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Dedication:

To our beloved families,whose unwavering support and encouragement have been our foundation throughout this journey.

To our parents,whose sacrifices, love, and guidance have always inspired us to strive for excellence.

To our friends and colleagues,whose camaraderie and understanding have provided us with strength and motivation.

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This work is dedicated to all of you,with deepest gratitude and heartfelt appreciation.

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Contents

Résumé

Une suite (a_n) de nombres réels est équidistribuée sur un intervalle si la probabilité de trouver des termes dans tout sous-intervalle est proportionnelle à la longueur de ce sous-intervalle.

On dit qu'elle est équidistribuée modulo 1 ou uniformément distribuée modulo 1 si la suite des parties fractionnaires de a_n, notée (a_n) ou $(a_n - \lfloor a_n \rfloor)$, est équidistribuée dans l'intervalle $[0, 1]$.

Pour tout nombre réel donné $r \geq 0$ et tout $\alpha > 0$, Koksma et H. Weyl ont respectivement prouvé que les ensembles E_α de tous les nombres réels positifs $r \geq 0$ et les ensembles W_r de tous les nombres réels positifs $\alpha > 0$, pour lesquels la suite $\alpha r^n{}_{n \in \mathbb{N}}$ n'est pas uniformément distribuée modulo 1, ont une mesure de Lebesgue nulle.

Dans ce mémoire, nous donnons certaines propriétés algébriques de certains ensembles E_α et montrons, entre autres choses, que les ensembles W_r sont non dénombrables.

Abstract

A sequence (a_n) of real numbers is equidistributed on an interval if the probability of finding any terms in any subinterval is proportional to the length of the subinterval. And is said to be equidistributed modulo 1 or uniformly distributed modulo 1 if the sequence of the fractional parts of a_n, denoted by (a_n) , is equidistributed in the interval $[0, 1]$. For any given real numbers $r \geq 0$ and $\alpha > 0$ Koksma and H. Weyl proved respectively that the setsE_α of all positive real numbers $r \geq 0$ and the sets W_r of all positive real numbers $\alpha > 0$, for which the sequence $\alpha r^n{}_{n\in\mathbb{N}}$ is not uniformly distributed modulo 1, have Lebesgue measure zero. In this memoir, we give some algebraic properties of certain sets E_αa and show, among other things, that the sets W_r are uncountable.

Rating

1. $\mathbb{N} = \{0, 1, 2 \ldots\}$: designates the set of natural integers.

2. $\mathbb{Z}$: designates the ring of rational or relative integers.

3. $\mathbb{Q}$: designates the body of rational numbers.

4. $\mathbb{R}$: designates the body of real numbers.

5. $\mathbb{C}$: designates the field of complex numbers.

6. A : designates a commutative and unitary ring.

7. $V \backslash \Omega$: designates the set of elements of V that are not in Ω.

8. $D(z_0, r)$: designates an open disk with center z_0 and radius r.

9. $\bar{D}(z_0, r)$: designates a closed disk with center z_0 and radius r..

10. log : denotes the neperian logarithm.

11. $E(x)$: designates the integer part of x..

12. $\{x\}$: is the fractional part of x.

13. $E'(x)$: designates the nearest integer to x i.e..

$$E'(x) = \begin{cases} E(x), \{x\} < \frac{1}{2} \\ E(x) + 1, \{x\} \geq \frac{1}{2}. \end{cases} \text{; so: } E'(1.4) = 1; E'(1.6) = 2$$

14. $\|x\|$: denotes the distance from the real number x to the nearest integer i.e..

$$\begin{aligned} \|x\| &= \left|x - E'(x)\right| = \min\{|x - n|, n \in \mathbb{Z}\} \\ &= \min\{\{x\}, 1 - \{x\}\} \end{aligned}$$

For two real numbers x_1, x_2 and an integer n, we have

$$\begin{gathered} \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|, \\ \|nx_1\| \leq |n| \, \|x_1\|. \end{gathered}$$

15. Given a real number x, we can write x as:

$$x = E'(x) + \varepsilon(x) \text{ with } \frac{-1}{2} \leq \varepsilon(x) < \frac{1}{2}.$$

16. For two integers n and k such that $0 \leq k \leq n$ we note

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

17. if $f(x)$ and $g(x)$ two functions defined on $[x_0, +\infty[$ and $g(x) > 0$ for $x \geq x_0$, so

$$\begin{aligned}
&f(x) = \mathrm{O}(g(x)) \text{ stands for } (\exists M > 0) \text{ so that} |f(x)| \leq M g(x)\, (\forall x \geq x_0) \\
&\qquad \left(\limsup \frac{|f(x)|}{g(x)} < \infty\right) (f(x) \text{ is a big O of } g(x)) \\
&f(x) = \mathrm{o}(g(x)) \text{ stands for } \lim_{x \to +\infty} \frac{|f(x)|}{g(x)} = 0 (f(x) \text{ is a small o of } g(x)) \\
&f(x) \sim g(x) \text{ stands fo } \lim_{x \to +\infty} \frac{|f(x)|}{g(x)} = 1 (f \text{ equivalent to } g \text{ or } V(+\infty))
\end{aligned}$$

These notations are also valid in a neighbourhood $V(x_0)$, avec $x_0 \neq +\infty$.

Introduction

The concept of uniform distribution modulo 1 was introduced by Hermann Weyl[7] in 1916. A real sequence $(x_n)_{n\in\mathbb{N}}$ is said to be uniformly distributed modulo 1 if, for any subinterval I of $[0,1)$, the proportion of its terms in I tends to the length of I as the sequence lengthens. Weyl's criterion states that a sequence is uniformly distributed modulo 1 if and only if the sum $\sum_{n=1}^{N} e^{2\pi i h x_n}$ is negligible compared to N for any h non-zero

Weyl demonstrated that arithmetic progressions are uniformly distributed modulo 1 if and only if their common difference is irrational. For geometric progressions, it has been shown that (λr^n) is uniformly distributed modulo 1 We also know (through theresult by J. F. Koksma[3]) that for almost all $\lambda > 0$ and almost all $r > 1$, except in sets of Lebesgue measure zero called "Weyl's exceptional sets" and "Koksma's exceptional sets."

No explicit number $r > 1$ is known for which (r^n) is uniformly distributed modulo 1 , although examples of $r > 1$ that are not uniformly distributed are known and are specific algebraic numbers, such as Pisot-Vijayaraghavan (P.V.) numbers and Salem numbers.1 were appeared in the article [1] by D. W. Boyd found geometric progressions with a transcendental ratio that are not uniformly distributed modulo 1 . It is well established that there exist transcendental real numbers $r > 1$ for which (r^n) is not uniformly distributed modulo 1 because Koksma's exceptional sets are uncountable.

"Our memory is organized as follows:

The first chapter: We provided definitions and reminders of the theories used in this memory, such as rational series, integer series, convergence radius, analytic functions, holomorphic functions, poles and residues, algebraic numbers and algebraic integers.

The second chapter: We define the concept of uniform distribution mod 1. We introduced uniformly distributed, Weyl's criterion, Lebesgue measure in R, Koksma's theorem, and equidistribution of geometric sequences and we gave explanatory explanations for them, and gave a remarkable example of a sequence uniformly distributed mod 1

The the last chapter: We gave some properties of exceptional sets of real numbers as the set of algebraic integers, then we characterized the subset S of algebraic integers and we gave the necessary and sufficient condition to belong to the set S, then we introduced the theorem (Weyl, 1916) and proved that the sets of Weyl are uncountable.

Chapter 1

Preliminaries

In this chapter, we give some reminders of definitions and necessary results relating to general algebra, formal series complex analysis and algebraic numbers which will be used throughout this memory.

1.1 Definitions

The structures of group, ring, field and vector space are covered first and are assumed to be familiar to the reader. A[X] is the ring of polynomials.

$$\sum_{k=0}^{+\infty} a_k X^k$$

with coefficients a_k in the ring A, whose addition and multiplication are defined below:

$$\sum_{k=0}^{n} a_k X^k + \sum_{k=0}^{n} b_k X^k = \sum_{k=0}^{n} (a_k + b_k) X^k,$$

$$(\sum_{k=0}^{n} a_k X^k)(\sum_{k=0}^{s} b_k X^k) = \sum_{k=0}^{s+n} c_k X^k, \text{ with } c_k = \sum_{p+q=k}^{s+n} a_p b_q.$$

Let A[[X]] be the ring of formal series

$$\sum_{k \geqslant 0} a_n X^n$$

into the indeterminate X with coefficients a_n in the ring A, whose addition and multiplication are those of the ring A [X], generalized.

Remark 1. *The ring A[X] of polynomials is a sub-ring of the ring A[[X]] of formal series; if A is a field, the ring A[[X]] has the external law:*

$$\lambda \sum_{k \geqslant 0} a_k X^k = \sum_{k \geqslant 0} (\lambda a_k) X^k \qquad where \qquad \lambda \in A,$$

is a vector space on A.

Proposition 1.1. *A formal series*

$$\sum_{k \geqslant 0} a_k X^k$$

of the ring A[[X]] is invertible in A[[X]] if and only if the coefficient a_0 of this series is invertible in the ring A.

Proof. The series $\sum_{k \geqslant 0} a_k X^k$ is invertible if and only if there exists a series $\sum_{k \geqslant 0} b_k X^k$ such that

$$\begin{cases} a_0 b_0 = 1 \\ a_0 b_1 + a_1 b_0 = 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots = 0 \\ a_0 b_n + \cdots \qquad\qquad + a_n b_0 = 0. \end{cases}$$

This system implies that a_0 is invertible.
On the other hand, if a_0 is invertible, we can compute $b_0 (b_0 = a_0^{-1})$, then $b_1 = (-a_1 a_0^{-1}) a_0^{-1}$. and all b_n by the following recurrence relation:

$$\begin{cases} b_0 = a_0^{-1} \\ b_n = -a_0^{-1} \sum_{p=0}^{n-1} a_{n-p} b_p, n \geq 1. \end{cases}$$

■

1.1.1 Field of rational fractions

Definition 1.1. *On note*

$$K(X) = \{P(X)/Q(X) \text{ where } P \text{ and } Q \text{ are in } A[X], Q \neq 0\}$$

the field of fractions of the ring $A[X]$ where A is an integral ring.

Rational series

Definition 1.2. *Let $\mathfrak{K}$ be a field and let $S(X)$ be a formal series of the ring $\mathfrak{K}[[X]]$.*
The series S is said to be rational if there exist two polynomials P and Q of the ring $\mathfrak{K}[X]$ with $Q(0) \neq 0$ such that:

$$S = P/Q$$

1.1.2 Rationality of formal series

Criterion 1.1. *A formal series*

$$S(X) = \sum_{n \geq 0} a_n X^n$$

of the ring $\mathfrak{K}[[X]]$ is rational if and only if there exist two integers s, n_0 and $(s+1)$ elements $q_0, q_1, \dots, q_s$ with $q_0 \neq 0$, of the field $\mathfrak{K}$ such that:

$$q_0 a_n + q_1 a_{n-1} + \ldots + q_s a_{n-s} = 0, \forall n \geq n_0.$$

Proof. If F is a rational series, we can write

$$F(X) = \sum_{n \geq 0} a_n X^n = \frac{P(X)}{Q(X)}$$

with
$$Q(X)=\sum_{n=0}^{s} q_nX^n, (P,Q)\in K[X]^2 \text{ and } \deg(P)=r.$$
The two formal series are equal
$$QF=P$$
and since
$$\begin{aligned}Q(X)F(X) &= \sum_{n=0}^{s} q_nX^n \times \sum_{n\geq 0} a_nX^n\\ &= \sum_{n\geq 0} c_nX^n \text{ with } c_n=\sum_{i=0}^{n} a_{n-i}q_i.\end{aligned}$$
Lets take $n_0=\sup(r+1,s)$ since $\deg(P)=r$ then $c_n=0$ for $n\geq n_0$, so we get
$$q_0a_n+q_1a_{n-1}+\ldots+q_sa_{n-s}=0, \forall n\geq n_0.$$
Conversely
$$Q(X)=\sum_{n=0}^{s} q_nX^n,$$
the relation given in the criterion, shows that QF is of degree $\leq n_0-1$, so F is a rational series.■

Lemma 1. *(Fatou) If S is a rational formal series of the ring $\mathbb{Z}[[X]]$, then there exist two coprime polynomials P and Q of the ring $\mathbb{Z}[X]$, such that :*
$$S=P/Q$$
with $Q(0)=1$.

Proof. Let be the following non-zero coprime polynomials,
$$P(X)=\sum_{n=0}^{s} p_nX^n \text{ and } Q(X)=\sum_{n=0}^{k} q_nX^n \text{ in } \mathbb{Z}[X],$$
and the fraction
$$F=P/Q.$$
If $F(X)\in\mathbb{Z}[[X]]$ then $Q(X)$ is invertible in $\mathbb{Z}[[X]]$, by Proposition 1.1, q_0 is invertible in $\mathbb{Z}$ we deduce that $q_0\in\{-1,1\}$ and since
$$Q(X)=q_0+q_1X+..+q_kX^k$$
then $Q(0)=1$ or $Q(0)=-1$.
In the case where $Q(0)=-1$, we take
$$F=-P/-Q$$
The lemma is proved. ■

1.2 Integer series

Definition 1.3. *Any series of the form*
$$\sum_{n\geq 0} a_nz^n$$
where a_n are scalars in any field and z a variable complex is called an integer series.

radius of convergence

Definition 1.4. *The radius of convergence of an integer series*

$$\sum_{n\geq 0} a_n z^n$$

is the number

$$\rho = \sup\left\{ r \in \mathbb{R}_+; \sum_{n\geq 0} |a_n| \, r^n < +\infty \right\} \in \overline{\mathbb{R}_+};$$

and convergence disk or convergence domain of the series is the ball $D(0;\rho)$.

Theorem 1.1. *The radius of convergence ρ of an integer series*

$$\sum_{n\geq 0} a_n z^n$$

is given by the formula

$$\frac{1}{\rho} = \limsup_{n\to+\infty} |a_n|^{\frac{1}{n}}$$

Remark 2. *Let $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 0} b_n z^n$ be two integer series whose radii of convergence R and R' respectively. Let R'' denote the radius of convergence of the sum of integer series.*

$$\sum_{n\geq 0} (a_n + b_n) \, z^n$$

then $R'' \geq \min(R, R')$.

1.2.1 Analytic functions

Definition 1.5. *Let Ω be an open of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a function. Let $t \in \Omega$. We say that f is analytic in t if there exists a number $r > 0$ such that the disk $D(t;r)$ is contained in Ω and an integer series*

$$\sum_{n\geq 0} a_n \omega^n$$

of radius of convergence $\rho \geq r$ such that, for $z \in D(t;r)$, we have

$$f(z) = \sum_{n\geq 0} a_n (z-t)^n.$$

We say that f is analytic on Ω if it is analytic at any point on Ω.

Proposition 1.2. *: [2] Let $f(z) = \sum_{n\geq 0} a_n z^n$ be the integer series whose radius of convergence $\rho \neq 0$. Let t be a point inside the disk of convergence. Then the integer series*

$$\sum_{n\geq 0} \frac{1}{n!} f^{(n)}(t) \omega^n$$

has a radius of convergence at least equal to $\rho - |t|$ and we have:

$$f(z) = \sum_{n\geq 0} \frac{1}{n!} f^{(n)}(t)(z-t)^n$$

for all z such that $|z-t| < \rho - |t|$.

Proof. Let $r_0 = |t|, \alpha_n = |a_n|$. Calculate the p-th derivative of f

$$f^{(p)}(t) = \sum_{q \geq 0} \frac{(p+q)!}{q!} a_{p+q} t^q$$

so that

$$\left| f^{(p)}(t) \right| \leq \sum_{q \geq 0} \frac{(p+q)!}{q!} \alpha_{p+q} r_0^q.$$

For $r_0 \leq r < \rho$, we have

$$\sum_{p \geq 0} \frac{1}{p!} \left| f^{(p)}(t) \right| (r - r_0)^p \leq \sum_{p,q} \frac{(p+q)!}{q!p!} \alpha_{p+q} r_0^q (r - r_0)^p$$

the terms of the series are positive, so we can see that

$$\sum_{p,q} \frac{(p+q)!}{q!p!} \alpha_{p+q} r_0^q (r - r_0)^p = \sum_{n \geq 0} \alpha_n \left(\sum_{p=0}^{n} \frac{n!}{p!(n-p)!} r_0^{n-p} (r - r_0)^p \right),$$

as

$$\sum_{p=0}^{n} \frac{n!}{p!(n-p)!} r_0^{n-p} (r - r_0)^p = (r - r_0 + r_0)^n$$

then

$$\sum_{p \geq 0} \frac{1}{p!} \left| f^{(p)}(t) \right| (r - r_0)^p \leq \sum_{p=0}^{n} \alpha_n p^n < +\infty.$$

So the radius of convergence of the series

$$\sum_{n \geq 0} \frac{1}{n!} f^{(n)}(t) \omega^n$$

is greater than or equal to $r - r_0$, since we can choose r arbitrarily close to ρ, then the radius of convergence is greater than or equal to $\rho - r_0$.

Let z be such that $|z - t| \leq \rho - r_0$, the series

$$\sum_{p,q} \frac{(p+q)!}{q!p!} a_{p+q} t^q (z - t)^p$$

converges absolutely. Fubini's inversion theorem states that its sum can be calculated by grouping terms arbitrarily. As before

$$\begin{aligned}\sum_{p,q} \frac{(p+q)!}{q!p!} a_{p+q} t^q (z-t)^p &= \sum_{n \geq 0} a_n \left(\sum_{p=0}^{n} \frac{n!}{p!(n-p)!} t^{n-p} (z-t)^p \right) \\ &= \sum_{n \geq 0} a_n z^n = f(z). \blacksquare\end{aligned}$$

Proposition 1.3. *The sum $S(z)$ of a convergent integer series*

$$\sum_{n \geq 0} a_n z^n,$$

of radius of convergence $\rho > 0$ is an analytic function in the disk $|z| < \rho$.

Proof. the series $S(z)$ is integer, so according to proposition 1.2, it is Taylor-developable at any point inside the disk of convergence and therefore analytic. $\blacksquare$

1.2.2 Holomorphic functions

Definition 1.6. *Ω be an open of $\mathbb{C}, a \in \Omega$. A function $f : \Omega \to \mathbb{C}$ is said to be holomorphic in a if the limit*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists in $\mathbb{C}$; it is said to be holomorphic in Ω if it is holomorphic at any point in Ω.
The set of holomorphic functions on an open Ω of $\mathbb{C}$ is denoted by $H(\Omega)$.

Remark 3. *Any analytic function on an open set Ω of $\mathbb{C}$ is holomorphic.*

1.2.3 Poles and residues

Definition 1.7. *Let Ω be an open of $\mathbb{C}$ and $z_0 \in \Omega$. If the function f is holomorphic on $\Omega \backslash \{z_0\}$, it has a Laurent expansion in z_0;*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n \left(z - z_0\right)^n$$

and the coefficient a_{-1} of $\left(z - z_0\right)^{-1}$ in this expansion is called the residual of f at z_0. Furthermore, if z_0 is a pole of order 1 then:

$$\operatorname{Re} s\left(f.z_0\right) = \lim_{z \to z_0} \left(z - z_0\right) f(z).$$

Definition 1.8. *The coefficient a_{-1} of the Laurent series expansion is called the residue of the function at the pole $z = z_0$. It is given by*

$$a_{-1} \equiv Res. = \lim_{z \to z_0} \frac{1}{(k-1)!} \left[\frac{d^{k-1}}{dz^{k-1}} \Big((z - z_0)^k f(z)\Big)\right]$$

(if z_0 is a pole of order k)

Definition 1.9. *Let Ω be an open of $\mathbb{C}, a \in \Omega$ and $f : \Omega \backslash \{a\} \to \mathbb{C}$ a holomorphic function. If there exists a function $g : \Omega \to \mathbb{C}$ and $n \in \mathbb{N}^*$ such that:*

$$g(a) \neq 0 \ \ et\ f(z) = \frac{g(z)}{(z-a)^n}, \forall z \in \Omega \backslash \{a\}$$

then a is a pole of order n of he function f.

Definition 1.10. *Let Ω be an open of $\mathbb{C}$. A function f is said to be meromorphic in Ω if there exists a discrete part $\digamma$ of Ω such that $f \in H(\Omega \backslash \digamma)$ and any point of $\digamma$ is a pole of f.*
The set of meromorphic functions on Ω is denoted by $M(\Omega)$.

Remark 4. *Any meromorphic function f is the quotient of two holomorphic functions h and g on Ω.*

$$f = \frac{g}{h}$$

such that the set of points $\digamma$ is the set of zero's of h.

1.3 Algebraic numbers and algebraic integers

1.3.1 Algebraic numbers

Definition 1.11. *The number $\alpha \in \mathbb{C}$ is said to be algebraic if it satisfies a polynomial equation*

$$x^n + a_1 x^{n-1} + \ldots + a_n$$

with rational coefficients $a_i \in \mathbb{Q}$. We denote the set of algebraic numbers by $\overline{\mathbb{Q}}$.

Examples:

1. $\alpha = \frac{1}{2}\sqrt{2}$ is algebraic, since it satisfies the equation $x^2 - \frac{1}{2} = 0$.

2. $\alpha = \frac{\sqrt{3}}{2} + 1$ is algebraic, since it satisfies the equation $(x-1)^3 = 2$, i.e.,

$$x^3 - 3x^2 + 3x - 3 = 0.$$

Lemma 2. *The number $\alpha \in \mathbb{C}$ is algebraic if and only if the vector space over $\mathbb{Q}$*

$$V = \langle 1, \alpha, \alpha^2, \ldots \rangle$$

is finite-dimensional.

Proof. Suppose $\dim_{\mathbb{Q}} V = d$. Then the $d+1$ elements $1, \alpha, \ldots, \alpha^d$ are linearly dependent over $\mathbb{Q}$; i.e., α satisfies an equation of degree $\leq d$.
Conversely, if

$$\alpha^n + a_1\alpha^{n-1} + \ldots + a_n = 0$$

then $\alpha^n = -a_1\alpha^{n-1} - \ldots - a_n \in \langle 1, \alpha, \ldots, \alpha^{n-1} \rangle$.
Now $\alpha^{n+1} = -a_1\alpha^n - \ldots - a_n\alpha \in \langle 1, \alpha, \ldots, \alpha^{n-1} \rangle$; and so successively

$$\alpha^{n+2}, \alpha^{n+3}, \ldots \in \langle 1, \alpha, \ldots, \alpha^{n-1} \rangle.$$

Thus $V = \langle 1, \alpha, \ldots, \alpha^{n-1} \rangle$ is finitely-generated.

1.3.2 Algebraic integers

Definition 1.12. *The number $\alpha \in \mathbb{C}$ is said to be an algebraic integer if it satisfies a polynomial equation*

$$x^n + a_1x^{n-1} + \ldots + a_n$$

with integer coefficients $a_i \in \mathbb{Z}$. We denote the set of algebraic integers by $\overline{\mathbb{Z}}$.

Remark. In algebraic number theory, an algebraic integer is often just called an integer, while the ordinary integers (the elements of $\mathbb{Z}$) are called rational integers.
Examples:
1. We have $\alpha = 3\sqrt{2} + 1 \in \overline{\mathbb{Z}}$, since α satisfies

$$(x-1)^2 = 18$$

i.e.,

$$x^2 - 2x - 17 = 0.$$

2. Again, $\alpha = \sqrt{2} + \sqrt{3} \in \overline{\mathbb{Z}}$, since α satisfies

$$(x - \sqrt{3})^2 = (x - 2\sqrt{3} + 3 = 2$$

i.e.,

$$x^2 - 2\sqrt{3}x + 1 = 0.$$

Hence

$$(x^2 + 1)^2 = 12x^2$$

i.e.,

$$x^4 - 10x^2 + 1 = 0.$$

Chapter 2

The modulo 1 distribution

The (mod1) distribution of a $(\varphi_n)_{n\in\mathbb{N}}$ sequence of real numbers is the $(\psi_n)_{n\in\mathbb{N}}$ obtained by reducing (mod1) each number φ_n to a number ψ_n belonging to a fixed interval of length 1; practically we consider one of the two intervals $[0,1]$ or $\left[-\frac{1}{2},\frac{1}{2}\right]$.

Historically, the problem of the distribution of a sequence was first raised by Lagrange, in connection with the calculation of the motion of the great planets.
More precisely, this calculation involved the distribution of the sequence $(n\alpha)_{n\in\mathbb{N}}, \alpha \in \mathbb{R}$.
In 1916, H. Weyl formalized the notion of uniformly distributed (mod1) of sequences.

Definition 2.1. *A sequence $(u_n)_{n\in\mathbb{N}}$ with values in $[0,1]$ is said to be equidistributed if, for any pair of real numbers $a<b$ of $[0,1]$, the sequence of integers of general term*

$$\varphi(n)=\operatorname{Card}\{k\in\mathbb{N}, k<n, a\leq u_k\leq b\}$$

is asymptotically equivalent to $n(b-a)$, i.e., $\varphi(n)$ will merge with $n(b-a)$, at infinity.
A real sequence $(u_n)_{n\in\mathbb{N}}$ is said to be equidistributed (mod1) *if the sequence of fractional parts $\{u_n\}_{n\in\mathbb{N}}$ of the elements of $(u_n)_{n\in\mathbb{N}}$ is equidistributed*

2.0.3 Weyl's criterion

The notion of equirÃĺpartition is closely linked to the Riemann integral. The following theorem is useful for proving Weyl's criterion.

Theorem 2.1. *A sequence $(u_n)_{n\in\mathbb{N}}$ with values in $[0,1]$ is uniformly distributed if and only if for any Riemann-integrable function f, we have:*

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\left(u_k\right)=\int_0^1 f(t)dt$$

Proof. Let $a<b$ be two real numbers in the interval $[0,1]$ and let $\chi_{[a,b]}$ be the indicator function for the interval $[a,b]$, i.e. the function

$$\begin{cases} x=1 \text{ if } x\in[a,b] \\ x=0 \text{ if not}\end{cases}$$

$$\int_0^1 \chi_{[a,b]}(t)dt=\int_a^b dt=b-a$$

or

$$\sum_{k=0}^{n-1} \chi_{[a;b]}(u_k) = \mathrm{Card}\{k \in \mathbb{N}, k < n, a \leq u_k \leq b\}$$

by definition of the function $\chi_{[a,b]}$. Therefore $(u_n)_{n\in\mathbb{N}}$ and uniformly distributed if and only if, for all real $a < b$

$$\lim_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{[a,b]}(u_k) = \int_0^1 \chi_{[a,b]}(t)dt$$

If the sequence $(u_n)_{n\in\mathbb{N}}$ is equirrect, we have the property announced for any function of type $\chi_{[a,b]}$, and we want this for any f Riemann-integrable function.

Since any staircase function can be written as a linear combination of segment-indicator functions, the property is still true for any staircase function. Finally, any Riemann-integrable function is the uniform limit of a sequence of staircase functions (see [6]). In other words, if f is a Riemann-integrable function, then for any $\varepsilon > 0$, we can find a staircase function φ_ε such that:

$$\forall x \in [0,1] \text{ we have } |f(x) - \varphi_\varepsilon(x)| < \varepsilon$$

Let f be such a function, for $\varepsilon > 0$ and φ_ε in steps verifying the approximation property of f, we have for all $n \geq 1 \left|\frac{1}{n}\sum_{k=0}^{n-1} f(u_k) - \frac{1}{n}\sum_{k=0}^{n-1}\varphi_\varepsilon(u_k)\right| < \frac{1}{n}\sum_{k=0}^{n-1}\varepsilon = \varepsilon$
and

$$\left|\int_0^1 f(t)dt - \int_0^1 \varphi_\varepsilon(t)dt\right| < \int_0^1 \varepsilon dt = \varepsilon$$

as we already have

$$\lim_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi_\varepsilon(u_k) = \int_0^1 \varphi_\varepsilon(t)dt$$

we deduce for $n \geq n_0$

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} \varphi_\varepsilon(u_k) - \int_0^1 \varphi_\varepsilon(t)dt\right| < \varepsilon$$

So for $n \geq n_0$ we have:

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f(u_k) - \int_0^1 f(t)dt\right| \leq \left|\frac{1}{n}\sum_{k=0}^{n-1} f(u_k) - \frac{1}{n}\sum_{k=0}^{n-1}\varphi_\varepsilon(u_k)\right| + \left|\frac{1}{n}\sum_{k=0}^{n-1}\varphi_\varepsilon(u_k) - \int_0^1 \varphi_\varepsilon(t)dt\right|$$
$$+ \left|\int_0^1 \varphi_\varepsilon(t)dt - \int_0^1 f(t)dt\right| < 3\varepsilon.$$

Since this is true for any real $\varepsilon > 0$, we deduce

$$\lim_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(u_k) = \int_0^1 f(t)dt.$$

Conversely, if the sequence $(u_n)_{n\in\mathbb{N}}$ verifies, for any f Riemann-integrable function.

$$\lim_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(u_k) = \int_0^1 f(t)dt$$

then this is true in particular for interval indicator functions (which are Riemann-integrable), and so the sequence is equi-distributed. ■

Corollary 2.1. *A sequence $(u_n)_{n\in\mathbb{N}}$ is equidistbutrd (mod 1) if and only if, for any Riemann integrable and 1-periodic function f, we have*

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(u_k\right)=\int_0^1 f(t)dt$$

Proof In fact, the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly distributed (mod1) if and only if the sequence of the general term $\{u_n\}$ is uniformly distributed in $[0,1]$, then for any f Riemann integrable function we have

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(\{u_k\}\right)=\int_0^1 f(t)dt.$$

If f is 1-periodic, then $f\left(\{u_n\}\right)=f\left(u_n\right)$ for all n, and so we have the result announced. Conversely, if for any Riemann integrable and 1-periodic function f we have

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(\{u_k\}\right)=\int_0^1 f(t)dt$$

then by Theorem 2.1 the sequence $(\{u_n\})_{n\in\mathbb{N}}$ is uniformly distributed , and therefore $(u_n)_{n\in\mathbb{N}}$ is uniformly distributed (mod1)■

Criterion 2.1. *(Weyl) A sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly distributed (mod 1) if and only if for any non-zero integer m,*

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\exp\left(2i\pi m u_k\right)=0$$

Proof. If the sequence $(u_n)_{n\in\mathbb{N}}$ is evenly spaced (mod1), then according to the previous corollary, for any f Riemann-integrable and 1-periodic function, we have:

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(u_k\right)=\int_0^1 f(t)dt$$

this is the case in particular for the function

$$x\to\exp(2i\pi mx)\text{ where }m\in\mathbb{Z}^*.$$

Reciprocally, if the sequence $(u_n)_{n\in\mathbb{N}}$ verifies, for any integer $m\in\mathbb{Z}^*$

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\exp\left(2i\pi m u_k\right)=0$$

and as $(x\to\exp(2i\pi mx)$ where $m\in\mathbb{Z}^*)$ is 1 -periodic we deduce that:

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\exp\left(2i\pi m\left\{u_k\right\}\right)=0$$

If f is a continuous and 1 -periodic function, then according to StoneWeierstrass' theorem (see[12]) there exists a sequence $(p_N)_{N\geq0}$ of trigonometric polynomials that converges uniformly to f.

(A trigonometric polynomial P is defined by $P(x) = \sum_{k=-n}^{k=n} \lambda_k \exp(ikx)$ where $\underset{-n\leq k\leq n}{(\lambda_k)} \in \mathbb{C}$, $x \in \mathbb{R}$). Then for $\varepsilon > 0$, we can find a trigonometric polynomial $p_{j,\varepsilon}(x) = \sum_{m=-N_j}^{N_j} \lambda_m \exp(2i\pi mx)$ such that:

$$\forall x \in [0,1] \text{ we have } |f(x) - p_{j,\varepsilon}(x)| < \varepsilon$$

For all $n \geq 1$, we have

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(\{u_k\}) - \frac{1}{n} \sum_{k=0}^{n-1} p_{j,\varepsilon}(\{u_k\}) \right| < \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon = \varepsilon$$

and

$$\left| \int_0^1 f(t)dt - \int_0^1 p_{j,\varepsilon}(t)dt \right| < \int_0^1 \varepsilon dt = \varepsilon$$

as

$$\int_0^1 p_{j,\varepsilon}(t)dt = \lambda_0$$

then

$$\left| \int_0^1 f(t)dt - \lambda_0 \right| < \varepsilon$$

and according to $(*)$ above

$$\lim_{n \longrightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=-N_j}^{N_j} \lambda_m \exp(2i\pi m \{u_k\}) = \lambda_0$$

then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=-N_j}^{N_j} \lambda_m \exp(2i\pi m \{u_k\}) - \lambda_0 \right| < \varepsilon$$

i.e. for $n \geq n_0$

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(\ \{u_k\}) - \int_0^1 f(t)dt \ \right| + \left| \int_0^1 f(t)dt - \lambda_0 \right| \quad < 3\varepsilon.$$

Since this is true for every real $\varepsilon > 0$, we derive

$$\lim_{n \longrightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\{u_k\}) = \int_0^1 f(t)dt.$$

Then the sequence $(\{u_n\})_{n\in\mathbb{N}}$ is uniformly distributed , so the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly distributed (mod1).$\blacksquare$

Corollary 2.2. *If α is an irrational number, then the sequence with the general term na is uniformly distributed* (mod1).

Proof. Let α be an irrational number.
We'll show that the sequence $(n\alpha)_{n\in\mathbb{N}}$ satisfies Weyl's criterion.
Notice that for two integers m and k we have

$$\exp(2i\pi mk\alpha) = (\exp(2i\pi m\alpha))^k$$

and since

$$m\alpha \notin \mathbb{Z} \text{ and } \exp(2i\pi m\alpha) \neq 1,$$

we get

$$\sum_{k=0}^{n-1} \exp(2i\pi mk\alpha) = \sum_{k=0}^{n-1} (\exp(2i\pi m\alpha))^k$$

$$= \frac{1-(\exp(2i\pi m\alpha))^n}{1-\exp(2i\pi m\alpha)}$$

thus

$$\left|\frac{1-(\exp(2i\pi m\alpha))^n}{1-\exp(2i\pi m\alpha)}\right| \leq \frac{1+|(\exp(2i\pi m\alpha))^n|}{|1-\exp(2i\pi m\alpha)|}$$
$$\leq \frac{2}{|1-\exp(2i\pi m\alpha)|}$$

We derive

$$\lim_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \exp(2i\pi mk\alpha) = 0.$$

So, according to Weyl's criterion, the sequence $(n\alpha)_{n\in\mathbb{N}}$ is equirrepar (mod1). ■

2.0.4 Measure of lebesgue in $\mathbb{R}$

Definition 2.2. *The exterior measure of any interval I with extrema $a < b$ is the positive real number $b - a$, denoted by $m^*(I)$.*

The exterior measure of an interval I is the same whether the interval is open, closed, or semi-open; in the following we consider open intervals. We extend the outer measure to all open $\mathbb{R}$ in the following way:

Definition 2.3. *Every open of $\mathbb{R}$ is a disjoint countable union of open intervals $]a_n, b_n[$ for $n \in \mathbb{N}$. This is unique. The exterior measure of such an open is then*

$$\sum_{n=0}^{+\infty} (b_n - a_n).$$

Definition 2.4. *Let $\mathcal{A} \subset \mathbb{R}$ be a bounded part. The outer measure of $\mathcal{A}$ is*

$$m^*(\mathcal{A}) = \inf\{m^*(\omega), \omega \text{ open and } \mathcal{A} \subset \Omega\}$$

If $\mathcal{A} \subset [a, b]$, the inner measure of $\mathcal{A}$ is

$$m_*(\mathcal{A}) = (b-a) - m^*([a,b]\backslash\mathcal{A}).$$

If $\mathcal{A}$ is unbounded, then the outer and inner measures of $\mathcal{A}$ are

$$m^*(\mathcal{A}) = \lim n \to +\infty m^*(\mathcal{A} \cap [-n,n]) \text{ and } m*(\mathcal{A}) = \lim n \to +\infty m*(\mathcal{A} \cap [-n,n])$$

We say that $\mathcal{A}$ is measurable if and only if if $m^*(\mathcal{A}) = m_*(\mathcal{A})$ *and we note $m(\mathcal{A})$ the common value. $m(\mathcal{A})$ is the Lebesgue measure of $\mathcal{A}$.*

2.0.5 Koksma's Theorem

Theorem 2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions continuously derivable on the interval $\lfloor a,b\rfloor$. Let (m,n)be $\in\mathbb{N}^2, m\neq n$.*

$$F_{m,n}(t)=f_m(t)-f_n(t).$$

Assume that the following conditions are satisfied for each pair (m,n).
(1) The derivative function $F'_{m,n}$ is monotone and non-zero for all $t\in[a,b]$.
(2) There exists an increasing sequence $(N_v)_{v\in\mathbb{N}}$ of integers satisfying

$$\lim_{v\longrightarrow+\infty}\frac{N_{v+1}}{N_v}=1$$

so that if for $N\geq 2$ we have

$$A_N=\frac{1}{N^2}\sum_{n=2}^{N}\sum_{m=1}^{n-1}\max\left\{\frac{1}{F'_{m,n}(a)},\frac{1}{F'_{m,n}(b)}\right\},$$

the series $\sum_{v\in\mathbb{N}}A_{N_v}$ is convergent. Then the series $(f_n(t))_{n\in\mathbb{N}}$ is evenly spaced (mod1) for almost all $t\in[a,b]$ (i.e., the set where the series is not evenly spaced (mod1) is of measure zero in the Lebesgue sense). To prove this theorem, we need the following two lemmas:

Lemma 3. *Let $(N_v)_{v\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with*

$$\lim_{v\longrightarrow+\infty}\frac{N_{v+1}}{N_v}=1;$$

then we can extract a subset $(N_{v_k})_{k\in\mathbb{N}}$ with

$$\lim_{k\rightarrow+\infty}\frac{N_{v_{k+1}}}{N_{v_k}}=1$$

so that the series $\sum_{k\in\mathbb{N}}\frac{1}{N_{v_k}}$ is convergent.

Proof For all $m\in\mathbb{N}^*$, let I_m be the interval $\left[m^2,(m+1)^2\right[$; We define the sequence $(N_{v_k})_{k\in\mathbb{N}^*}$ as follows: If the interval I_m contains at least two terms of the sequence $(N_v)_{v\in\mathbb{N}}$, let N_{v_k} be the smaller and $N_{v_{k+1}}$ the larger. If it contains a single term, let N_{v_k} be that term. Now two terms N_{v_k} and $N_{v_{k+1}}$ are either consecutive in the sequence $(N_v)_{v\in\mathbb{N}}$, or they satisfy

$$1<\frac{N_{v_{k+1}}}{N_{v_k}}<\frac{(m+1)^2}{m^2}$$

then

$$\lim_{k\longrightarrow+\infty}\frac{N_{v_{k+1}}}{N_{v_k}}=1.$$

Let be the interval

$$B_t=\bigcup_{m=1}^{t-1}I_m=\left[1,t^2\right[$$

we have that the interval B_t contains at most $2(t-1)$ terms of the sequence $(N_{v_k})_{k\in\mathbb{N}^*}$ so for all $t\geq 2$ we have

$$N_{v_{2t-1}}\geq t^2 \text{ and } N_{v_{2t}}\geq t^2$$

and like

$$\sum_{k\in\mathbb{N}^*}\frac{1}{N_{v_k}}=\sum_{t\in\mathbb{N}^*}\left(\frac{1}{N_{v_{2t-1}}}+\frac{1}{N_{v_{2t}}}\right)\in\ \leq\sum_{t\in\mathbb{N}^*}\frac{2}{t^2}$$

The convergence of the series $\sum_{t\in\mathbb{N}^*}\frac{2}{t^2}$ implies that the series $\sum_{k\in\mathbb{N}^*}\frac{1}{N_{v_k}}$ is convergent. ■

Lemma 4. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that the series $\sum_{n\geq 0} u_n$ is convergent. Then there exists an increasing series $(\gamma_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty}\gamma_n = +\infty$, so that the series $\sum_{n\geq 1} u_n\gamma_n$ is convergent.*

Proof. Let $S = \sum_{n\geq 0} u_n$ and for any integer n, $R_n = \sum_{k\geq n+1} u_k$. The sequence defined by

$$\gamma_n = \frac{\sqrt{S}}{\sqrt{R_{n-1}} + \sqrt{R_n}}, n \geq 1$$

and an ascending sequence with $\lim_{n\to+\infty}\gamma_n = +\infty$. We have

$$\begin{aligned}\sum_{k=1}^{n} u_k\gamma_k &= \sum_{k=1}^{n} u_k \frac{\sqrt{S}}{\sqrt{R_{k-1}} + \sqrt{R_k}} = \sqrt{S}\sum_{k=1}^{n} u_k \frac{\sqrt{R_{k-1}} - \sqrt{R_k}}{R_{k-1} - R_k} \\ &= \sqrt{S}\sum_{k=1}^{n}\left(\sqrt{R_{k-1}} - \sqrt{R_k}\right) = \sqrt{S}\left(\sqrt{S} - \sqrt{R_n}\right).\end{aligned}$$

It follows that the series $\sum_{n\geq 1} u_n\gamma_n$ is convergent. ■

Proof (of Theorem 2.2) *Assume that the conditions of Theorem 2.2 are satisfied. Let*

$$\sigma_h(N,t) = \frac{1}{N}\sum_{n=1}^{N}\exp\left(2i\pi h f_n(t)\right) \text{ where } h \in \mathbb{Z}^*$$

We now construct a zero-measure part E of the interval $[a,b]$ in the Lebesgue sense, such that for any $t \in [a,b]\backslash E$,

$$\lim_{N\to+\infty}\sigma_h(N,t) = 0, \forall h \in \mathbb{Z}^*$$

We have:

$$\begin{aligned}|\sigma_h(N,t)|^2 &= \left(\frac{1}{N}\sum_{n=1}^{N}\exp\left(2i\pi h f_n(t)\right)\right)\left(\frac{1}{N}\sum_{n=1}^{N}\exp\left(-2i\pi h f_n(t)\right)\right) \\ &= \frac{1}{N} + \frac{1}{N^2}\sum_{n=2}^{N}\sum_{m=1}^{n-1}\left[\exp\left(2i\pi h\left(f_m(t) - f_n(t)\right)\right) + \exp\left(2i\pi h\left(f_n(t) - f_m(t)\right)\right)\right] \\ &= \frac{1}{N} + \frac{2}{N^2}\sum_{n=2}^{N}\sum_{m=1}^{n-1}\cos\left(2\pi h F_{m,n}(t)\right)\end{aligned}$$

then

$$\int_a^b |\sigma_h(N,t)|^2\,dt = \frac{b-a}{N} + \frac{2}{N^2}\sum_{n=2}^{N}\sum_{m=1}^{n-1}\int_a^b \cos\left(2\pi h F_{m,n}(t)\right)dt$$

We set

$$I_{m,n}(h) = \int_a^b \cos\left(2\pi h F_{m,n}(t)\right)dt$$

Knowing that the sign of the derivative $F'_{m,n}$ remains constant on the interval $[a,b]$, then the function $F_{m,n}$ has an inverse function $\Phi_{m,n}$, we have

$$I_{m,n}(h) = \int_\alpha^\beta \Phi'_{m,n}(u)\cos(2\pi hu)du \text{ with } \alpha = F_{m,n}(a), \beta = F_{m,n}(b)$$

Since the function $\Phi'_{m,n}$ is monotone, the second formula of the mean theorem gives

$$\begin{aligned}|I_{m,n}(h)| &\leq \frac{1}{\pi|h|}\max\left(\left|\Phi'_{m,n}(\alpha)\right|,\left|\Phi'_{m,n}(\beta)\right|\right)\\ &= \frac{1}{\pi|h|}\max\left(\frac{1}{\left|F'_{m,n}(a)\right|},\frac{1}{\left|F'_{m,n}(b)\right|}\right)\end{aligned}$$

and then

$$\begin{aligned}\int_a^b |\sigma_h(N,t)|^2\,dt &\leq \frac{b-a}{N}+\frac{2}{N^2}\sum_{n=2}^{N}\sum_{m=1}^{n-1}\frac{1}{\pi|h|}\max\left(\frac{1}{\left|F'_{m,n}(a)\right|},\frac{1}{\left|F'_{m,n}(b)\right|}\right)\\ &\leq \frac{b-a}{N}+\frac{1}{N^2}\sum_{n=2}^{N}\sum_{m=1}^{n-1}\max\left(\frac{1}{\left|F'_{m,n}(a)\right|},\frac{1}{\left|F'_{m,n}(b)\right|}\right).\end{aligned}$$

So

$$\int_a^b |\sigma_h(N,t)|^2\,dt \leq \frac{b-a}{N}+A_N$$

Thanks to condition (2) and Lemma 3, we can successively extract a sequence of $(N_v)_{v\in\mathbb{N}}$, again denoted $(N_v)_{v\in\mathbb{N}}$, such that the series

$$\sum_{v\in\mathbb{N}}\left(\frac{b-a}{N_v}+A_{N_v}\right)$$

converges.
According to Lemma 4, there exists an increasing $(\gamma_v)_{v\in\mathbb{N}}$ such that the series

$$\sum_{v\in\mathbb{N}}\left(\frac{b-a}{N_v}+A_{N_v}\right)\gamma_v$$

converges.

We now define the set E.

To do this, let's put: $\forall v\in\mathbb{N}, \forall h\in\mathbb{Z}^*$

$$E_v(h)=\{t\in[a,b]/\,|\sigma_h(N_v,t)|>\frac{1}{\sqrt{\gamma_v}}\}$$

The Lebesgue measure of $E_v(h)$ satisfies

$$m\left(E_v(h)\right)\leq\gamma_v\int_a^b\left|\sigma_h\left(N_v,t\right)\right|^2dt\leq\gamma_v\left(\frac{b-a}{N_v}+A_{N_v}\right)$$

Let's say: $\forall h\in\mathbb{Z}^*$

$$F_u(h)=\cup_{v=u+1}^{+\infty}E_v(h).$$

It is a countable collection of measurable sets, so it is measurable and the sequence $(F_u(h))_{u\in\mathbb{N}}$ is a decreasing sequence of sets.

Let $m\left(F_u(h)\right.$ be the measure of $F_u(h)$, then we have

$$m\left(F_u(h) \leq \sum_{v=u+1}^{+\infty} m\left(E_v(h)\right) \leq \sum_{v=u+1}^{+\infty} \gamma_v\left(\frac{b-a}{N_v}+A_{N_v}\right)\right.$$

and So

$$\lim_{u \longrightarrow+\infty} m\left(F_u(h)=0\right.$$

The set $\cap_{u=1}^{+\infty} F_u(h)$ has Lebesgue measure zero, and the same is true for the set E defined by

$$E=\cup_{h \in \mathbb{Z}^*}\left[\cap_{u=1}^{+\infty} F_u(h)\right]$$

because E is a countable union of sets of measure zero.

If t does not belong to E, then for any $h \in \mathbb{Z}^*, t$ does not belong to $\cap_{u=1}^{+\infty} F_u(h)$ and there exists some $\mu_1(h)$ such that for $\mu \geq \mu_1(h)$ we have

$$\left|\sigma_h\left(N_\mu, t\right)\right| \leq \frac{1}{\sqrt{\gamma_\mu}}$$

it follows that

$$\lim_{u \longrightarrow+\infty} \sigma_h\left(N_\mu, t\right)=0.$$

For any integer $N \geq N_0$, there exists μ such that

$$N_\mu \leq N<N_{\mu+1}$$

then we have

$$\begin{aligned}
\left|\sigma_h(N, t)\right| & =\left|\frac{1}{N} \sum_{n=1}^N \exp \left(2 i \pi h f_n(t)\right)\right| \\
& \leq \frac{1}{N_\mu}\left|\sum_{n=1}^N \exp \left(2 i \pi h f_n(t)\right)\right| \\
& \leq \frac{1}{N_\mu}\left|\sum_{n=1}^{N_{\mu+1}} \exp \left(2 i \pi h f_n(t)\right)\right| \\
& \leq \frac{1}{N_\mu}\left|\sum_{n=1}^{N_\mu} \exp \left(2 i \pi h f_n(t)\right)\right|+\frac{1}{N_\mu} \sum_{n=N_\mu+1}^{N_{\mu+1}}\left|\exp \left(2 i \pi h f_n(t)\right)\right| \\
& \leq \sigma_h\left(N_\mu, t\right)+\frac{N_{\mu+1}-N_\mu}{N_\mu}
\end{aligned}$$

we get whatever $h \in \mathbb{Z}^*$

$$\lim_{N \rightarrow+\infty} \sigma_h(N, t)=0$$

The sequence $\left(f_n(t)\right)_{n \in \mathbb{N}}$ is therefore evenly spaced $(\bmod 1)$.$\blacksquare$

Remark 5. *In Koksma's theorem, we replace condition (2) with the following stronger condition (3):*

(3) There exists a real $K>0$ such that

$$\left|F'_{m,n}(t)\right| \geq K, \forall t \in [a,b]$$

Let t be in $[a,b]$. For any integer $N > 1$, order the sequence $(f'_n(t))_{1\leq n\leq N}$ in ascending order. In the new order we have

$$f'_n(t) - f'_{n-1}(t) \geq K \text{ where } 1 \leq n \leq N$$

and therefore $\forall t \in [a,b]$

$$f'_m(t) - f'_n(t) = \sum_{i=n+1}^{m} f'_i(t) - f'_{i-1}(t) \geq \sum_{i=n+1}^{m} K$$
$$\geq K(m-n) \text{ where } 1 \leq n < m \leq N$$

This gives us

$$\sum_{n=1}^{m-1} \frac{1}{\left|F'_{m,n}(t)\right|} \leq \frac{1}{K} \sum_{n=1}^{m-1} \frac{1}{m-n}$$
$$= \frac{1}{K} \sum_{n=1}^{m-1} \frac{1}{n} \leq \frac{1}{K}\left(1 + \sum_{n=2}^{N} \int_{n-1}^{n} \frac{1}{t} dt\right)$$

$$\leq \frac{1}{K}\left(1 + \int_{1}^{N} \frac{1}{t} dt\right) \leq \frac{1}{K}(1 + \log N)$$

and then

$$A_N \leq \frac{N(1+\log N)}{KN^2} = \frac{(1+\log N)}{KN}.$$

We now define the sequence $(N_v)_{v>0}$ by

$$N_v = v^2$$

then we have

$$\lim_{v\to+\infty} \frac{N_{v+1}}{N_v} = 1$$

It follows that

$$A_{Nv} \leq \frac{(1+2\log v)}{Kv^2}$$

If we assume for $x \geq 1$ that

$$f(x) = \frac{(1+2\log x)}{x^2}$$

we know that

$$f^{prime}(x) = -\frac{4\log x}{x^3} < 0$$

then the series

$$\sum_{v\in\mathbb{N}^*}\frac{(1+2\log v)}{Kv^2}$$

is of the same kind as the integral

$$\int_1^{+\infty}\frac{(1+2\log x)}{Kx^2}dx$$

and since

$$\int_1^{+\infty}\frac{(1+2\log x)}{Kx^2}dx=\left[-\frac{(1+2\log x)}{Kx}\right]_1^{+\infty}+\int_1^{+\infty}\frac{2}{Kx^2}dx$$
$$=\frac{3}{K}$$

Therefore $\sum_{v\in\mathbb{N}}A_{N_v}$ *is convergent, so condition (2) is satisfied.*

Theorem 2.3. *Let* λ *be a non-zero real number; the sequence* $(\lambda\alpha^n)_{n\in\mathbb{N}}$ *is equiregular (mod 1) for almost all real numbers* $\alpha>1$.

Proof. *Assuming* $t>1, f_n(t)=\lambda t^n\,(n\in\mathbb{N}^*)$, *we have*

$$f_n'(t)-f_m'(t)=\lambda\left(nt^{n-1}-mt^{m-1}\right)\ \text{for } n\neq m$$

where

$$\left|f_n'(t)-f_m'(t)\right|\geq|\lambda|>0$$

Applying Koksma's theorem, we deduce that the sequence $(\lambda t^n)_{n\in\mathbb{N}}$ *is* (mod1) *equirreparable for almost all* $t\in[k,k+1]\,(k\in\mathbb{N}^*)$.

Let $E_k\subset[k,k+1]$ *be the set of real numbers* α *such that: the sequence* $(\lambda\alpha^n)_{n\in\mathbb{N}}$ *is not ordered* (mod1), *then* E_k *has Lebesgue measure zero. Let's posit*

$$E=\cup_{k\in\mathbb{N}^*}E_k$$

Since a countable union of sets of Lebesgue measure zero is of measure zero, the sequence $(\lambda t^n)_{n\in\mathbb{N}}$ *is equirect* (mod1) *for almost all real* $t>0$.■

Chapter 3

Some properties of exceptional sets

3.1 Characterization of subset S of algebraic integers

We're going to introduce a special set of numbers, which we'll call U. To do this, we'll need the following theorem:.

Theorem 3.1. *[7] Let α be a real > 1. Suppose there exists a real $\lambda \geq 1$, such that*

$$\|\lambda\alpha^n\| \leq \frac{1}{2e\alpha(\alpha+1)(1+\log\lambda)}, \forall n \in \mathbb{N} \tag{3.1}$$

Then α is an algebraic integer; its conjugates have modulus less than or equal to 1 and λ belongs to the field $Q(\alpha)$, generated on Q by α.

Remark 6. *We see that the sequence $(\lambda\alpha^n)_{n\in\mathbb{N}}$ verifying condition (3.1) above is non-equivalent ($\bmod 1)$: indeed since*

$$2e\alpha(\alpha+1)(1+\log\lambda) > 8$$

then

$$\|\lambda\alpha^n\| < \frac{1}{8}, \forall n \in \mathbb{N}$$

so the sequence of integers of the general term

$$\varphi(n) = \mathrm{Card}\left\{k \in \mathbb{N}, k < n, \frac{1}{8} \leq \{u_k\} \leq \frac{7}{8}\right\}, \forall n \in \mathbb{N}$$

is zero, so $\varphi(n)/n \underset{n\to\infty}{\longrightarrow} 0 \neq \frac{7}{8} - \frac{1}{8} = \frac{3}{4}$ and the sequence $(\lambda\alpha^n)_{n\in\mathbb{N}}$ is not equal to $(\bmod 1)$.

To prove this theorem, we need some auxiliary results.

Let's say:

$$u_n = E'(\lambda\alpha^n), \varepsilon_n = \varepsilon(\lambda\alpha^n) \ \ or u_n + \varepsilon_n = \lambda\alpha^n, \forall n \in \mathbb{N},$$

Let's introduce the linear form V_n defined on $\mathbb{R}^{s+1}$ by

$$V_n(x) = \sum_{i=0}^{s} u_{n+i}x_i \ \ or\ s \in \mathbb{N}^*$$

and suppose there exists $a = (a_0, \dot{;} a_s) \in \mathbb{Z}^{s+1} \backslash (0)$ *and* $\mathbf{A} \in \mathbb{N}^*$ *so that*

$$\sup_{0 \le i \le s} |a_i| \le \mathbf{A}.$$

Lemma 5. *if* $V_n(a) = 0$ *and*

$$|\varepsilon_i| < \frac{1}{(s+1)(\alpha+1)\mathbf{A}}, \forall i \in \mathbb{N} \tag{3.2}$$

then $V_{n+1}(a) = 0$.

Proof. We have:

$$\begin{aligned} |V_{n+1}(a) - \alpha V_n(a)| &= \left| \sum_{i=0}^{s} a_i \left(u_{n+1+i} - \alpha u_{n+i} \right) \right| \\ &= \left| \sum_{i=0}^{s} a_i \left(\left(\lambda \alpha^{n+1+i} - \varepsilon_{n+1+i} \right) - \alpha \left(\lambda \alpha^{n+i} - \varepsilon_{n+i} \right) \right) \right| \end{aligned}$$

or

$$|V_{n+1}(a) - \alpha V_n(a)| \le \sum_{i=0}^{s} \left| a_i \left(\varepsilon_{n+1+i} - \alpha \varepsilon_{n+i} \right) \right|.$$

Or

$$|a_i \left(\varepsilon_{n+1+i} - \alpha \varepsilon_{n+i} \right)| \le \mathbf{A} \left(|\varepsilon_{n+1+i}| + \alpha \left| \varepsilon_{n+i} \right| \right)$$

$$< \frac{(1+\alpha)\mathbf{A}}{(s+1)(\alpha+1)\mathbf{A}} = \frac{1}{(s+1)}$$

so

$$|V_{n+1}(a) - \alpha V_n(a)| < \sum_{i=0}^{s} \frac{1}{(s+1)} = 1.$$

Since by hypothesis, $V_n(a) = 0$, we obtain $|V_{n+1}(a)| < 1$ and since $V_{n+1}(a)$ is an integer, then $V_{n+1}(a) = 0$. ■

Lemma 6. *if*

$$|\varepsilon_i| < \frac{1}{(s+1)(\alpha+1)\mathbf{A}}, \forall i \in I\!N$$

and

$$\mathbf{A} \ge 2\lambda^{\frac{1}{S}} \alpha - 1 \tag{3.3}$$

then there exists $a \in \mathbb{Z}^{s+1} \backslash (0)$ *such that* $V_0(a) = 0$.

Proof. Let V_0 be the linear form defined by

$$V_0(x) = \sum_{i=0}^{s} u_i x_i$$

and let be the set

$$D_{(\mathbf{A},s)} = \left\{ x = (x_0, \ldots, x_s) \in Z^{s+1}, 0 \le x_i \le \mathbf{A} \right\}.$$

We have:

$$\text{Card} \left(D_{(\mathbf{A},s)} \right) = (\mathbf{A}+1)^{s+1}$$

and

$$0 \leq V_0(x) = \sum_{i=0}^{s} \left(\lambda\alpha^i - \varepsilon_i\right) x_i$$
$$\leq \sum_{i=0}^{s} \left(\lambda\alpha^i + |\varepsilon_i|\right) x_i \leq \sum_{i=0}^{s} \left(\lambda\alpha^s + |\varepsilon_i|\right) \mathbf{A}.$$

Then, according to (3.2)

$$0 \leq V_0(x) \leq (s+1)\mathbf{A}\left(\lambda\alpha^s + \frac{1}{(s+1)(\alpha+1)\mathbf{A}}\right)$$

$$= (s+1)(\mathbf{A}+1)\lambda\alpha^s - (s+1)\lambda\alpha^s + \frac{1}{\alpha+1}.$$

We also have

$$\alpha \geq 1, s \geq 1 \text{ et } \lambda \geq 1,$$

then

$$\frac{1}{\alpha+1} \leq \frac{1}{2} \text{ and } -(s+1)\lambda\alpha^s \leq -2$$

or

$$-(s+1)\lambda\alpha^s + \frac{1}{\alpha+1} < -1$$

so

$$0 \leq V_0(x) < (s+1)(\mathbf{A}+1)\lambda\alpha^s - 1$$

and according to (3.3)

$$(\mathbf{A}+1)^{s+1} \geq 2^s(\mathbf{A}+1)\lambda\alpha^s \geq (s+1)(\mathbf{A}+1)\lambda\alpha^s$$

or

$$0 \leq V_0(x) < (\mathbf{A}+1)^{s+1} - 1.$$

Since the linear form used above is defined on the set $D_{(\mathbf{A},s)}$ to $(\mathbf{A}+1)^{s+1}$ elements and takes its values from the whole $\left\{0,\ldots,(\mathbf{A}+1)^{s+1}-2\right\}$ to $(\mathbf{A}+1)^{s+1}-1$ elements, the drawer principle ensures that there are two different points b and b_0 in

$D_{(\mathbf{A},s)}$ such as $V_0(b) = V_0(b_0)$. So, for $a = b - b_0$we obtain that $a \in \mathbb{Z}^{s+1}\backslash(0)$ with $\sup_{0\leq i\leq s}|a_i| \leq \mathbf{A}$ and $V_0(a) = 0$. ■

Definitions of numbers S and A [4]

Thanks to condition (3.1), we can determine S and A and, consequently, build a a dans $\mathbb{Z}^{s+1}\backslash(0)$ satisfying the condition

$$V_n(a) = 0, \forall n \in \mathbb{N}.$$

Lets ask

$$s-1 \leq \log\lambda < s \text{ and } \mathbf{A} < 2\lambda^{\frac{1}{s}}\alpha \leq \mathbf{A}+1$$

and consider the continuous and derivable function on $[s-1; s]$:

$$\varphi(x) = 1 - \frac{x}{s} + \log\frac{1+x}{1+s}.$$

We have:

$$\varphi(s-1) = \frac{1}{s} - \log\left(1+\frac{1}{s}\right) > 0, \varphi(s) = 0$$

and

$$\varphi'(x) = -\frac{1}{s} + \frac{1}{x+1} \leq 0 \text{ with } x \geq s-1$$

so φ is strictly decreasing in the interval $[s-1; s]$, or

$$\varphi(s-1) \geq \varphi(\log\lambda) > \varphi(s) = 0.$$

so

$$\varphi(\log\lambda) = 1 - \frac{\log\lambda}{s} + \log\frac{1+\log\lambda}{1+s} > 0$$

done

$$\frac{\log\lambda}{s} + \log(1+s) < 1 + \log(1+\log\lambda)$$

Passing to the exponential, we find

$$(s+1)\lambda^{\frac{1}{s}} < e(1+\log\lambda) \tag{3.4}$$

Let's now check that condition (3.1) and the definition of A imply conditions (3.2) and (3.3). We have:

$$2\lambda^{\frac{1}{s}}\alpha \leq \mathbf{A}+1 \text{ that is to say } 2\lambda^{\frac{1}{s}}\alpha - 1 \leq \mathbf{A}\text{,or (3.3).}$$

Also

$$\mathbf{A} < 2\lambda^{\frac{1}{s}}\alpha, \text{ d'oÃź } \mathbf{A}(s+1) < 2\lambda^{\frac{1}{s}}\alpha(s+1),$$

or according to (3.4)

$$\mathbf{A}(s+1) < 2e\alpha(1+\log\lambda) \tag{3.5}$$

Furthermore, according to (3.1)

$$2e\alpha(1+\log\lambda) < \frac{1}{(\alpha+1)\,|\varepsilon_n|}, \tag{3.6}$$

it follows from (3.5) and (3.6) that

$$|\varepsilon_n| < \frac{1}{(s+1)(\alpha+1)\mathbf{A}},$$

which is fine (3.2).

Proof.(of theorem 3.1) From the two previous lemmas we can find an $a = (a_i)_{0\leq i\leq s}$, with non-zero a_i in $\mathbb{Z}$, such that

$$a_0u_n + a_1u_{n+1} + \ldots + a_su_{n+s} = 0, \forall n \in \mathbb{N}.$$

Then according to criterion 1.1.1 and lemma 1.1.1 the series

$$\sum_{n\succeq 0} u_n z^n$$

is rational, hence the existence of two polynomials B and Q prime to each other, with $Q(0) = 1$, with integer coefficients such that

$$\frac{B(z)}{Q(z)} = \sum_{n\succeq 0} u_n z^n = \sum_{n\succeq 0} (\lambda\alpha^n - \varepsilon_n)\, z^n$$

$$= \sum_{n\succeq 0} \lambda\alpha^n z^n - \sum_{n\succeq 0} \varepsilon_n z^n.$$

We have:

$$|\varepsilon_n|^{\frac{1}{n}} \leq \left(\frac{1}{2}\right)^{\frac{1}{n}}$$

then

$$\lim \sup_{n\longrightarrow+\infty} |\lambda\alpha^n|^{\frac{1}{n}} = \alpha \text{ and } \limsup_{n\longrightarrow+\infty} |\varepsilon_n|^{\frac{1}{n}} \leq 1.$$

Thus, the series

$$\sum_{n\succeq 0} \lambda\alpha^n z^n \text{ and } \sum_{n\succeq 0} \varepsilon_n z^n$$

are convergent on disks $D(0.1/\alpha)$ and $D(0.1)$ respectively, then

$$\frac{B(z)}{Q(z)} = \frac{\lambda}{1-\alpha z} - \sum_{n\succeq 0} \varepsilon_n z^n, \forall z \in D(0, 1/\alpha).$$

The series,

$$\sum_{n \succeq 0} \varepsilon_n z^n$$

and being analytic on $D(0,1)$, the function f such that

$$f(z) = \frac{B(z)}{Q(z)}$$

has the number $1/\alpha$ as its unique pole (of order 1) on $D(0,1)$.
Thus the polynomial $Q(z)$ has the unique zero $1/\alpha$ in the disk $D(0,1)$, the number α and an algebraic integer; the conjugates $1/\alpha_i$ of $1/\alpha$, being outside $D(0,1)$ in $\mathbb{C}$, the α_i are of modulus ≤ 1.
Moreover, the residue of f in $\frac{1}{\alpha}$ is

$$\operatorname{Res}\left(f, \frac{1}{\alpha}\right) = \lim_{z \longrightarrow \frac{1}{\alpha}} \left(z - \frac{1}{\alpha}\right)\left(\frac{\lambda}{1-\alpha z} - \sum_{n \succeq 0} \varepsilon_n z^n\right) = -\frac{\lambda}{\alpha}, \tag{3.7}$$

on the other,

$$\operatorname{Res}\left(f, \frac{1}{\alpha}\right) = \lim_{z \to \frac{1}{\alpha}} \left(z - \frac{1}{\alpha}\right)\frac{B(z)}{Q(z)} = \lim_{z \to \frac{1}{\alpha}} \frac{B(z)}{\frac{Q(z)}{z - \frac{1}{\alpha}}} = \frac{B\left(\frac{1}{\alpha}\right)}{Q'\left(\frac{1}{\alpha}\right)}. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\lambda = -\alpha \frac{B\left(\frac{1}{\alpha}\right)}{Q'\left(\frac{1}{\alpha}\right)}.$$

So the number λ is indeed an element of $Q(\alpha)$. ■

3.2 Sab set of integer numbers

Definition 3.1. *[5] A Pisot number is any real algebraic integer greater than 1 whose conjugates have modulus strictly less than 1 .*
The set of Pisot numbers is denoted **S**; *it is a subset of the set* **U** *introduced earlier.*

Remark 7. *Any relative integer greater than 1 belongs to* **S**.

3.2.1 Of theus numbers

Proposition 3.1. *Let $\theta \in$* **S**. *The sequence $(\|\theta^n\|)_{n \in I\!N}$ converges to 0 .*

Proof. Let $P(X) = X^s + c_{s-1}X^{s-1} + \ldots + c_0$ be the irreducible polynomial in $\mathbb{Z}[X]$ such that $P(\theta) = 0$.

Consider its companion matrix

$$\mathfrak{C} = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & . & 0 & -c_2 \\ : & : & . & . & : \\ 0 & 0 & .. & 1 & c_{s-1} \end{pmatrix}.$$

It's a matrix with integer coefficients, and its characteristic polynomial is P.
Since P is split in $\mathbb{C}[X]$, we can triangularize $\mathfrak{C}$ and obtain the following matrix:

$$\mathfrak{D} = \begin{pmatrix} \theta & a_{1,2} & .. & a_{1,s-1} & a_{1,s} \\ 0 & \theta_2 & .. & . & a_{2,s} \\ 0 & 0 & . & : & : \\ : & : & . & . & a_{s-1,s} \\ 0 & 0 & .. & 0 & \theta_s \end{pmatrix}.$$

Or $(\theta_j)_{j\in\{2,..,s\}}$ are the conjugates of θ and by construction θ and $(\theta_j)_{j\in\{2,..,s\}}$ are the eigenvalues of the matrix. We then obtain that:

$$\mathrm{Tr}(\mathfrak{C}) = \mathrm{Tr}(\mathfrak{D}) \text{ and } \mathrm{Tr}\left(\mathfrak{C}^n\right) = \mathrm{Tr}\left(\mathfrak{D}^n\right).$$

We deduce that:

$$\mathrm{Tr}\left(\mathfrak{D}^n\right) \in \mathbb{Z} \text{ and } \mathrm{Tr}\left(\mathfrak{D}^n\right) = \theta^n + \sum_{j=2}^{s} \theta_j^n.$$

Note that

$$d = \sup_{2\leq j\leq s} |\theta_j| < 1,$$

we have

$$\left|\mathrm{Tr}\left(\mathfrak{D}^n\right) - \theta^n\right| \leq \sum_{j=2}^{s} \left|\theta_j^n\right| \leq (s-1)d^n.$$

Now

$$\lim_{n\to+\infty} (s-1)d^n = 0$$

therefore θ^n tends to wards an integer, which means by definition of ε_n that $\|\theta^n\|$ will tend towards 0. ■

This Proposition leads us to the following question: could we find a necessary and sufficient condition for membership of **S** through a convergence of the same type as in proposition 3.1?

To answer this question, we will first demonstrate a proposition borrowed from complex analysis.

Proposition 3.2. *Let φ be a meromorphic function on an open Ω containing $\bar{D}(0,1)$. Suppose φ has a Taylor series expansion in 0 on the disk $D(0,1)$*

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \textit{ with } \lim_{n\to+\infty} a_n = 0.$$

Then φ has no pole on the circle $C(0,1)$.

Proof. From Proposition 1.1.3, the function φ is analytic on $D(0,1)$, so we deduce that the radius of convergence of the series is $R \geq 1$.

1^{eer} **case:** if $R > 1$ then φ has no pole on the circle $C(0,1)$.

2^{eme} **case:** if $R = 1$ then φ has at least one singular point on the circle $C(0,1)$. (otherwise, $R > 1$ and by hypothesis, this is not the case).

We can assume, without loss of generality, that this singularity lies at $z = 1$. Let $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0, |a_n| < \varepsilon$. So for $0 < r < 1$, we have:

$$|\varphi(z)| \leq \left| \sum_{n=0}^{n_0-1} a_n z^n \right| + \sum_{n=n_0}^{+\infty} |a_n| \, r^n \leq M + \varepsilon \frac{r^{n_0}}{1-r},$$

with M a constant. We therefore have:

$$|\varphi(z)|(1-r) \leq M(1-r) + \varepsilon$$

hence

$$\lim_{r \to 1,} |\varphi(r)|(1-r) = 0.$$

This contradicts the fact that 1 is a pole of φ. Indeed, if 1 is a pole of order $m \geq 1$ of the function φ, we have on a neighborhood of 1

$$\varphi(r) = \frac{g(r)}{(r-1)^m} \text{ where } g(1) \neq 0$$

and the limit gives us that $\lim_{r\to 1} \frac{g(r)}{(1-r)^{m-1}} = 0$ then $g(1) = 0$. This is absurd, so we have $R > 1$.$\blacksquare$

We can now state the following theorem:

Theorem 3.2. *An algebraic real number theta > 1 belongs to* **S** *if and only if there exists a real $\lambda \neq 0$ such that*

$$\lim_{n \to +\infty} \|\lambda \theta^n\| = 0$$

Proof. Since θ is algebraic, we can find a polynomial with integer coefficients that cancels out at θ. Let $\sum_0^s q_i X^i$ be this polynomial. Then we have:

$$\sum_0^s q_i \theta^i = 0 \text{ and } \sum_0^s q_i \lambda \theta^{n+i} = 0, \forall n \in \mathbb{N}.$$

By decomposing $\lambda\theta^{n+i}$ into the sum of $u_{n+i} = E^{prime}\left(\lambda\theta^{n+i}\right)$ and $\varepsilon_{n+i} = \varepsilon\left(\lambda\theta^{n+i}\right)$, we can write

$$\sum_{0}^{s} q_i u_{n+i} = -\sum_{0}^{s} q_i \varepsilon_{n+i}, \forall n \in \mathbb{N}.$$

We have, by hypothesis,

$$\lim_{n \to +\infty} \varepsilon_n = 0.$$

So, for $n \geq n_0$ we have:

$$\left|\sum_{0}^{s} q_i u_{n+i}\right| < 1$$

And consequently

$$\sum_{0}^{s} q_i u_{n+i} = 0 \text{ where } n \geq n_0$$

then the series

$$\sum_{n \geq 0} u_n X^n$$

is rational. As in the proof of theorem 3.1.8, the set is equal B/Q with B and Q polynomials with integer coefficients. with B and Q polynomials with integer coefficients, prime to each other with $Q(0) = 1$. We then have:

$$\frac{B(z)}{Q(z)} = \sum_{n \geq 0} u_n z^n = \frac{\lambda}{1 - \theta z} - \sum_{n \geq 0} \varepsilon_n z^n, \forall z \in D(0, 1/\theta)$$

Using the previous property, from the condition

$$\lim_{n \longrightarrow +\infty} \varepsilon_n = 0$$

it follows that the series $\sum_{n \geq 0} \varepsilon_n z^n$ has no pole on $\bar{D}(0,1)$. Q has a unique zero in $\bar{D}(0,1)$ and $\theta \in \mathbf{S}$, the other part of the demonstration is given by theorem 3.1. ■

3.3 Appendix: Table of Pisot numbers below 1.6

The table below gives the 12 Pisot numbers below 1.6 in ascending order and their minimal polynomials.

	Number of Pisots	Polyn $\tilde{A}'$ me minimal
1	1.3247179572447460260	$x^3 - x - 1$
2	1.3802775690976141157	$x^4 - x^3 - 1$
3	1.4432687912703731076	$x^5 - x^4 - x^3 + x^2 - 1$
4	1.4655712318767680267	$x^3 - x^2 - 1$
5	1.5015948035390873664	$x^6 - x^5 - x^4 + x^2 - 1$
6	1.5341577449142669154	$x^5 - x^3 - x^2 - x - 1$
7	1.5452156497327552432	$x^7 - x^6 - x^5 + x^2 - 1$
8	1.5617520677202972947	$x^6 - 2x^5 + x^4 - x^2 + x - 1$
9	1.5701473121960543629	$x^5 - x^4 - x^2 - 1$
10	1.5736789683935169887	$x^8 - x^7 - x^6 + x^2 - 1$
11	1.5900053739013639252	$x^7 - x^5 - x^4 - x^3 - x^2 - x - 1$
12	1.5911843056671025063	$x^9 - x^8 - x^7 + x^2 - 1$

3.4 On the exceptional Weyl set

We denote μ the Lebesgue measure on $\mathbb{R}$. Further, given $(x_n)_{n\in\mathbb{N}}$ a sequence of real numbers, N a positive integer and E a subset of $[0,1[$, we set:

$$A\left(E;N;(x_n)\right) := \#\left\{n \in \mathbb{N} \mid n \leq N \; and < x_n > \in E\right\}.$$

For $r > 1$, let E_r, D_r and W_r denote respectively: the set of the real positive numbers λ satisfying

$$\|\lambda r^n\| \leq \frac{1}{r-1}(\forall n \in \mathbb{N})$$

the set of the real positive numbers λ for which the sequence $(\lambda r^n)_{n\in\mathbb{N}}$ is not dense modulo 1 and the set of the real positive numbers λ for which the sequence $(\lambda r^n)_{n\in\mathbb{N}}$ is not uniformly distributed modulo 1.

Theorem 3.3. ***(Weyl, 1916) :*** *Let $x > 1$ be a real number. Then for almost any real ξ, the sequence $\{xix^n\}$ is equidistributed.*

Theorem 3.4. *[6] For any real number $r > 1$, the set E_r is uncountably infinite.*

proof. When $r \leq 3$, it is obvious that $E_r =]0, +\infty[$ is indeed uncountably infinite.
Let's assume that $r > 3$. To show that the set E_r in the theorem is uncountably infinite, we will construct an injective function σ from $\mathbb{N}$ to $0, 1$ into E_r.
Initially, to any $f \in \{0,1\}^{\mathbb{N}}$, we associate the sequence of positive integers $u_n(f)$ defined by:

$$u_0(f) = 1$$
$$u_{n+1}(f) = \lfloor r \cdot u_n(f) \rfloor + f(n) \quad (\forall n \in \mathbb{N}) \tag{1}$$

Given $f \in \{0,1\}^{\mathbb{N}}$ fixed, set $u_n = u_n(f)$ for all $n \in \mathbb{N}$. Equation (1) implies that we have:

$$ru_n - 1 < u_{n+1} \leq ru_n + 1 \quad (\forall n \in \mathbb{N})$$

Then, using this last double inequality, we can easily verify that the two real sequences $(x_n)_n$ and $(y_n)_n$ defined by :

$$\begin{cases} x_n := \dfrac{u_n}{r^n} - \dfrac{1}{r^n(r-1)} \\ y_n := \dfrac{u_n}{r^n} + \dfrac{1}{r^n(r-1)} \end{cases} \quad (\forall n \in \mathbb{N})$$

are adjacent, more precisely: $(x_n)_n$ is increasing, $(y_n)_n$ is decreasing, and $x_n - y_n \to 0$ as n tends to infinity. These two sequences thus converge to the same limit $\lambda = \lambda(f)$ (depending on f), which necessarily satisfies:

$$x_n \leq \lambda \leq y_n \quad (\forall n \in \mathbb{N})$$

This gives:

$$|\lambda r^n - u_n| \leq \frac{1}{r-1} \quad (\forall n \in \mathbb{N}) \tag{2}$$

Now since r is assumed to > 3, we have $\frac{1}{r-1} < \frac{1}{2}$, and equation (2) shows that for any $n \in \mathbb{N}$, the integer u_n is the integer closest to the real number λr^n. Consequently, we have:

$$\|\lambda r^n\| = |\lambda r^n - u_n| \leq \frac{1}{r-1}$$

that clearly demonstrates that $\lambda = \lambda(f)$ belongs the set E_r of the theorem. We have thus established a application

$$\sigma : \{0,1\}^{\mathbb{N}} \to E_r$$

which associates any $f \in \{0,1\}^{\mathbb{N}}$ with the real number $\lambda(f)$. By equipping the set $\{0,1\}^{\mathbb{N}}$ with the usual lexicographic order and the set E_r with the induced order from the usual order of $\mathbb{R}$, we will show in the following that σ is strictly increasing with respect to these orders, which will imply its injectivity and conclude this demonstration. Let f and g be two arbitrary elements of $\{0,1\}^{\mathbb{N}}$ such that $f < g$ in the lexicographic order. Therefore, there exists $k \in \mathbb{N}$ such that we have $f(i) = g(i)$ for $0 \leq i \leq k-1$ and $f(k) < g(k)$. Hence, we certainly have $f(k) = 0$ and $g(k) = 1$. Consequently, a simple recurrence starting from the very definition of the sequences $(u_n(f))_n$ and $(u_n(g))_n$ shows that we have:

$$\begin{aligned} &u_n(f) = u_n(g) \quad \text{for } n \in \{0, \ldots, k\} \text{ and} \\ &u_{k+1}(f) - u_{k+1}(g) = f(k) - g(k) = -1 \end{aligned}$$

Hence:

$$\begin{aligned} \sigma(f) - \sigma(g) &= \frac{\sigma(f) r^{k+1} - u_{k+1}(f)}{r^{k+1}} - \frac{\sigma(g) r^{k+1} - u_{k+1}(g)}{r^{k+1}} - \frac{1}{r^{k+1}} \\ &\leq \frac{\left|\sigma(f) r^{k+1} - u_{k+1}(f)\right|}{r^{k+1}} + \frac{\left|\sigma(g) r^{k+1} - u_{k+1}(g)\right|}{r^{k+1}} - \frac{1}{r^{k+1}} \\ &\leq \frac{2}{r^{k+1}(r-1)} - \frac{1}{r^{k+1}} \quad \text{(using (2) With } n = k+1 \text{ For } f \text{ and } g) \\ &= \frac{3-r}{r^{k+1}(r-1)} < 0 \quad (\text{car}\, r > 3) \end{aligned}$$

This gives $\sigma(f) < \sigma(g)$, demonstrating that σ is strictly increasing as required. The proof is complete.■

Corollary 3.1. *For any real numberr* $\in \left[3, +\infty\right[\cup \left\{2, \frac{5}{2}\right\}$, *"The set D_r is infinite and uncountable.*

proof We distinguish the following four cases:"

• Case $r > 3$: In this case, the result of corollary 3.1.1is an immediate consequence of that of Theorem 3.4 . Indeed, let $r > 3$ be a fixed real number and λ be an arbitrary element of the set E_r from Theorem 3.4 . By the very definition of E_r, each term of the sequence $(\lambda r^n)_{n \in \mathbb{N}}$ is congruent modulo 1 to some real number in the closed interval $\left[-\frac{1}{r-1}, \frac{1}{r-1}\right]$. Since this latter interval has a length of $\frac{2}{r-1} < 1$ (because $r > 3$), its complement in $]-1/2, 1/2\,[$ is indeed a non-empty open set disjoint from the set of representatives (in $[-1/2, 1/2[$) of the modulo 1 classes of the terms of the sequence $(\lambda r^n)_n$. Consequently, the sequence $(\lambda r^n)_{n \in \mathbb{N}}$ is not dense modulo 1. This implies that $E_r \subset D_r$ and since E_r is infinite and uncountable (according to Theorem 3.4), then the same holds (a fortiori) for D_r.

• Case $r = 2$: It is easily verified (by distinguishing the cases of even and odd n) that any real number $\lambda > 0$ can be written in the form

$$\lambda = \sum_{i=0}^{\infty} \frac{a_i}{4^i}$$

(with $a_i \in \{0, 1\}$ for all $i \in \mathbb{N}$) satisfying $< \lambda 2^n > \leq \frac{2}{3} (\forall n \in \mathbb{N}$). Such a $\lambda$is therefore an element of D_r. Since the set

$$\left\{ \sum_{i=0}^{\infty} \frac{a_i}{4^i} \,\middle|\, a_i = 0.or1 \text{ } \textit{for all } i \right\}$$

is evidently infinite and uncountable, then D_r is consequently infinite and uncountable.

• Case $r = 3$: It is immediately verified that any real number $\lambda > 0$ can be written in the form $\lambda = \sum_{i=0}^{\infty} \frac{a_i}{3^i}$ (with $a_i \in \{0, 1\}$ for all $i \in \mathbb{N}$)satisfying $< \lambda 3^n \rangle \leq \frac{1}{2} (\forall n \in \mathbb{N})$. Such a $\lambda$is therefore an element of D_r. Since the set

$$\left\{ \sum_{i=0}^{\infty} \frac{a_i}{3^i} \middle| \, a_i = 0.or1 \; for \; all \; i \right\}$$

is evidently infinite and uncountable, then the same holds a fortiori for D_r.

• Case $r = \frac{5}{2}$: Let $s := r^3 > 15$. We will show in what follows that D_r contains E_s. Since E_s is infinite and uncountable (according to Theorem 3.4), we will conclude that D_r is also infinite and uncountable. Let λ be an arbitrary element of E_s. By the very definition of the set E_s, we have:

$$\left\| \lambda r^{3n} \right\| = \| \lambda s^n \| \leq \frac{1}{s-1} \quad (\forall n \in \mathbb{N})$$

This implies that the sequence $\left(< \lambda r^{3n} \rangle \right)_{n \in \mathbb{N}}$ traverses the union of the two intervals $\left[0, \frac{1}{s-1} \right]$ and $\left[1 - \frac{1}{s-1}, 1 \right[$ whose sum of lengths is $\frac{2}{s-1}$. Now, using the elementary fact asserting that:

"When the fractional part of a real number x traverses a finite union of intervals whose sum of lengths is $\leq \alpha (\alpha > 0)$, then, given $p, q \in \mathbb{N}^*$, the fractional part of the real number $\frac{p}{q} x$ traverses a finite union of intervals whose sum of lengths is" $\leq p\alpha$",

we deduce that the sequence$\left(\langle \lambda r^{3n-1} \rangle \right)_{n \in \mathbb{N}^*}$ traverses a finite union of intervals whose sum of lengths is $\leq \frac{4}{s-1}$ and that the sequence $\left(< \lambda r^{3n-2} \rangle \right)_{n \in \mathbb{N}^*}$traverses a finite union of intervals whose sum of lengths is $\leq \frac{4}{s-1}$ and that the sequence $(< \lambda r^n \rangle)_{n \in \mathbb{N}}$ traverses a finite union of intervals whose sum of lengths is $\leq \frac{2}{s-1} + \frac{4}{s-1} + \frac{8}{s-1} = \frac{14}{s-1} < 1$ (because $s > 15$). It follows from this that there exists a non-empty open $\subset [0, 1[$ which does not meet the set

$$\{ < \lambda r^n > | \, n \in \mathbb{N} \}$$

Therefore the sequence $(\lambda r^n)_{n \in \mathbb{N}}$ is not dense modulo 1, i.e. $\lambda \in D_r$. The inclusion $E_s \subset D_r$ is thus proved, which completes the proof of the corollary for this case and ends this demonstration.■

Corollary 3.2. *For any real number $r > 1$, the set W_r is infinite and uncountable.*

proof Given a real number $r > 1$, let's choose an integer $k \geq 1$ such that

$$r^k > 2k + 1$$

According to Theorem 3.4, the set E_{r^k} is infinite and uncountable. We will show in the following that this latter set is included in W_r, which will consequently imply the uncountable infinitude of the set W_r. Let λ be an arbitrary element of E_{r^k}. For any positive integer n that is a multiple of k, say $n = km$ for some $m \in \mathbb{N}$, we have:

$$\| \lambda r^n \| = \left\| \lambda \left(r^k \right)^m \right\| \leq \frac{1}{r^k - 1} \, (\text{car} \, \lambda \in E_{r^k})$$

In other words :

$$< \lambda r^n > \notin]_{\frac{1}{r^k-1}}, 1 - \frac{1}{r^k - 1}[$$

Therefore, we conclude that for

$$I =]\frac{1}{r^k - 1}, 1 - \frac{1}{r^k - 1}[$$

we have :

$$\lim \sup_{N \to +\infty} \frac{A\left(I; N; (\lambda r^n)\right)}{N} \leq 1 - \frac{1}{k} < 1 - \frac{2}{r^k - 1} = \mu(I) \left(\text{car}\, r^k > 2k + 1\right).$$

This implies that the sequence $(\lambda r^n)_{n\in\mathbb{N}}$ is not equidistributed modulo 1, hence $\lambda \in W_r$. Therefore, we indeed have $E_{r^k} \subset W_r$, completing this demonstration. ■

If we restrict the assumption of Corollary 3.1 to $r > 2$, a similar approach to that of the proof of Theorem 3.4 allows us to show that the set of real numbers $\lambda > 0$ for which the sequence $(\lambda r^n)_{n\in\mathbb{N}}$ is not dense modulo 1 is infinite; however, it does not indicate whether this set is countable or uncountable.

Theorem 3.5. *For any real number $r > 2$, the set D_r is infinite.*

proof.- We associate with every $k \in \mathbb{N}$, the sequence of positive integers $(u_n(k))_{n\in\mathbb{N}}$ defined by:

$$u_0(k) = 1$$
$$u_{n+1}(k) = \lfloor r \cdot u_n(k) \rfloor + k \quad (\forall n \in \mathbb{N})$$

This last relation implies that we have:

$$ru_n(k) + k - 1 < u_{n+1}(k) \leq ru_n(k) + k \quad (\forall k, n \in \mathbb{N})$$

This allows us to easily verify that for any $k \in \mathbb{N}$, the two sequences with general terms

$$\alpha_n(k) := \frac{u_n(k)}{r^n} + \frac{k-1}{r^n(r-1)}$$

and

$$\beta_n(k) := \frac{u_n(k)}{r^n} + \frac{k}{r^n(r-1)}$$

are respectively strictly increasing and decreasing. Furthermore, since (for all $k \in \mathbb{N}$)$\alpha_n(k) - \beta_n(k) \to 0$ when n ttends to infinity, these two sequences $(\alpha_n(k))_n$ and $(\beta_n(k))_n$ are adjacent, and consequently, they have the same limit $\lambda(k)$ satisfying:

$$\alpha_n(k) < \lambda(k) \leq \beta_n(k)(\forall n \in \mathbb{N})$$

. This implies:

$$\frac{k-1}{r-1} < \lambda(k) r^n - u_n(k) \leq \frac{k}{r-1} \quad (\forall k, n \in \mathbb{N}) \tag{3}$$

This shows that for any $k \in \mathbb{N}$, each term of the sequence $(\lambda(k)r^n)_{n \in \mathbb{N}}$ is congruent modulo 1 to some real number in the interval $\left]\frac{k-1}{r-1}, \frac{k}{r-1}\right]$; but since this interval has a length of $\frac{1}{r-1} < 1$ (because $r > 2$), then the sequence $(\lambda(k)r^n)_{n \in \mathbb{N}}$ is not dense modulo 1 for any value of $k \in \mathbb{N}$.

Finally, by setting $n = 0$ in (3), we see that

$$\lambda(k) \in \left]1 + \frac{k-1}{r-1}, 1 + \frac{k}{r-1}\right]$$

(for all $k \in \mathbb{N}$). Since the intervals

$$\left]1 + \frac{k-1}{r-1}, 1 + \frac{k}{r-1}\right]$$

(for $k \in \mathbb{N}$) are evidently pairwise disjoint, the real numbers $\lambda(k)$ (for $k \in \mathbb{N}$) are pairwise distinct, and consequently, the set of real numbers $\lambda > 0$ for which the sequence $(\lambda r^n)_{n \in \mathbb{N}}$ is not dense modulo 1 is indeed infinite. This completes the proof..■

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