
&\geq -24(y-x) - 0.(x-y).
\end{aligned}$$

with $\gamma(t) \leq x \leq y \leq \delta(t)$, for all $t \in I$.
Hence the assumption (H_3) holds with $M = 24$ and $N = 0$. By Theorem 4.2.5, the nonlinear system (4.14) has the extremal solution $(x^*, y^*) \in C^{\frac{1}{3}}([1,2]) \times C^{\frac{1}{3}}([1,2])$, such that $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ on $[1,2]$, which can be obtained by taking limits from the iterative sequences:

$$\begin{aligned}
x_{n+1}(t) &= \int_1^t s^{\frac{-2}{3}} e^{72(s^{\frac{1}{3}} - t^{\frac{1}{3}})} \left[\frac{s(2-x_n(s))^2 - y_n^2(s)}{\sqrt[3]{s}} + 24(x_n(s))\right] ds, \quad t \in I, \\
y_{n+1}(t) &= \int_1^t s^{\frac{-2}{3}} e^{72(s^{\frac{1}{3}} - t^{\frac{1}{3}})} \left[s^{\frac{2}{3}}(2-y_n(s))^3 - s^{-\frac{1}{3}} x_n^2(s) + 24y_n(s)\right] ds + 0.5e^{72(1-t^{\frac{1}{3}})}, \ t \in I.
\end{aligned}$$

Conclusion

In this work, we have considered the existence of extremal solutions for nonlinear conformable fractional differential equation involving integral boundary condition, and for a coupled system of nonlinear first order ordinary differential equations with initial conditions. Also, we present the existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations with initial conditions.
These results will be obtained by using the monotone iterative technique combined with the method of upper and lower solutions..

Bibliography

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66.

[2] T. Abdeljawad, M. AlHorani and R. Khalil, *Conformable fractional semigroups of operators*, J. Semig. Theory. Appl. **2015** (2015), Art. ID 7, 9 pp.

[3] S. Ali, K. Shah, H. Khan, M. Arif, S. Mahmood, Monotone iterative techniques and Ulam-Hyers stability analysis for nonlinear fractional order differential equations with integral boundary value conditions, *Eur. J. Pure Appl. Math* , **12**(2)(2019), 432-447.

[4] D.R. Anderson and R.I. Avery, *Fractional-order boundary value problem with Sturm-Liouville boundary conditions*, Electr. J. Differ. Equ. **2015** (2015), no. 29, 10 pp.

[5] H. Batarfi, J. Losada, J.J. Nieto and W. Shammakh, *Three-Point Boundary Value Problems for Conformable Fractional Differential Equations*, J. Funct. Spaces. **2015** (2015), Art. ID 706383, 6 pp.

[6] B. Bendouma, A. Cabada and A. Hammoudi, *Existence of solutions for conformable fractional problems with nonlinear functional boundary conditions*, Malaya Journal of Mathematik. **7** (2019), no. 4, 700-708.

[7] B. Bendouma, *Monotone Iterative Technique for a Coupled System of Nonlinear Conformable Fractional Dynamic Equations on Time Scales*, Jordan J. Math. Stat. JJMS 2023, **16**, 41-55.

[8] A. Cabada, *The monotone method for first-order problems with linear and nonlinear boundary conditions*, Appl. Math. Comput. **63** (1994), 163–188.

[9] H. Chen, S. Meng, Y. Cui, Monotone iterative technique for conformable fractional differential equations with deviating arguments, *Hindawi Discrete Dynamics in Nature and Society*, **2020**, Article ID 5827127, 9 pages, 2020.

[10] W. S. Chung, *Fractional Newton mechanics with conformable fractional derivative*, J. Comput. Appl. Math. **290** (2015), 150–158.

[11] A. Granas and J. Dugundji, *Fixed Point Theory.* Springer Monographs in Mathematics. Springer, New York 2003.

[12] A. Gökdoğan, E. Ünal and E. Çelik, *Existence and Uniqueness Theorems for Sequential Linear Conformable Fractional Differential Equations*, 2015.

[13] O.S. Iyiola and E.R. Nwaeze, *Some new results on the new conformable fractional calculuswith application using D'Alambert approach*, Progr. Fract. Dier. Appl. **2**(2016), (2), 115–122.

[14] T. Jankowski, *Differential Equations with Integral Boundary Conditions*, Journal of Computational and Applied M athematics, **147**(2002), 1–8.

[15] U.N. Katugampola, *A new fractional derivative with classical properties*, preprint, 2014.

[16] R. Khaldi and A. Guezane-Lakoud, *Lyapunov inequality for a boundary value problem involving conformable derivative*, Prog. Fract. Differ. Appl. 2017.

[17] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. **264** (2014), 65–70.

[18] A. Kilbas, M.H. Srivastava and J.J. Trujillo, *Theory and Application of Fractional Differential Equations*, North Holland Mathematics Studies 204, 2006.

[19] S. Liu, H. Wang, X. Li, H. Li, *Extremal iteration solution to a coupled system of nonlinear conformable fractional differential equations,* Journal of Nonlinear Sciences and Applications, **10**(9) (2017), 5082-5089.

[20] S. Liu, H. Li, *Extremal system of solutions for a coupled system of nonlinear fractional differential equations by monotone iterative method,* J.Nonlinear Sci. Appl. 9 (2016) 3310-3318.

[21] R.L. Magin, *Fractional calculus in Bioengineering*, CR in Biomedical Engineering **32** (2004), no. 1, 1–104.

[22] S. Meng, Y. Cui, Extremal solution to conformable fractional differential equations involving integral boundary condition, *Mathematics*, **7**, Article ID 186, 2019.

[23] K.S. Miller and B. Ross, *Fractional difference calculus*, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications, Nihon University, Koriyama, Japan, (1989), 139-152.

[24] D. O'Regan, *Existence theory for nonlinear ordinary differential equations*, Kluwer Academic Publishers Group, Dordrecht, (1997).

[25] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer Academic Publisher, Dordrecht, 2002.

[26] I. Podlubny, *Fractional Differential Equations*, Academic Press: San Diego CA, (1999).

[27] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives. Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, ,Switzerland, 1993.

[28] K. Shugui, C. Huiqing, Y. Yaqing and G. Ying, *Existence and uniqueness of the solutions for the fractional initial value problem*, Electr. J. Shanghai Normal University (Natural Sciences), **45**(2016), no. 3, 313-319.

[29] A. Shi and S. Zhang, *Upper and lower solutions method and a fractional differential equation boundary value problem*, Electr. J. Qual. Theory. Differ. Equ. **30** (2009), 13 pages.

[30] S. Zhang and X. Su, *The existence of a solution for a fractional differential equation with nonlinear boundary conditions considered using upper and lower solutions in reverse order*, Comput. Math. Appl. **62** (2011), no. 3, 1269-1274.