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ON HADAMARD AND HADAMARD-TYPE INTEGRALS AND APPLICATIONS

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Dedication

*In the name of **ALLAH**, Most Gracious, Most Merciful. Praise be to **ALLAH** who gave me strength, inspiration and prudence to bring this thesis to a close. Peace be upon his mssanger Muhammad and his honorable family.*

We dedicate this project to our dear parents for their unwavering support throughout this academic journey.

To our sisters for their presence and encouragement.

To our teachers for their valuable teaching and wise advice.

This work is the result of our combined efforts, and we are deeply grateful to them.

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Contents

Abstract

In this memory,we will focus on the study of Hermite Hadamard type integral inequalities and some applications for some special means, also we quote Fractional integrals identities.

Keywords: Hadamard fractional integrals. Hadamard-type fractional integral, convex function, Hermite-Hadamard inequality.

Introduction

The theory of inequalities has emerged as an interesting area to explore in recent years. It also constitutes an important subject of research where several mathematical situations call for these inequalities. However, integral inequalities have experienced significant development, and new techniques or even new methods have emerged, contributing to solving numerous important problems in the theory of approximation and numerical analysis, where error estimation is required. Moreover, the importance of these integral inequalities largely arises in probability theory, real analysis, complex analysis, numerical analysis.... One very interesting inequality that is widely studied in the literature is due to Hermite and Hadamard, who discovered it independently (discovered by Charles Hermite in 1883 and proved by Jacques Hadamard in 1893). It is now known as the Hermite-Hadamard inequality, which can be considered the first fundamental result for convex functions with a natural geometric interpretation and many applications. It provides an estimate of the average value of a convex function over a bounded interval. This famous result is interpreted as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

In recent years, many researchers have paid much attention to convexity theory due to its great utility in various fields of pure and applied sciences. The theory of convex functions and inequalities are closely related. The concept of convex functions has indeed found an important place in abundant literature developed on this subject, and for more details, one can consult Mitrinović, Pečarić, and Fink .

Many mathematicians have devoted their efforts to generalizing and refining this inequality and extending it to different classes of functions.

The object of this work is to present Hadamard and Hadamard type integrals and inequality of Hermite Hadamard .

The brief consist of four chapters divided as follows: In the first chapter are defined some function spaces and some special function,Then we report fractional integrals of Reiman Liouville ,Hadamard and Hadamard-Type,and some inequalities of Holder,Young...

The second chapter affected Inequalities of Hermite Hadamard type for convex functuion with apllication .

The third chapter is devoted on Fractional integrals identities and Hadamard-type fractional integration in the space X_c^p.

The last chapter affected Hermite Hadamard inequalities involving Hadamard Fractionals integrals and some application to some special means.

We conclude this modest work by a general conclusion and an interesting bibliography.

Notations

We note

$\mathbb{N}$: Set of natural integres.

$\mathbb{R}$: Set of real numbers.

$\|.\|_\infty$: Infinity norm.

$\|.\|_E$: Norm of the Banach space E.

L_p: The lebesgue Space of p-integrable function's over $[a, b]$.

X_p^c: The weighted space.

RL_a^α: Left-sided fractional integral in the sense of Rieman Liouville.

H_a^α: Left-sided fractional integral in the sense of Hadamard.

${}^\mu H_a^\alpha$: Left-sided fractional integral in the sense of Hadamard type.

Chapter 1

Preliminaries

1.1 Some Function Spaces

Let $[a,b](-\infty < a < b < \infty)$ be a finite or infinite interval in $\mathbb{R}$,

1.1.1 The L_p spaces

Definition 1.1.1. [2] Let $p \in \mathbb{R}, 1 \leq p \leq \infty$; we denote the Lebesgue space by:

$$L_p([a,b]) = \left\{ f : [a,b] \to \mathbb{C}, f \quad \text{measurable and} \|f\|_p < \infty \right\}$$

whose norm is defined by:

$$\begin{cases} \|f\|_{L_p([a,b])} = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}, 1 \leq p < \infty, \\ \\ \|f\|_{L_\infty([a,b])} = ess \sup_{t \in [a,b]} |f(t)|. \end{cases}$$

Theorem 1 *The space $(Lp)([a,b], \|.\|_p)$ is a Banach space.*

1.1.2 The L_c^p Spaces

Definition 1.1.2. [1] Let $p \in \mathbb{R}$ be with $1 \leq p < \infty, c \in \mathbb{R}$, we denote by $L_c^p[a,b]$.The weighted Lebesgue space consisting of those real valued Lebesgue measurable function $f : [a, b] \to \mathbb{R}$ with $\|f\|_{L_c^p} < \infty$, where

$$\begin{cases} \|f\|_{L_c^p([a,b])} = \left(\int_a^b |e^{ct} f(t)|^p dt \right)^{\frac{1}{p}}, 1 \le p \le \infty, c \in \mathbb{R}, \\ \\ \|f\|_{L_c^\infty([a,b])} = ess \sup_{t \in [a,b]} |e^{ct} f(t)|, c \in \mathbb{R}. \end{cases}$$

Theorem 2 *The* (L_c^p) *Spaces is a Banach space.*

1.1.3 The X_c^p Spaces

Definition 1.1.3. [1] Let $p \in \mathbb{R}$ with $1 \le p \le \infty, c \in \mathbb{R}$, we denote by $X_c^p([a, b])$ the weighted space with the power weight and consist of those Lebesgue measurable functions f on $[a, b]$ with $\|f\|_{X_c^p} < \infty$, where

$$\begin{cases} \|f\|_{X_c^p([a,b])} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, 1 \le p < \infty, c \in \mathbb{R}, \\ \\ \|f\|_{X_c^\infty([a,b])} = ess \sup_{t \in [a,b]} |t^c f(t)|, c \in \mathbb{R}. \end{cases}$$

In particular,for $c = 0$, we denote $X_0^p([a, b]) = X^p([a, b])$ and the norm is defined as:

$$\begin{cases} \|f\|_{X^p([a,b])} = \left(\int_a^b |f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, (1 \le p < \infty) \\ \\ \|f\|_{X^\infty([a,b])} = ess \sup_{t \in [a,b]} |f(t)|. \end{cases}$$

$$\|f\|_{X_c^p([a,b])} \le \|f\|_{X^p([a,b])} \le \|f\|_{L^p([a,b])}, 1 \le p < \infty, c < 0$$

Thus

$$L^p(a, b) \subseteq X^p(a, b) \subseteq X_c^p(a, b), 1 \le p < \infty, c < 0.$$

If $c = \frac{1}{p}$, $X_c^p([a, b])$ consides with $L_p([a, b])$.

Theorem 3 *The space* $(X_c^p, \|.\|_{X_c^p})$ *is a Banach space.*

1.2 Some fondamental inequalities

1.2.1 Young's Inequality

Theorem 4 *[8][16] Let $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for all complex numbers x and y,*

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}. \tag{1.1}$$

1.2.2 Arthimitic Mean-Geometric Mean Inequality

Theorem 5 *[16] Let $p, q > 0$ and $p + q = 1$, then for all $x, y > 0$,*

$$px + qy \geq x^p y^q. \tag{1.2}$$

1.2.3 Holder's Inequality

Theorem 6 *[9][16] Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.If f and g are real functions defined on $[a, b]$ and if $|f|^p, |g|^q$ are integrable functions on $[a, b]$,$q \geq 1$ then*

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}.$$

1.2.4 Power mean inequality

Theorem 7 *[14][16] Let $q \geq 1$ .If f and g are real function defined on $[a, b]$ and if $|f|, |f||g|^q$ are integrable functions on $[a, b]$ then*

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)||g(x)|^q dx\right)^{\frac{1}{q}}. \tag{1.3}$$

1.2.5 Minkowsky's inequality

Theorem 8 *[16] If $p \geq 1$ and $\int_a^b |f|^p < \infty, \int_a^b |g|^p < \infty$, then*

$$\left(\int_a^b |f(x) + g(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}.$$

1.2.6 Jensen's inequality

Theorem 9 *[10][16] Jensen's inequality states that if* $\varphi : [0, \infty[\to \mathbb{R}$ *is convex then*

$$\varphi\left(\int f d\mu\right) \leq \int \varphi(f(s))d\mu(s) \tag{1.4}$$

for all probability measures μ *and all non-negative,* μ*-integrablle functions* f *.If* φ *is concave the inequality (1.4) is reversed .*

1.3 Convex Function

Definition 1.3.1. [9] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be covex if:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \tag{1.5}$$

for all $x, y \in I$ and $t \in [0, 1]$

The function f is said concave if (1.5)is reversed.

Example 1.1. The function $f(x) = e^x$ is a convex function on $\mathbb{R}$. The function $f(x) = x^2$ is a convex function on $\mathbb{R}$

Proposition 1.3.1.
1. *A function* f *is convex on* $I \to \mathbb{R}$ *if and only if* $f'(t)$ *is increasing.*
2. *If* $f, g : I \to \mathbb{R}$ *are two convex functions then* $f + g$ *is also convex .*
3. *If* f, g *are two convex functions on* $[a, b]$ *then* $(f \circ g)$ *is not necessarily convex. Anecessary condition is that* f*is increasing.*

1.4 Fubini's Theorem

Theorem 10 *[3] Let* μ *and* ν *be* σ*-finite outer measures on* X *and* Y *respectively.*

(a) For any non-negative $\mu \times \nu$*-mesurable function* f*.*

$$x \mapsto f(x, y)\, is\, \mu - mesurable\, for\, \nu - a.e.y,\, and$$

$$y \mapsto \int_X f(x,y)d\mu(x) \, is \, \nu - measurable.$$

More over,

$$\int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y) = \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y.)$$

(b) (a) holds for $f \in L^1(\mu,\nu)$.

Lemma 1.4.1. *[11] For* $0 < \sigma \leq 1$ *and* $0 < a < b$,*we have*

$$|a^\sigma - b^\sigma| \leq (a-b)^\sigma.$$

Lemma 1.4.2. *([11]) For all* $\lambda, \nu, \kappa > 0$ *,then for any* $t > 0$,*we have*

$$t^{1-\nu}\int_0^t (t-s)^{\nu-1}s^{\lambda-1}e^{-\kappa s}ds \leq max\left\{1, 2^{1-\nu}\right\}\Gamma(\lambda)\left(1+\frac{\lambda}{\nu}\right)\kappa^{-\lambda}.$$

1.5 Some Concepts on Fractional Calculus

1.5.1 Gamma function

Definition 1.5.1. [3] For every $z \in \mathbb{C}, Re(z) > 0$ The Gamma function is defined as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1}\exp^{-t} dt, Re(z) > 0.$$

Proposition 1.5.1. *[3] The Gamma function generalizes the factorial function, as mentioned. we have*

1. $\Gamma(n+1) = \int_0^\infty t^n \exp^{-t} dt = n!, n \in \mathbb{N}$,
2. $\Gamma(z+1) = z\Gamma(z)$, *for* $z > 0$,
3. $\Gamma(z+n) = z(z+1)(z+2)...(z+n-1)\Gamma(z), z > 0.$

1.5.2 Betta function

Definition 1.5.2. [3] The Betta function defined by:

$$\beta(a,b) = \int_0^1 t^{(a-1)}(1-t)^{b-1}, a > 0, b > 0.$$

Proposition 1.5.2. *[3]*

1. $\beta(a,b) = \beta(b,a)$,
2. $a\beta(a,b+1) = \beta(a+1,b)$,
3. $\Gamma(a)\Gamma(b) = \Gamma(a+b)\beta(a,b), \forall a,b, a > 0, b > 0.$

1.5.3 Incomplete Gamma function

Definition 1.5.3. [4] The Incomplete Gamma function definded for $\nu > 0$ and $x > 0$ by :

$$\gamma(\nu,x) = \int_0^x t^{\nu-1}e^{-t}dt.$$

1.5.4 Riemann-Liouville fractional integrals

Definition 1.5.4. [5][6] Riemann-Liouville fractional integrals of order $\alpha > 0$ for a continuous function $f : [a,b] \to \mathbb{R}$ are defined by:

$$\begin{cases} (R^{\alpha}_{a+}f)(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f(s)ds, 0 \leq a < t, \\ (R^{\alpha}_{b-}f)(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^b (s-t)^{\alpha-1} f(s)ds, 0 \leq t < b, \\ (R^{0}_{a+}f)(t) = f(t). \end{cases}$$

Proposition 1.5.3. *[5][6] Let $f : [a,b] \to \mathbb{R}$ be a continuous function*
Then,for every α, β,we have:

$$(R^{\alpha}_{a+}R^{\beta}_{a}f)(t) = (R^{\alpha+\beta}_{a+}f)(t),$$

and

$$(R^{\alpha}_{a+}R^{\beta}_{a}f)(t) = (R^{\beta}_{a+}R^{\alpha}_{a}f)(t)$$

is called semi-group property.

1.5.5 Hadamard fractional integrals

Definition 1.5.5. [5][6] Hadamard fractional integrals of order $0 < \alpha$, for a continuous function $f : [a, b] \to \mathbb{R}$ are defined by:

$$\begin{cases} (H^{\alpha}_{a+}f)(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, 0 < a < t, \\ (H^{\alpha}_{b-}f)(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_t^b \left(\ln \frac{s}{t}\right)^{\alpha-1} f(s)\frac{ds}{s}, 0 < t < b, \\ (H^{0}_{a+}f)(t) = f(t), (H^{0}_{b-}f)(t) = f(t). \end{cases}$$

Exemples 1.2. 1. For $f(t) = t$ and $\alpha = 1$, we have

$$(H^{1}_{a+}f)(t) = \frac{1}{\Gamma(1)} \int_a^t ds = s\Big|_a^t = t - a.$$

$$(H^{1}_{b-}f)(t) = \int_t^b ds = s\Big|_t^b = b - t.$$

2. For $f(t) = t^2$ and $\alpha = 2$,we have

$$\begin{aligned} (H^{2}_{a+}f)(t) &= \frac{1}{\Gamma(2)} \int_a^t \ln \frac{t}{s} \times sds, \\ &= \int_a^t s(\ln t - \ln s)ds, \\ &= \ln t \int_a^t sds - \int_a^t s \ln sds, \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \ln t \int_a^t sds = \frac{1}{2}\left(t^2 \ln t - a^2 \ln t\right).$$

$$I_2 = -\int_a^t s \ln sds.$$

Integrating by parts the term I_2, we have

$$\begin{aligned} I_2 &= -\frac{1}{2}s^2 \ln s\Big|_a^t + \frac{1}{2}\int_a^t sds \\ &= \frac{1}{2}\left(\frac{1}{2}t^2 - \frac{1}{2}a^2 - t^2 \ln t + a^2 \ln a\right) \end{aligned}$$

$$\begin{aligned} I_1 + I_2 &= \frac{1}{2}\left[\frac{1}{2}t^2 - \frac{1}{2}a^2 + (\ln a - \ln t)\right] \\ &= \frac{1}{2}\left(\frac{1}{2}t^2 + (\ln a - \ln t - \frac{1}{2})a^2\right) \end{aligned}$$

3. For $f(t) = t^{n+1}, \alpha = 1$

$$(H_{a+}f)(t) = \int_a^t s^n ds = \frac{1}{n+1} s^{n+1}\Big|_a^t = \frac{1}{n+1}(t^{n+1} - a^{n+1})$$

.

$$(H_{b-}f)(t) = \int_t^b s^n ds = \frac{1}{n+1} s^{n+1}\Big|_t^b = \frac{1}{n+1}(b^{n+1} - t^{n+1}).$$

Proposition 1.5.4. *[5] Let f be a continous function.Then,for every $\alpha, \beta > 0$ we have:*

$$(H_{a+}^{\alpha} H_{a+}^{\beta} f)(t) = H_{a+}^{\alpha+\beta} f(t), \tag{1.6}$$

is called semi-group property.

Proof 1 *We have*

$$\begin{aligned}
(H_{a+}^{\alpha} H_{a+}^{\beta} f)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \ln\left(\frac{t}{s}\right)^{\alpha-1} f(s) H_{a+}^{\beta} \frac{ds}{s} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \ln\left(\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s \ln\left(\frac{s}{u}\right)^{\beta-1} f(u) \frac{du}{u} \right) \frac{ds}{s} \quad (a \leq u \leq s, a \leq s \leq t), \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(u) \left(\int_u^t \ln\left(\frac{t}{s}\right)^{\alpha-1} \ln\left(\frac{s}{u}\frac{ds}{s}\right)^{\beta-1} \right) \frac{du}{u} \quad (a \leq u \leq s \leq t)
\end{aligned}$$

we use variable substitution $\rho = \frac{\ln\left(\frac{s}{u}\right)}{\ln\left(\frac{t}{u}\right)}$ *,through the Beta function, we obtain*

$$\begin{aligned}
(H_{a+}^{\alpha} H_{a+}^{\beta} f)(u) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \\
&\int_a^t f(u) \left(\int_0^1 \left(\ln\left(\frac{t}{u}\right) - \left(\rho \ln\frac{t}{u}\right)\right)^{\alpha-1} \left(\rho \ln\frac{t}{u}\right)^{\beta-1} \ln\frac{t}{u} d\rho \right) \frac{du}{u} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(u) \left(\ln\left(\frac{t}{u}\right)^{\alpha+\beta-1} \int_0^1 (1-\rho)^{\alpha-1} \rho^{\beta-1} d\rho \right) \frac{du}{u} \\
&= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \ln\left(\frac{t}{u}\right)^{\alpha+\beta-1} f(u) \frac{du}{u} \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \ln\left(\frac{t}{u}\right)^{\alpha+\beta-1} f(u) \frac{du}{u} \\
&= (H_{a+}^{\alpha+\beta} f)(t).
\end{aligned} \tag{1.7}$$

1.5.6 The operator A

Definition 1.5.6. [1] In order to establish the connection between Riemann–Liouville fractional integral and Hadamard fractional integral, we have to introduce an elementary operator. For a real-valued function $f(t)$ defined almost everywhere on $\mathbb{R}^+$, the operator A is defined as follows:

$$(Af)(t) = f(e^t).$$

Then for a function g defined almost everywhere $[a, b]$, its inverse A^{-1} has the form

$$(A^{-1}g)(t) = g(\ln t).$$

Using these two operators, we establish the connection between Hadamard fractional integral and Riemann–Liouville fractional integral, which can be shown by the relation:

$$(H^{\alpha}_{a+}f)(t) = (A^{-1}R^{\alpha}_{\ln a}Af)(t). \tag{1.8}$$

Proof 2 *We also*

$$(Af)(t) = f(e^t),$$

we prove that

$$(H^{\alpha}_{a+}f)(t) = (A^{-1}R^{\alpha}_{\ln a+}Af)(t),$$

we have

$$\begin{aligned}(A^{-1}R^{\alpha}_{\ln a+}Af)(t) &= (R^{\alpha}_{\ln a+}Af)(\ln t),\\ &= \frac{1}{\Gamma(\alpha)}\int_{\ln a}^{\ln t}(\ln t - s)^{\alpha-1}Af(s)ds,\\ &= \frac{1}{\Gamma(\alpha)}\int_{\ln a}^{\ln t}(\ln t - s)^{\alpha-1}f(e^s)ds. \end{aligned} \tag{1.9}$$

we use a change of variable : Let $u = e^s$ where $s = \ln u$ and $ds = \dfrac{du}{u}$,we have

$$\begin{aligned}(R^{\alpha}_{\ln a+}Af)(\ln t) &= \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(\ln t - \ln u)^{\alpha-1}\frac{f(u)}{u}du,\\ &= \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(\ln\frac{t}{u}\right)^{\alpha-1}\frac{f(u)}{u}du,\\ &= \left(H^{\alpha}_{a+}f\right)(t). \end{aligned} \tag{1.10}$$

1.5.7 Hadamard-type fractional integrals

Definition 1.5.7. [7][4] The Hadamard-type fractional integrals with order ($\alpha \in \mathbb{C}\, with\, Re(\alpha) > 0$) and parameter $\mu \in \mathbb{C}$ of a given function f are defined as

$$\begin{cases} ({}^{\mu}H^{\alpha}_{a+}f)(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t \left(\frac{s}{t}\right)^{\mu}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds, \\ ({}^{\mu}H^{\alpha}_{b-}f)(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_t^b \left(\frac{t}{s}\right)^{\mu}\left(\ln\frac{s}{t}\right)^{\alpha-1}\frac{f(s)}{s}ds, \end{cases} \tag{1.11}$$

where $t \in (a, b)$ and $a > b$ in $\mathbb{R}$.

Remark 11 *If $\mu = o$,we obtain*

$$({}^{0}H^{\alpha}_{a+}f)(t) = (H^{\alpha}_{a+}f)(t).$$

Proposition 1.5.5. *[4] Let $\alpha > 0, \beta > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ and let $\mu \in \mathbb{R}$ and $c \in \mathbb{R}$ be such that $\mu \geq c$. Then for $f \in X^p_c(a, b)$ the semi group property holds*

$$({}^{\mu}H^{\alpha}_{a+}H^{\beta}_{a+}f)(t) = ({}^{\mu}H^{\alpha+\beta}_{a+}f)(t). \tag{1.12}$$

Proof 3 *Firt we prove (1.12) for functions f .Applying Fubini's theorem we find*

$$\begin{aligned} ({}^{\mu}H^{\alpha}_{a+}H^{\beta}_{a+}f)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{u}{t}\right)^{\mu}\left(\ln\frac{t}{u}\right)^{\alpha-1} \\ &\times \left[\frac{1}{\Gamma(\beta)}\int_a^u \left(\frac{s}{u}\right)^{\mu}\left(\ln\frac{u}{s}\right)^{\beta-1} f(s)\frac{ds}{s}\right]\frac{du}{u} \\ &= \frac{t^{-\mu}}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t s^{\mu-1}f(s)ds\int_s^t \left(\ln\frac{t}{u}\right)^{\alpha-1}\left(\ln\frac{u}{s}\right)^{\beta-1}\frac{du}{u}. \end{aligned} \tag{1.13}$$

The inner integral is evaluated by the change of variable $\tau = \ln(u/s)/\ln(t/s)$:

$$\int_s^t \left(\ln\frac{t}{u}\right)^{\alpha-1}\left(\ln\frac{u}{s}\right)^{\beta-1}\frac{du}{u} = \left(\ln\frac{t}{s}\right)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Substituting this relation into (1.13), we have

$$\begin{aligned} ({}^{\mu}H^{\alpha}_{a+}H^{\beta}_{a+}f)(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t \left(\frac{u}{t}\right)^{\mu}\left(\ln\frac{t}{u}\right)^{\alpha+\beta-1}\frac{du}{u} \\ &= ({}^{\mu}H^{\alpha+\beta}_{a+}f)(t). \end{aligned}$$

1.5.8 The multiplication Operator M

Definition 1.5.8. [7] In order to establish the connection between Hadamard fractional integrals and Hadamard-type fractional , we have to introduce an elementary operator. For a real-valued functions f, g defined almost everywhere on $\mathbb{R}^+$, the operator M is defined as follows:

$$(M_{g(t)}f)(t) = f(t)g(t).$$

and

$$(M_{t^\mu}f)(t) = t^\mu f(t).$$

Using these operator,we establish the connection between Hatamard-type fractional integral and Hadamard fractional integral,wich can be shown by the relation:

$$({}^\mu H^\alpha_{a+}f)(t) = (M^{-1}_{t^\mu}H^\alpha_{a+}M_{t^\mu}f)(t)$$

So Hadamard-type fractional calculus is the conjugation of Hadamard fractional calculus with multiplication by t^μ:

$$({}^\mu H^\alpha_{a+}f)(t) = (t^{-\mu}H^\alpha_{a+}t^\mu f)(t)$$

Proof 4 *We have*

$$\begin{aligned}
(t^{-\mu}H^\alpha_{a+}t^\mu f)(t) &= t^{-\mu}\frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} s^\mu\frac{f(s)}{s}ds \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} t^{-\mu}s^\mu\frac{f(s)}{s}ds \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{s}{t}\right)^{\mu}\left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}ds \\
&= ({}^\mu H^\alpha_{a+}f)(t).
\end{aligned}$$

1.6 Special means

Definition 1.6.1. [11] Let's consider the following special means for arbitrary real numbers $a, b, a \neq b$ as follows :

. The arithmitic mean :

$$A(a,b) = \frac{a+b}{2}, a, b \in \mathbb{R}_+.$$

. The geometric mean:

$$G(a,b) = \sqrt{ab}, a, b \in \mathbb{R}_+.$$

. The harmonic mean:

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}, a, b \in \mathbb{R} - \{0\}.$$

. The logarithmic mean:

$$L(a,b) = \frac{b-a}{\ln b - \ln a}, a, b \in \mathbb{R}_+.$$

. The generalized logarithmic mean:

$$L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, p \neq -1, 0, a, b > 0.$$

Lemma 1.6.1 *For $a, b \in \mathbb{R}_+, a \neq b$, we have:*

$$H(a,b) \leq G(a,b) \leq A(a,b).$$

Proof 5 *We start with the left side, using*

$$\left(\frac{1}{a}+\frac{1}{b}\right)^2 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{ab},$$

and

$$\begin{aligned} \frac{4}{ab} &\leq \left(\frac{1}{a}+\frac{1}{b}\right)^2, \\ \frac{4}{\left(\frac{1}{a}+\frac{1}{b}\right)^2} &\leq ab, \\ \frac{2}{\frac{1}{a}+\frac{1}{b}} &\leq \sqrt{ab}, \\ &= H(a,b) \leq \sqrt{ab}. \end{aligned} \tag{1.14}$$

we move to the right side, using:

$$(a+b)^2 = a^2 + b^2 + 2ab$$

, *and*

$$
\begin{aligned}
4ab &\leq (a+b)^2, \\
ab &\leq \frac{(a+b)^2}{4}, \\
\sqrt{ab} &\leq \frac{a+b}{2}, \\
G(a,b) &\leq A(a,b). \qquad (1.15)
\end{aligned}
$$

From(1.14)and(1.15),we have

$$
\begin{aligned}
\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} &\leq \frac{a+b}{2}, \\
H(a,b) \leq G(a,b) &\leq A(a,b). \qquad (1.16)
\end{aligned}
$$

Chapter 2

Inequalities of Hermite-Hadamard type for convex function with applications

2.1 Classical Hermite Hadamard inequality(H-H)

Theorem 12 *[11] If $f : [a, b] \to \mathbb{R}$ is a convex function on the interval I , we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Proof 6 *Set f is a convex function on $[a, b]$, then we have :*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{a}{2}+\frac{b}{2}+\frac{t}{2}-\frac{t}{2}\right) \\ &= f\left(\frac{1}{2}(a+b-t)+\frac{1}{2}t\right) \\ &\leq \frac{1}{2}f(a+b-t)+\frac{1}{2}f(t). \end{aligned} \tag{2.2}$$

Set $t = sa + (1-s)b$, *for* $s \in [0,1]$, *we obtain*

$$\begin{aligned} f(a+b-t) + f(t) =& f(a+b-sa+(1-s)b) + f(sa+(1-s)b) \\ \leq& f((1-s)a+sb) + f(sa+(1-s)b) \\ \leq& f(a) + f(b). \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f(a+b-t) + \frac{1}{2}f(t) \leq \frac{f(a)+f(b)}{2},$$

From the convexity of the function f*,we have*

$$f(sa+(1-s)b) \leq sf(a) + (1-s)f(b).$$

Integre inequality on $[0,1]$, *we obtain*

$$\int_0^1 f(sa+(1-s)b)ds \leq f(a)\int_0^1 sds + f(b)\int_0^1 (1-s)ds, \tag{2.4}$$

since

$$\int_0^1 sds = \int_0^1 (1-s)ds = \frac{1}{2}.$$

By change of variable $t = sa + (1-s)b$, *we have*

$$\int_0^1 f(sa+(1-s)b)ds = \frac{1}{b-a}\int_a^b f(t)dt.$$

for (2.4), we obtain the right inequality of (2.1) , from the convexity of the function f, *we have also*

$$\begin{aligned} \frac{1}{2}\left[f(sa+(1-s)b) + f((1-s)a+sb)\right] \geq& f\left[\frac{sa+(1-s)b+(1-s)a+sb}{2}\right] \\ =& f\left(\frac{a+b}{2}\right), \end{aligned}$$

integre inequality on $[0,1]$, *we obtain the left inequality of (2.1)*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \leq& \frac{1}{2}\left[\int_0^1 f(sa+(1-s)b)ds + \int_0^1 f((1-s)a+sb)ds\right] \\ =& \frac{1}{b-a}\int_a^b f(t)dt. \end{aligned}$$

So the double inequality (2.1) is verified.

2.2 Hermite-Hadamard type inqualities for convex functions

Lemma 2.2.1 *[13] Let $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I \to \mathbb{R}$ be differentiable. If $f' \in L^1(a, b)$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{2}\int_0^1 (1-2s)f'(sa+(1-s)b)ds.$$

Proof 7 *Set $I_1 = \int_0^1 (1-2s)f'(sa+(1-s)b)ds$,interating by parts, we get*

$$\begin{aligned} I_1 &= \int_0^1 (1-2s)f'(sa+(1-s)b)ds \\ &= \frac{f(sa+(1-s)b)}{a-b}(1-2s)\Big|_0^1 + 2\int_0^1 \frac{f(sa+(1-s)b)}{a-b}ds \\ &= \frac{f(a)+f(b)}{a-b} - \frac{2}{b-a}\int_a^b f(t)dt. \end{aligned}$$

Theorem 13 *[13] Let $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I \to \mathbb{R}$ be differentiable and $|f'|$ is convex on $[a, b]$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|}{8} \tag{2.5}$$

Proof 8 *From Lemma 2.2.1 and the convexity of $|f'|$ on $[a, b]$, we obtain*

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| &= \left|\frac{b-a}{2}\int_0^1 (1-2s)f'(sa(1-s)b)ds\right| \\ &\leq \frac{b-a}{2}\int_0^1 |1-2s||f'(sa+(1-s)b)|ds \\ &\leq \frac{b-a}{2}\int_0^1 |1-2s|\left[s|f'(a)|+(1-s)|f'(b)|\right]ds \\ &\leq \frac{(b-a)|f'(a)|+|f'(b)|}{2}\int_0^1 |1-2s|sds \\ &\leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}, \end{aligned} \tag{2.6}$$

where has it been used

$$\int_0^1 |1-2s|(1-s)ds = \int_0^1 |1-2s|sds = \int_0^{\frac{1}{2}} |1-2s|sds + \int_{\frac{1}{2}}^1 |2s-1|sds = \frac{1}{4}.$$

Theorem 14 *[13] Let $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I \to \mathbb{R}$ be differentiable and $p > 1$, If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \quad (2.7)$$

Proof 9 *From Lemma2.2.1 and Holder's Inequality,we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}\int_0^1 |1-2s||f'(sa+(1-s)b)|ds$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2s|^p ds\right)^{\frac{1}{p}}\left(\int_0^1 |f'(sa+(1-s)b)|^q ds\right)^{\frac{1}{q}}, \quad (2.8)$$

and $\frac{1}{p} + \frac{1}{q} = 1.$
From the convexity of $|f'|^q$, we have

$$\int_0^1 |f'(sa+(1-s)b)|^q ds \leq \int_0^1 [s|f'(a)|^q + (1-s)|f'(b)|^q]\, ds$$
$$= \frac{|f'(a)|^q + |f'(b)|^q}{2}, \quad (q = \frac{p}{p-1}) \quad (2.9)$$

on the other hand,we have

$$\int_0^1 |1-2s|^p = \int_0^{\frac{1}{2}}(1-2s)^p ds + \int_{\frac{1}{2}}^1 (2s-1)^p ds = 2\int_0^{\frac{1}{2}}(1-2s)^p ds = \frac{1}{p+1}. \quad (2.10)$$

A combination of (2.8) and (2.10) immediality gives the required inequality (2.7). The proof is complete.

Theorem 15 *[13] Let $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I \to \mathbb{R}$ be differentiable and $|f'|^q$ is convex on $[a, b]$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}. \quad (2.11)$$

Proof 10 *From Lemma2.2.1*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}\int_0^1 |1-2s||f'(sa+(1-s)b)|ds, \quad (2.12)$$

and by the power-mean inequalities

$$\int_0^1 |1-2s||f'(sa+(1-s)b)|ds \leq \left(\int_0^1 |1-2s|ds\right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2s||f'(sa+(1-s)b)|^q ds\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ *is convex,we have*

$$\int_0^1 |1-2s||f'(sa+(1-s)b)|^q ds \leq \int_0^1 |1-2s|[s|f'(a)|^q + (1-s)|f'(b)|^q]ds = \frac{|f'(a)|^q + |f'(b)|^q}{4}.$$

Since $\int_0^1 |1-2s|ds = \frac{1}{2}$ *,we have from (2.12) and the displayed inequality following it that*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}$$

.

Lemma 2.2.2 *[13] Let* $a, b \in \mathbb{I}$ *with* $a > b$ *and* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *be differentiable ,If* $f' \in L(a,b)$ *,then*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{4}\left[\int_0^1 sf'\left(s\frac{a+b}{2} + (1-s)a\right)ds + \int_0^1 (s-1)f'\left(sb + (1-s)\frac{a+b}{2}\right)ds\right]. \quad (2.13)$$

Proof 11 *Integrating by parts ,we have*

$$\int_0^1 sf'\left(s\frac{a+b}{2} + (1-s)a\right)ds = \frac{2}{b-a}sf\left(s\frac{a+b}{2} + (1-s)a\right)\Big|_0^1 - \frac{2}{b-a}\int_0^1 f\left(s\frac{a+b}{2} + (1-s)a\right)ds = \frac{2}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_0^1 f\left(s\frac{a+b}{2} + (1-s)a\right)ds.$$

By change of variables we obtain $t = s\frac{a+b}{2} + (1-s)a \Longrightarrow dt = \frac{(b-a)}{2}ds$ *then*

$$\int_0^1 sf'\left(s\frac{a+b}{2} + (1-s)a\right)ds = \frac{2}{(b-a)}f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(t)dt$$

Set

$$I_1 = \frac{2}{(b-a)} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2} \int_a^{\frac{a+b}{2}} f(t)dt.$$

Similarly

$$\int_0^1 (s-1) f'\left(sb + (1-s)\frac{a+b}{2}\right) ds = \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2} \int_{\frac{a+b}{2}}^b f(t),$$

set

$$I_2 = \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2} \int_{\frac{a+b}{2}}^b f(t)dt,$$

then

$$I_1 + I_2 = \frac{4}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2} \int_a^b f(t)dt,$$

hence

$$\begin{aligned}\frac{b-a}{4}[I_1 + I_2] &= \frac{b-a}{4}\left[\frac{4}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2} \int_a^b f(t)dt\right] \\ &= f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt.\end{aligned}$$

Theorem 16 *[13] Let $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I \to \mathbb{R}$ is differentiable and $|f'|^q$ is convex on $[a, b]$. Then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{2.14}$$

Proof 12 *Our starting point is the identity*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt = \frac{1}{b-a} \int_a^b S(t) f'(t)dt, \tag{2.15}$$

where

$$S(t) = \begin{cases} t-a, & t \in \left[a, \frac{a+b}{2}\right), \\ t-b, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

An argument parallel to that of 2.11 but with (2.15)in place of Lemma 2.2.1 gives the desired result.

We now derive comparable results to (2.11) and (2.14)with a concavity property instead of convexity.

Theorem 17 *[13] Let $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I \to \mathbb{R}$ is differentiable and $|f'|^q (q \geq 1)$ is concave on $[a, b]$. Then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{2}\right) \right|, \tag{2.16}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{4}\right) \right|. \tag{2.17}$$

Proof 13 *First, note that*

$$\begin{aligned} |f'(\lambda t + (1-\lambda)z)|^q \geq \lambda |f'(t)|^q + (1-\lambda)|f'(z)|^q \\ \geq (\lambda |f'(t)| + (1-\lambda)|f'(z)|)^q. \end{aligned}$$

by the convexity of$|f'|^q$ and the power-mean inequality. Hence,

$$f'(\lambda t + (1-\lambda)z) \geq \lambda |f'(t)| + (1-\lambda)|f'(z)|,$$

so $|f'|$ is also concave.

Accordingly by the Jensen integral inequality, we have

$$\begin{aligned} \int_0^1 |1-2s| f'(sa + (1-s)b)ds &\leq \left(\int_0^1 |1-2s|ds \right) \left| f'\left(\frac{\int_0^1 |1-2s|(sa+(1-s)b)ds}{\int_0^1 |1-2s|ds} \right) \right|, \\ &= \frac{1}{2} \left| f' \frac{a+b}{2} \right|. \end{aligned}$$

By (2.12) we have (2.16). Similarly using (2.15) we can prove (2.17).

2.2.1 Applications to special means

Proposition 2.2.1. *[13]* Let $a, b \in \mathbb{R}, a < b$ and $n \in \mathbb{N}, n \geq 2$.Then

$$|A(a^n, b^n) - L_n(a, b)^n| \leq \frac{n(b-a)}{4} A(|a|^{n-1}, |b|^{n-1}). \tag{2.18}$$

Proof 14 *Using Theorem (2.5) with $f(t) = t^n, t \in [a, b]$, we obtain*

$$\begin{aligned} \left| \frac{a^n + b^n}{2} - \frac{1}{b-a} \int_a^b t^n dt \right| &= \left| \frac{a^n + b^n}{2} - \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right| \\ &\leq \frac{(b-a)(n|a|^{n-1} + n|b|^{n-1})}{8} \\ &\leq \frac{n(b-a)}{4} \times \frac{|a|^{n-1} + |b|^{n-1}}{2} \end{aligned}$$

Proposition 2.2.2. *[13] Let $a, b \in \mathbb{R}, a < b$ and $n \in \mathbb{N}, n \geq 2$ .Then for all $p > 1$*

$$|A(a^n, b^n) - L_n(a,b)| \leq \frac{n(b-a)}{2(p+1)^{\frac{1}{p}}} \left[A\left(|a|^{\frac{(n-1)p}{(p-1)}}, |b|^{\frac{(n-1)p}{(p-1)}}\right)\right]^{\frac{(p-1)}{p}}. \tag{2.19}$$

Proof 15 *Using (2.7) with $f(t) = t^n, t \in [a, b]$, we obtain*

$$\left|\frac{a^n + b^n}{2} - \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}} \left[\frac{n\left(|a|^{\frac{(n-1)p}{(p-1)}} + n|b|^{\frac{(n-1)p}{(p-1)}}\right)}{2}\right]$$

Proposition 2.2.3. *[13] Let $a, b \in \mathbb{R}, a < b$ and $0 \notin [a, b]$.Then*

$$|A(a^{-1}, b^{-1}) - L^{-1}(a,b)| \leq \frac{(b-a)}{4} A(|a|^{-2}, |b|^{-2}). \tag{2.20}$$

Proof 16 *Using (2.5)with $f(t) = \frac{1}{t}, t \in [a, b]$ we obtain*

$$\begin{aligned}\left|\frac{\frac{1}{a} + \frac{1}{b}}{2} - \frac{1}{b-a}\int_a^b \frac{1}{t}dt\right| &= \left|\frac{a^{-1} + b^{-1}}{2} - \frac{\ln b - \ln a}{b-a}\right| \\ &\leq \frac{(b-a)(|\frac{1}{a}|^2 + |\frac{1}{b}|^2)}{8} \\ &\leq \frac{(b-a)}{4} \times \frac{|a|^{-2} + |b|^{-2}}{2}\end{aligned}$$

Proposition 2.2.4. *[13] Let $a, b \in \mathbb{R}, a < b$ and $0 \notin [a, b]$ .Then for all $p > 1$*

$$|A(a^{-1}, b^{-1}) - L^{-1}(a,b) \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}} \left[A\left(|a|^{\frac{-2p}{(p-1)}}, |b|^{\frac{-2p}{(p-1)}}\right)\right]^{\frac{(p-1)}{p}} \tag{2.21}$$

Proof 17 *The proof is immediate from (2.7) with $f(t) = \frac{1}{t}, t \in [a, b]$.*

Proposition 2.2.5. *[13] Let $a, b \in \mathbb{R}, a < b$ and $n \in \mathbb{Z}, n \geq 2$ .Then for all $q \geq 1$*

$$|A(a^n, b^n) - L_n(a,b)^n| \leq \frac{|n|(b-a)}{4} \left[A\left(|a|^{(n-1)q}, |b|^{(n-1)q}\right)\right]^{\frac{1}{q}} \tag{2.22}$$

and

$$|A(a,b)^n - L_n(a,b)^n| \leq \frac{|n|(b-a)}{4} \left[A\left(|a|^{(n-1)q}, |b|^{(n-1)q}\right)\right]^{\frac{1}{q}}. \tag{2.23}$$

Proof 18 *Using (2.11) and (2.14) with* $f(t) = t^n, t \in \mathbb{R}, n \in \mathbb{Z}, n \geq 2$ *we obtain*

$$\left| \frac{a^n + b^n}{2} - \frac{b^{n+1} - a^{n+1}}{b-a} \right| \leq \frac{b-a}{4} \left[\frac{|n||a|^{(n-1)q} + |n||b|^{(n-1)q}}{2} \right]^{\frac{1}{q}}$$
$$\leq \frac{|n|(b-a)}{4} \left[\frac{|a|^{(n-1)q} + |b|^{(n-1)q}}{2} \right]^{\frac{1}{q}}$$

and

$$\left| \left(\frac{a+b}{2} \right)^n - \frac{b^{n+1} - a^{n+1}}{b-a} \right| \leq \frac{|n|(b-a)}{4} \left[\frac{|a|^{(n-1)q} + |b|^{(n-1)q}}{2} \right]^{\frac{1}{q}}$$

Proposition 2.2.6. *[13] Let* $a, b \in \mathbb{R}, a < b$ *and* $0 \notin [a, b], n \geq 2$ *.Then for all* $q \geq 1$

$$|A(a^{-1}, b^{-1}) - L^{-1}(a, b) \leq \frac{(b-a)}{4} \left[A \left(|a|^{-2q}, |b|^{-2q} \right) \right]^{\frac{1}{2}} \tag{2.24}$$

and

$$|A(a, b)^{-1} - L^{-1}(a, b)| \leq \frac{(b-a)}{4} \left[A \left(|a|^{-2q}, |b|^{-2q} \right) \right]^{\frac{1}{2}} \tag{2.25}$$

Proof 19 *The result follows from (2.11) and (2.14) with* $f(t) = \dfrac{1}{t}$.

Chapter 3

Some fractional integrals identities

3.1 Identities involving Hadamard fractional integrals

3.1.1 Properties of Hadamard fractional integrals

Lemma 3.1.1 *[11] Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $0 < a < b$, If $f' \in L^1[a,b]$, then the following equality for fractional integrals holds :*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[H^{\alpha}_{a+}f(b) + H^{\alpha}_{b-}f(a)]$$
$$= \frac{\ln b - \ln a}{2}\int_0^1 [(1-s)^{\alpha} - s^{\alpha}]\, e^{s\ln a+(1-s)\ln b} f'\left(e^{s\ln a+(1-s)\ln b}\right) ds$$
$$= \frac{\ln b - \ln a}{2}\int_0^1 [(1-s)^{\alpha} - s^{\alpha}]\, a^s b^{1-s} f'\left(a^s b^{1-s}\right) ds. \tag{3.1}$$

Proof 20 *Denote*

$$I = \int_0^1 [(1-s)^{\alpha} - s^{\alpha}]\, e^{\ln b - s(\ln b - \ln a)} f'\left(e^{\ln b - s(\ln b - \ln a)}\right) ds$$
$$= I_1 + I_2, \tag{3.2}$$

where

$$I_1 = \int_0^1 (1-s)^{\alpha}\, e^{\ln b - s(\ln b - \ln a)} f'\left(e^{\ln b - s(\ln b - \ln a)}\right) ds,$$
$$I_2 = -\int_0^1 s^{\alpha} e^{\ln b - s(\ln b - \ln a)} f'\left(e^{\ln b - s(\ln b - \ln a)}\right) ds.$$

Integrating by parts the term I_1 with s over $[0,1]$, we have

$$I_1 = \int_0^1 (1-s)^\alpha\, e^{\ln b - s(\ln b - \ln a)} f'\left(e^{\ln b - s(\ln b - \ln a)}\right) ds$$
$$= (1-s)^\alpha \frac{f\left(e^{\ln b - s(\ln b - \ln a)}\right)}{\ln a - \ln b}\Bigg|_0^1 - \frac{\alpha}{\ln b - \ln a}\int_0^1 (1-s)^{\alpha-1} f\left(e^{\ln b - s(\ln b - \ln a)}\right) ds$$

by change of variable $u = e^{\ln b - s(\ln b - \ln a)}$ and $s = \dfrac{\ln b - \ln u}{\ln b - \ln a}$,we have

$$I_1 = \frac{f(b)}{\ln b - \ln a} + \frac{\alpha}{(\ln a - \ln b)^2}\int_b^a \left(\frac{\ln u - \ln a}{\ln b - \ln a}\right)^{\alpha-1} f(u)\frac{du}{u}$$
$$= \frac{f(b)}{\ln b - \ln a} - \frac{\alpha}{(\ln b - \ln a)^{\alpha+1}}\int_a^b (\ln u - \ln a)^{\alpha-1} f(u)\frac{du}{u}$$
$$= \frac{f(b)}{\ln b - \ln a} - \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}} H^\alpha_{b-} f(a). \tag{3.3}$$

Similarly, we get I_2

$$I_2 = -\int_0^1 s^\alpha e^{\ln b - s(\ln b - \ln a)} f'\left(e^{\ln b - s(\ln b - \ln a)}\right) ds$$
$$= s^\alpha \frac{f(e^{\ln b - s(\ln b - \ln a)})}{\ln b - \ln a}\Bigg|_0^1 - \frac{\alpha}{\ln b - \ln a}\int_0^1 s^{\alpha-1} f\left(e^{\ln b - s(\ln b - \ln a)}\right) ds$$
$$= \frac{f(a)}{\ln b - \ln a} - \frac{\alpha}{(\ln b - \ln a)^{\alpha+1}}\int_a^b (\ln b - \ln u)^{\alpha-1} f(u)\frac{du}{u}$$
$$= \frac{f(a)}{\ln b - \ln a} - \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}} H^\alpha_{a+} f(b). \tag{3.4}$$

Substiting (3.3) and (3.4) into (3.2), it follows that:

$$I = \frac{f(a)+f(b)}{\ln b - \ln a} - \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}}[H^\alpha_{a+} f(b) + H^\alpha_{b-} f(a)]. \tag{3.5}$$

Thus,by multiplying both sides by $\dfrac{\ln b - \ln a}{2}$ in (3.5), we deduce (3.1) immediately.

Remark 18 *Lemma 3.1.1 is a result of the following Lemma:*

Lemma 3.1.2 *Let $f : [x,y] \to \mathbb{R}$ be a differentiable mapping on (x,y) with $x < y$. If $f' \in L[x,y]$, then the following equality for fractional integrals holds*

$$\frac{f(x)+f(y)}{2} - \frac{\Gamma(\alpha+1)}{2(y-x)^\alpha}[R^\alpha_{x+} f(y) + R^\alpha_{y-} f(x)]$$
$$= \frac{y-x}{2}\int_0^1 [(1-s)^\alpha - s^\alpha] f'(sx + (1-s)y) ds. \tag{3.6}$$

Proof 21 *Take $x = \ln a, y = \ln b$ and using the operator A,we obtain*

$$\frac{Af(\ln a) + Af(\ln b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[R^{\alpha}_{\ln a+}Af(\ln b) + R^{\alpha}_{\ln b-}Af(\ln a)]$$
$$= \frac{\ln b - \ln a}{2}\int_0^1 [(1-s)^{\alpha} - s^{\alpha}]Af'(s\ln a + (1-s)\ln b)ds,$$

then

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[H^{\alpha}_{a+}f(b) + H^{\alpha}_{b-}f(a)]$$
$$= \frac{\ln b - \ln a}{2}\int_0^1 [(1-s)^{\alpha} - s^{\alpha}]e^{s\ln a + (1-s)\ln b}f'(e^{s\ln a + (1-s)\ln b})ds.$$

Example 3.1. Let $a = 1, b = e, \alpha = 2, f(t) = t^2$.Then all the assumptions in Lemma (3.1.1) are satisfied. Clearly,

the left-sided term of (3.1)$\Longleftrightarrow \dfrac{1+e^2}{2} - \dfrac{1}{2}e^2 + \dfrac{1}{2} = 1,$

the right-sided term of (3.1) $\Longleftrightarrow$

$$\int_0^1 (1-2s)e^{2(1-s)}ds$$
$$= \int_0^1 e^{2(1-s)}ds - 2\int_0^1 se^{2(1-s)}ds$$
$$= \frac{1}{2}(e^2 - 1) - 2\left(\frac{1}{4}e^2 - \frac{3}{4}\right) = 1$$

Lemma 3.1.3 *[11] Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $0 < a < b$, If $f' \in L^1[a,b]$, then the following equality for fractional integrals holds :*

$$\frac{(\ln t - \ln a)^{\alpha} + (\ln b - \ln t)^{\alpha}}{\ln b - \ln a}f(t) - \frac{\Gamma(\alpha+1)}{\ln b - \ln a}\left[H^{\alpha}_{t-}f(a) + H^{\alpha}_{t+}f(b)\right]$$
$$= \frac{(\ln t - \ln a)^{\alpha+1}}{\ln b - \ln a}\int_0^1 s^{\alpha}e^{s\ln t + (1-s)\ln a}f'\left(e^{s\ln t + (1-s)\ln a}\right)ds$$
$$- \frac{(\ln b - \ln t)^{\alpha+1}}{\ln b - \ln a}\int_0^1 s^{\alpha}e^{s\ln t + (1-s)\ln b}f'\left(e^{s\ln t + (1-s)\ln b}\right)ds$$
$$= \frac{(\ln t - \ln a)^{\alpha+1}}{\ln b - \ln a}\int_0^1 s^{\alpha}t^s a^{1-s}f'\left(t^s a^{1-s}\right)ds$$
$$- \frac{(\ln b - \ln t)^{\alpha+1}}{\ln b - \ln a}\int_0^1 s^{\alpha}t^s b^{1-s}f'\left(t^s b^{1-s}\right)ds$$

for any $t \in (a,b)$.

Proof 22 *Integrating by parts, we can state*

$$\begin{aligned}&\int_0^1 s^{\alpha} e^{s\ln t+(1-s)\ln a} f'\left(e^{s\ln t+(1-s)\ln a}\right) ds\\ &= s^{\alpha}\frac{f\left(e^{s\ln t+(1-s)\ln a}\right)}{\ln t-\ln a}\Bigg|_0^1 - \frac{\alpha}{\ln t-\ln a}\int_0^1 s^{\alpha-1} f\left(e^{s\ln t+(1-s)\ln a}\right) ds\end{aligned}$$

by change of variables $u = e^{s\ln t+(1-s)\ln a}$ *and* $s = \dfrac{\ln u-\ln a}{\ln t-\ln a}$ *,we have*

$$\begin{aligned}&= \frac{f(t)}{\ln t-\ln a} - \frac{\alpha}{(\ln t-\ln a)^{\alpha+1}}\int_a^t (\ln u-\ln a)^{\alpha-1} f(u)\frac{du}{u}\\ &= \frac{f(t)}{\ln t-\ln a} - \frac{\Gamma(\alpha+1)}{(\ln t-\ln a)^{\alpha+1}} H_{t-}^{\alpha} f(a),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\int_0^1 s^{\alpha} e^{s\ln t+(1-s)\ln b} f'\left(e^{s\ln t+(1-s)\ln b}\right) ds\\ &= s^{\alpha}\frac{f\left(e^{s\ln t+(1-s)\ln b}\right)}{\ln t-\ln b}\Bigg|_0^1 - \frac{\alpha}{\ln t-\ln b}\int_0^1 s^{\alpha-1} f(e^{s\ln t+(1-s)\ln b}) ds\end{aligned}$$

by change of variables $u = e^{\ln b - s(\ln b-\ln t)}$ *and* $s = \dfrac{\ln b-\ln u}{\ln b-\ln t}$ *, we have*

$$\begin{aligned}&= \frac{f(t)}{\ln t-\ln b} - \frac{\alpha}{(\ln b-\ln t)^{\alpha+1}}\int_b^t (\ln b-\ln u)^{\alpha-1} f(u)\frac{du}{u}\\ &= \frac{f(t)}{\ln t-\ln b} + \frac{\Gamma(\alpha+1)}{(\ln b-\ln t)^{\alpha+1}} H_{t+}^{\alpha} f(b).\end{aligned} \tag{3.8}$$

Multiplying both sides of (3.7) and (3.8) by $\dfrac{(\ln t-\ln a)^{\alpha+1}}{\ln b-\ln a}$ *and* $\dfrac{(\ln b-\ln t)^{\alpha+1}}{\ln b-\ln a}$*,respectively, we have*

$$\begin{aligned}&\frac{(\ln t-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 s^{\alpha} e^{s\ln t+(1-s)\ln a} f'\left(e^{s\ln t+(1-s)\ln a}\right) ds\\ &= \frac{(\ln t-\ln a)^{\alpha} f(t)}{\ln b-\ln a} - \frac{\Gamma(\alpha+1)}{\ln b-\ln a} H_{t-}^{\alpha} f(a),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\frac{(\ln b-\ln t)^{\alpha+1}}{\ln b-\ln a}\int_0^1 s^{\alpha} e^{s\ln t+(1-s)\ln b} f'\left(e^{s\ln t+(1-s)\ln b}\right) ds\\ &= -\frac{(\ln b-\ln t)^{\alpha} f(t)}{\ln b-\ln a} + \frac{\Gamma(\alpha+1)}{\ln b-\ln a} H_{t+}^{\alpha} f(b).\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we obtain the desired result .

Lemma 3.1.4 *[11] Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $0 < a < b$, If $f' \in L^1[a,b]$, then the following equality for fractional integrals holds :*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}[H^\alpha_{a+}f(b)+H^\alpha_{b-}f(a)]-f\left(\frac{a+b}{2}\right)\\
&=\frac{b-a}{2}\int_0^1 kf'\left(sa+(1-s)b\right)ds\\
&-\frac{\ln b-\ln a}{2}\int_0^1[(1-s)^\alpha-s^\alpha]e^{s\ln a+(1-s)\ln b}f'\left(e^{s\ln a+(1-s)\ln b}\right)ds\\
&=\frac{b-a}{2}\int_0^1 kf'\left(sa+(1-s)b\right)ds\\
&-\frac{\ln b-\ln a}{2}\int_0^1[(1-s)^\alpha-s^\alpha]a^sb^{1-s}f'\left(a^sb^{1-s}\right)ds,
\end{aligned} \tag{3.11}$$

where

$$k=\begin{cases}1, & 0\le s\le \frac{1}{2},\\ \\ -1, & \frac{1}{2}\le s\le 1.\end{cases}$$

Proof 23 *Denote*

$$\begin{aligned}
I&=\frac{b-a}{2}\int_0^{\frac{1}{2}}f'\left(sa+(1-s)b\right)ds-\frac{b-a}{2}\int_{\frac{1}{2}}^1 f'\left(sa+(1-s)b\right)ds\\
&-\frac{\ln b-\ln a}{2}\int_0^1(1-s)^\alpha e^{s\ln a+(1-s)\ln b}f'\left(e^{s\ln a+(1-s)\ln b}\right)ds\\
&+\frac{\ln b-\ln a}{2}\int_0^1 s^\alpha e^{s\ln a+(1-s)\ln b}f'\left(e^{s\ln a+(1-s)\ln b}\right)ds\\
&=I_1+I_2+I_3+I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1&=\frac{b-a}{2}\int_0^{\frac{1}{2}}f'(sa+(1-s)b)ds,\\
I_2&=-\frac{b-a}{2}\int_{\frac{1}{2}}^1 f'(sa+(1-s)b)ds,\\
I_3&=-\frac{\ln b-\ln a}{2}\int_0^1(1-s)^\alpha e^{\ln b-s(\ln b-\ln a)}f'\left(e^{\ln b-s(\ln b-\ln a)}\right)ds,\\
I_4&=\frac{\ln b-\ln a}{2}\int_0^1 s^\alpha e^{\ln b-s(\ln b-\ln a)}f'\left(e^{\ln b-s(\ln b-\ln a)}\right)ds.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} I_1 &= \frac{b-a}{2}\int_0^{\frac{1}{2}} f'(sa+(1-s)b)ds \\ &= -\frac{f(sa+(1-s)b)}{2}\Big|_0^{\frac{1}{2}} = \frac{1}{2}\Big[f(b)-f\Big(\frac{a+b}{2}\Big)\Big], \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_2 &= -\frac{b-a}{2}\int_{\frac{1}{2}}^{1} f'(sa+(1-s)b)ds \\ &= \frac{f(sa+(1-s)b)}{2}\Big|_{\frac{1}{2}}^{1} = \frac{1}{2}\Big[f(a)-f\Big(\frac{a+b}{2}\Big)\Big], \end{aligned} \tag{3.13}$$

$$\begin{aligned} I_3 &= -\frac{\ln b-\ln a}{2}\int_0^1 (1-s)^{\alpha} e^{\ln b - s(\ln b-\ln a)} f'\left(e^{\ln b - s(\ln b-\ln a)}\right) ds \\ &= (1-s)^{\alpha}\frac{f(e^{\ln b - s(\ln b-\ln a)})}{2}\Big|_0^1 + \frac{\alpha}{2}\int_0^1 (1-s)^{\alpha-1}\left(e^{\ln b - s(\ln b-\ln a)}\right) ds \end{aligned}$$

by change of variables $u = e^{\ln b - s(\ln b-\ln a)}$ *and* $s = \dfrac{\ln b-\ln u}{\ln b-\ln a}$, *we have*

$$\begin{aligned} &= -\frac{f(b)}{2} + \frac{\alpha}{2(\ln a-\ln b)}\int_b^a \left(\frac{\ln u-\ln a}{\ln b-\ln a}\right)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(b)}{2} + \frac{\alpha}{2(\ln b-\ln a)^{\alpha}}\int_a^b (\ln u-\ln a)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(b)}{2} + \frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}} H^{\alpha}_{b-}f(a), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} I_4 &= \frac{\ln b-\ln a}{2}\int_0^1 s^{\alpha} e^{\ln b - s(\ln b-\ln a)} f'\left(e^{\ln b - s(\ln b-\ln a)}\right) ds \\ &= -s^{\alpha}\frac{f(e^{lnb - s(\ln b-\ln a)})}{2}\Big|_0^1 + \frac{\alpha}{2}\int_0^1 s^{\alpha-1} f\left(e^{\ln b - s(\ln b-\ln a)}\right) ds \\ &= -\frac{f(a)}{2} + \frac{\alpha}{2(\ln a-\ln b)}\int_a^b \left(\frac{\ln b-\ln u}{\ln b-\ln a}\right)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(a)}{2} + \frac{\alpha}{2(\ln b-\ln a)^{\alpha}}\int_a^b (\ln b-\ln u)^{\alpha-1} f(u)\frac{du}{u} \\ &= -\frac{f(a)}{2} + \frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}} H^{\alpha}_{a+}f(b). \end{aligned} \tag{3.15}$$

From (3.12),(3.13),(3.14),(3.15),we obtain the result.

3.2 Hadamard-type fractional integral in X_c^p

In this section we show that the Hadamard-type fractional integration operator ${}^{\mu}H_{a+}^{\alpha}$ is defined on $X_c^p(a,b)$ for $\mu \geq c$.

Theorem 19 *[4] Let $\alpha > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ and let $\mu \in \mathbb{R}$ and $c \in \mathbb{R}$ be such that $\mu \geq c$, Then the operator ${}^{\mu}H_{a+}^{\alpha}$ is bounded in $X_c^p(a,b)$ and*

$$\|{}^{\mu}H_{a+}^{\alpha}\|_{X_c^p} \leq K\|f\|_{X_c^p}, \tag{3.16}$$

where

$$K = \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{b}{a}\right)^{\alpha} \tag{3.17}$$

for $\mu = c$,
while

$$K = \frac{1}{\Gamma(\alpha)}(\mu-c)^{-\alpha}\gamma\left[\alpha, (\mu-c)\ln\left(\frac{a}{b}\right)\right] \tag{3.18}$$

for $\mu < c$.

Proof 24 *First consider the case $1 \leq p < \infty$ .Since $f(s) \in X_c^p(a,b)$,then $s^{c-\frac{1}{p}}f(s) \in L_p(a,b)$ and we can apply the generalized Minkowsky inequality ,we have*

$$\begin{aligned}
\|{}^{\mu}H_{a+}^{\alpha}\|_{X_c^p} &= \left(\int_a^b t^{cp}\left|\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{s}{t}\right)^{\mu}\left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}\right|\frac{dt}{t}\right)^{\frac{1}{p}} \\
&= \left(\int_a^b \left|\frac{1}{\Gamma(\alpha)}\int_1^{\frac{t}{a}} t^{c-\frac{1}{p}}u^{-\mu}(\ln u)^{\alpha-1} f\left(\frac{t}{u}\right)\frac{du}{u}\right|^p dt\right)^{\frac{1}{p}} \\
&\leq \int_1^{\frac{b}{a}} u^{-\mu-1}(\ln u)^{\alpha-1}\left(\int_{at}^b t^{cp}\left|f\left(\frac{t}{u}\right)\right|^p\frac{dt}{t}\right)^{\frac{1}{p}} ds \\
&= \int_1^{\frac{b}{a}} u^{c-\mu-1}(\ln u)^{\alpha-1}\left(\int_a^{\frac{b}{u}} |t^c f(t)|^p\frac{dt}{t}\right)^{\frac{1}{p}} du
\end{aligned}$$

and hence

$$\|{}^{\mu}H_{a+}^{\alpha}f\|_{X_c^p} \leq M\|f\|_{X_c^p},$$

where

$$M = \int_1^{\frac{b}{a}} u^{c-\mu-1}(\ln u)^{\alpha-1}du.$$

Direct calculation show that M coincides with K given in (3.17) and (3.18), when $\mu = c$ and $\mu > c$, respectively. Thus (3.16) is proved for $1 \leq p \leq \infty$.
Let now $p = \infty$ we have

$$|t^c \left({}^{\mu}H^{\alpha}_{a+}f\right)(s)| \leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{s}{t}\right)^{\mu-c} \left(\ln \frac{s}{t}\right)^{\alpha-1} |s^c f(s)| \frac{ds}{s}$$

and thus

$$|t^c \left({}^{\mu}H^{\alpha}_{a+}f\right)(s)| \leq K(t)\|f\|_{X^{\infty}_c}, \tag{3.19}$$

where

$$K(t) = \int_1^{\frac{t}{a}} u^{-(\mu-c)} (\ln u)^{\alpha-1} \frac{du}{u}.$$

When $\mu = c$, then for any $a \leq t \leq b$

$$K(t) = \frac{1}{\Gamma(\alpha+1)} \left(\ln \frac{t}{a}\right)^{\alpha} \leq \frac{1}{\Gamma(\alpha+1)} \left(\ln \frac{b}{a}\right)^{\alpha}. \tag{3.20}$$

If $\mu > c$, then making the change of variable $(\mu - c)u = y$ and taking the incomplete Gamma function into account we find

$$K(t) = \frac{1}{\Gamma(\alpha)} (\mu - c)^{-\alpha} \gamma \left[\alpha, (\mu - c) \ln \left(\frac{t}{a}\right)\right].$$

$\gamma(\nu, t)$ is increasing function and thus

$$K(t) \leq \frac{1}{\Gamma(\alpha)} (\mu - c)^{-\alpha} \gamma \left[\alpha, (\mu - c) \ln \left(\frac{b}{a}\right)\right] \tag{3.21}$$

for any $a \leq t \leq b$. It follows from (3.19) and (3.21)that for any $a \leq t \leq b$

$$|t^c \left({}^{\mu}H^{\alpha}_{a+}f\right)(s)| \leq K\|f\|_{X^{\infty}_c}, \tag{3.22}$$

where K is given by (3.17) and (3.18) when $\mu = c$ and $\mu > c$, respectivly.

Corollary 20 *[4] Let $\alpha > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ and let $\mu \in \mathbb{R}$ be such that $\mu \geq \frac{1}{p}$. Then the operator ${}^{\mu}H^{\alpha}_{a+}$ is bouded in $L_p(a, b)$ and*

$$\|{}^{\mu}H^{\alpha}_{a+}f\|_p \leq K_1\|f\|_p,$$

where K_1 *is given by* (3.17) *for* $\mu = \dfrac{1}{p}$, *while*

$$K_1 = \frac{1}{\Gamma(\alpha)} \left(\mu - \frac{1}{p}\right)^{-\alpha} \gamma \left[\alpha, \left(\mu - \frac{1}{p}\right) \ln\left(\frac{b}{a}\right)\right]$$

for $\mu > \dfrac{1}{p}$.

Theorem 21 *[4] Let* $\alpha > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ *and let* $c \leq 0$. *Then the operator* H^{α}_{a+} *is buonded in* $X^p_c(a,b)$ *and*

$$\|H^{\alpha}_{a+} f\|_{X^p_c} \leq K_2 \|f\|_{X^p_c},$$

where

$$K_2 = \frac{1}{\Gamma(\alpha+1)} \left(\ln \frac{b}{a}\right)^{\alpha}$$

for $c = 0$, *while*

$$K_2 = \frac{1}{\Gamma(\alpha)} (-c)^{-\alpha} \gamma \left[\alpha, -c \ln\left(\frac{b}{a}\right)\right]$$

for $c < 0$.

Chapter 4

Hermite-Hadamard inequalities

4.1 Hermite-Hadamard's inequalities for Hadamard fractional integrals

Theorem 22 *[12][11] Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a \leq b$ and $f \in L^1[a, b]$. If f is a nondecreasing and convex function on $[a, b]$, then the following double inequalities for fractional integrals holds:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}\left[H^{\alpha}_{a+}f(b) + H^{\alpha}_{b-}f(a)\right] \leq f(b). \tag{4.1}$$

Proof 25 *Since f is a nondecreasing and convex function on $[a, b]$, we have for $x, y \in [a, b]$:*

$$f\left(\sqrt{xy}\right) \leq f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{4.2}$$

Set $x = e^{\ln b - s(\ln b - \ln a)}$ and $y = e^{\ln a + s(\ln b - \ln a)}$ for $0 < s < 1$, then

$$2f\left(\sqrt{ab}\right) = 2f\left(\sqrt{e^{\ln b + \ln a}}\right) \leq f\left(e^{\ln b - s(\ln b - \ln a)}\right) + f\left(e^{\ln a + s(\ln b - \ln a)}\right). \tag{4.3}$$

Multiplying both sides of (4.3) by $s^{\alpha-1}$, then integrating the resulting inequality with respect to s over $[0, 1]$.we obtain

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\sqrt{ab}\right) = \frac{2}{\alpha}f\left(\sqrt{e^{\ln b+\ln a}}\right) \\
&\leq \int_0^1 s^{\alpha-1} f\left(e^{\ln b - s(\ln b - \ln a)}\right) ds + \int_0^1 s^{\alpha-1} f\left(e^{\ln a + s(\ln b - \ln a)}\right) ds. \\
&\leq J_1 + J_2
\end{aligned}$$

where

$$J_1 = \int_0^1 s^{\alpha-1} f\left(e^{\ln b - s(\ln b - \ln a)}\right) ds$$

by change of variable $u = e^{\ln b - s(\ln b - \ln a)}$, $s = \dfrac{\ln b - \ln u}{\ln b - \ln a}$, *we have*

$$J_1 = \frac{1}{\ln a - \ln b} \int_b^a \left(\frac{\ln b - \ln u}{\ln b - \ln a}\right)^{\alpha-1} f(u) \frac{du}{u}$$

and

$$J_2 = \int_0^1 s^{\alpha-1} f\left(e^{\ln a + s(\ln b - \ln a)}\right) ds$$

by change of variables $v = e^{\ln a + s(\ln b - \ln a)}$, $s = \dfrac{\ln v - \ln a}{\ln b - \ln a}$, *we have*

$$J_2 = \frac{1}{\ln b - \ln a} \int_a^b \left(\frac{\ln v - \ln a}{\ln b - \ln a}\right)^{\alpha-1} f(v) \frac{dv}{v}$$

then

$$J_1 + J_2 = \frac{\Gamma(\alpha)}{(\ln b - \ln a)^{\alpha}} [H^{\alpha}_{a+} f(b) + H^{\alpha}_{b-} f(a)],$$

wich impllies that

$$f(\sqrt{ab}) \leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}} [H^{\alpha}_{a+} f(b) + H^{\alpha}_{b-} f(a)]. \tag{4.4}$$

On the other hand, note that f *is nondecreasing, we have*

$$f(e^{\ln b - s(\ln b - \ln a)}) + f(e^{\ln a + s(\ln b - \ln a)}) \leq 2f\left(e^{\ln b}\right) = 2f(b). \tag{4.5}$$

Then multiplying both sides of (4.5) by s^{α} *and integrating the resulting inequality with respect to* s *over* $[0, 1]$, *we obtain*

$$\begin{aligned}
&\int_0^1 s^{\alpha-1} f(e^{\ln b - s(\ln b - \ln a)}) ds + \int_0^1 s^{\alpha-1} f(e^{\ln a + s(\ln b - \ln a)}) ds \\
&= \frac{\Gamma(\alpha)}{(\ln b - \ln a)^{\alpha}} [H^{\alpha}_{a+} f(b) + H^{\alpha}_{b-} f(a)] \\
&\leq 2f(e^{\ln b}) \int_0^1 s^{\alpha-1} ds \\
&= \frac{2}{\alpha} f(b),
\end{aligned}$$

wich yieds

$$\frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha}} [H^{\alpha}_{a+} f(b) + H^{\alpha}_{b-} f(a)] \leq 2f(b).$$

Remark 23 *The Theorem 22 is a result of the following Theorem:*

Theorem 24 *[11] Let $f : [x, y] \to \mathbb{R}$ be a positive function with $0 \leq x \leq y$ and $f \in L^1[x, y]$. If f is a convex function on $[x, y]$, then the following double inequalities for fractional integrals holds:*

$$f\left(\frac{x+y}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[R^{\alpha}_{x+}f(y) + R^{\alpha}_{y-}f(x)\right] \leq \frac{f(x)+f(y)}{2}, \tag{4.6}$$

with $\alpha > 0$.

Proof 26 *Take $x = \ln a$ and $y = \ln b$ and using the operator A, we obtain*

$$Af\left(\frac{\ln a + \ln b}{2}\right)$$
$$\leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[R^{\alpha}_{\ln a+}Af(\ln b) + R^{\alpha}_{\ln b-}Af(\ln a)] \leq \frac{Af(\ln a) + Af(\ln b)}{2}$$

then

$$f(e^{\frac{\ln a + \ln b}{2}}) \leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[H^{\alpha}_{a+}f(b) + H^{\alpha}_{b-}f(a)] \leq \frac{f(e^{\ln a}) + f(e^{\ln b})}{2}$$
$$f(a^{\frac{1}{2}}b^{\frac{1}{2}}) \leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[H^{\alpha}_{a+}f(b) + H^{\alpha}_{b-}f(a)] \leq \frac{f(a) + f(b)}{2}$$

thus

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[H^{\alpha}_{a+}f(b) + H^{\alpha}_{b-}f(a)] \leq f(b).$$

Example 4.1. Let $a = 1, b = e, \alpha = 2$ and $f(t) = t^2$.Then all the assumptons in (4.1) are satisfied. Clearly,

$$H^2_{1+}f(e) = \int_1^e (1 - \ln s)sds = \int_0^1 ue^{2(1-u)}du = \frac{1}{4}e^2 - \frac{3}{4},$$
$$H^2_{e-}f(1) = \int_1^e s\ln sds = \int_0^1 ue^{2u}du = \frac{1}{4}e^2 + \frac{1}{4}. \tag{4.7}$$

Thus,
(4.1)$\Longleftrightarrow e < \frac{\Gamma(3)}{2(\ln e - \ln 1)^2}[H^2_{1+}f(e) + H^2_{e-}f(1)] = \frac{1}{2}e^2 - \frac{1}{2} < e^2.$
We can obtain the following explicit estimate for some $\alpha \in [0, 1]$.

4.1.1 Hermite-Hadamard type inequalities for Hadamard-fractional integrals

Theorem 25 *[12][11] Let $f : [a, b] \to \mathbb{R}$ be differentiable mapping on (a, b) with $0 < a < b$, If $\alpha \in [0, 1], f' \in L^1[a, b]$ and is nondecreasing, then the following equality for fractional integrals holds :*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[H^{\alpha}_{a+}f(b)+H^{\alpha}_{b-}f(a)]\right| \\
&\leq \frac{b(\ln b-\ln a)}{2}\left[\frac{\alpha+2}{\alpha+1}\left(\frac{\ln b-\ln a}{2}\right)^{-1}+\frac{\sqrt{\frac{a}{b}}}{2(\alpha+1)}\right]|f'(b)|. \qquad (4.8)
\end{aligned}$$

Proof 27 *Using Lemma 3.1.1 and the nondecreasing property of f', we find*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[H^{\alpha}_{a+}f(b)+H^{\alpha}_{b-}f(a)]\right| \\
&\leq \frac{\ln b-\ln a}{2}\int_0^1\left|(1-s)^{\alpha}-s^{\alpha}\right|e^{\ln b-s(\ln b-\ln a)}\left|f'(b)\right|ds \\
&= \frac{b(\ln b-\ln a)|f'(b)|}{2}\int_0^{\frac{1}{2}}\left[(1-s)^{\alpha}-s^{\alpha}\right]e^{-s(\ln b-\ln a)}ds \\
&+ \frac{b(\ln b-\ln a)|f'(b)|}{2}\int_{\frac{1}{2}}^1\left[s^{\alpha}-(1-s)^{\alpha}\right]e^{-s(\ln b-\ln a)}ds \\
&= \frac{b(\ln b-\ln a)|f'(b)|}{2}(k_1+k_2). \qquad (4.9)
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \int_0^{\frac{1}{2}}\left[(1-s)^{\alpha}-s^{\alpha}\right]e^{-s(\ln b-\ln a)}ds, \\
k_2 &= \int_{\frac{1}{2}}^1\left[s^{\alpha}-(1-s)^{\alpha}\right]e^{-s(\ln b-\ln a)}ds.
\end{aligned}$$

Calculating k_1, we have :

$$\begin{aligned}
k_1 &= \int_0^{\frac{1}{2}} \Big[(1-s)^\alpha - s^\alpha\Big] e^{-s(\ln b - \ln a)} ds \\
&\leq \int_0^{\frac{1}{2}} (1-2s)^\alpha e^{-s(\ln b - \ln a)} ds. \\
&= \frac{1}{2}\int_0^1 (1-s)^{(\alpha+1)-1} e^{-\frac{\ln b - \ln a}{2}s} ds \\
&\leq \max\Big\{1, 2^{-\alpha}\Big\}\left(1 + \frac{1}{\alpha+1}\right)\left(\frac{\ln b - \ln a}{2}\right)^{-1} \\
&\leq \frac{\alpha+2}{\alpha+1}\left(\frac{\ln b - \ln a}{2}\right)^{-1}, \qquad (4.10)
\end{aligned}$$

where Lemma 1.4.1 and Lamma 1.4.2 are used. Calculating k_2, we have:

$$\begin{aligned}
k_2 &= \int_{\frac{1}{2}}^1 \Big[s^\alpha - (1-s)^\alpha\Big] e^{-s(\ln b - \ln a)} ds \\
&\leq \int_{\frac{1}{2}}^1 (2s-1)^\alpha e^{-s(\ln b - \ln a)} ds \\
&= \frac{1}{2}\int_1^2 (s-1)^\alpha e^{-\frac{\ln b - \ln a}{2}s} ds \\
&= \frac{1}{2} e^{-(\ln b - \ln a)} \int_0^1 (1-\tau)^\alpha e^{\frac{\ln b - ln a}{2}\tau} d\tau \\
&\leq \frac{1}{2} e^{-(\ln b - I\ln a)} \int_0^1 (1-\tau)^\alpha d\tau \\
&= \frac{\sqrt{\frac{a}{b}}}{2(\alpha+1)} \qquad (4.11)
\end{aligned}$$

where Lemma 1.4.1 are used.

Theorem 26 *[12][11] Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $0 < a < b$, If $f' \in L^1[a,b]$ and is nondecreasing, then the following equality for fractional integrals holds :*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}[H^\alpha_{a+}f(b) + H^\alpha_{b-}f(a)]\right| \\
&\leq \frac{b(\ln b - \ln a)}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)|f'(b)|.
\end{aligned}$$

Proof 28 *Using Lamma 3.1.1 and the nondecresing property of f', one can obtain:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[H^{\alpha}_{a+}f(b)+H^{\alpha}_{b-}f(a)]\right|\\
&\leq \frac{\ln b-\ln a}{2}\int_0^1 |(1-s)^{\alpha}-s^{\alpha}|e^{\ln b-s(\ln b-\ln a)}|f'(b)|ds.\\
&\leq \frac{b(\ln b-\ln a)|f'(b)|}{2}\int_0^{\frac{1}{2}}[(1-s)^{\alpha}-s^{\alpha}]ds\\
&+\frac{b(\ln b-\ln a)|f'(b)|}{2}\int_{\frac{1}{2}}^{1}[s^{\alpha}-(1-s)^{\alpha}]ds\\
&=\frac{b(\ln b-\ln a)|f'(b)}{2}\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\\
&=\frac{b(\ln b-\ln a)}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)|f'(b)|. \qquad (4.12)
\end{aligned}
$$

where $e^{\ln b-s(\ln b-\ln a)}\leq e^{\ln b}=b$ is used, The proof is completed.

Remark 27 *The Theorem 26 is result of the following Theorem:*

Theorem 28 *Let $f:[x,y]\to\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $|f'|$ is convex on $[x,y]$, then following equality for integrals holds*

$$
\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)}[R^{\alpha}_{x+}f(y)+R^{\alpha}_{y-}(x)]\right|\leq\frac{y-x}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)|f'(x)+f'(y)|.
$$

Proof 29 *Take $x=\ln a, y=\ln b$ and using the operator A we obtain*

$$
\begin{aligned}
&\left|\frac{Af(\ln a)+Af(\ln b)}{2}-\frac{\Gamma\alpha+1}{2(\ln b-\ln a)^{\alpha}}[R^{\alpha}_{\ln a+}Af(\ln b)+R^{\alpha}_{\ln b+}Af(\ln a)]\right|\\
&\leq\frac{\ln b-\ln a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)|Af'(\ln a)+Af'(\ln b)|
\end{aligned}
$$

then

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma\alpha+1}{2(\ln b-\ln a)^{\alpha}}[H^{\alpha}_{a+}f(b)+H^{\alpha}_{b-}f(a)]\right|\\
&\leq\frac{\ln b-\ln a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)|af'(a)+bf'(b)|\\
&\leq\frac{b(\ln b-\ln a)}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)|f'(b)|. \qquad (4.13)
\end{aligned}
$$

4.1.2 Applications to some special means

Proposition 4.1.1. *[12] Let $a, b \in \mathbb{R}^+, a < b$. Then*

$$|A(a,b) - L(a,b)| \leq \frac{b(\ln b - \ln a)}{8}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right] \tag{4.14}$$

and

$$|A(a,b) - L(a,b)| \leq \frac{b(\ln b - \ln a)}{4} \tag{4.15}$$

Proof 30 *Applying Theorem (25) and Theorem (26) respectively , for $f(x) = x$ and $\alpha = 1$, one can obtain:*

$$\left|\frac{a+b}{2} - \frac{2(b-a)}{2(\ln b - \ln a)}\right| \leq \frac{b(\ln b - \ln a)}{2}\left[\frac{3}{2}\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \frac{1}{4}\sqrt{\frac{a}{b}}\right] \times \frac{4}{4}$$
$$\leq \frac{b(\ln b - \ln a)}{8}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right],$$

the proof of 4.14 is completed.
And

$$\left|\frac{a+b}{2} - \frac{b-a}{\ln b - \ln a}\right| \leq \frac{b(\ln b - \ln a)}{2}\left(1 - \frac{1}{2}\right)$$
$$\leq \frac{b(\ln b - \ln a)}{4}, \tag{4.16}$$

the proof of 4.15 is completed.

Proposition 4.1.2. *[12] Let $a, b \in \mathbb{R}^+, a < b$ and $n \in \mathbb{Z}, |n| \geq 2$. Then*

$$\left|A(a^{n+1}, b^{n+1}) - \frac{b-a}{\ln b - \ln a} L_n^n(a,b)\right|$$
$$\leq \frac{(n+1)b^{n+1}(\ln b - lna)}{8}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right], \tag{4.17}$$

and

$$\left|A(a^{n+1}, b^{n+1}) - \frac{b-a}{\ln b - \ln a} L_n^n(a,b)\right|$$
$$\leq \frac{(n+1)b^{n+1}(\ln b - lna)}{4}. \tag{4.18}$$

Proof 31 *Applying Theorem (25) and Theorem (26) respectively , for $f(x) = x^{n+1}$ and $\alpha = 1, x \in \mathbb{R}, n \in \mathbb{Z}, |n| \geq 2$, one can obtain:*

$$\begin{aligned}
&\left|\frac{a^{n+1}+b^{n+1}}{2} - \frac{b^{n+1}-a^{n+1}}{(n+1)(\ln b - \ln a)}\right| \\
&\leq \frac{b(\ln b - \ln a)}{2}\left[\frac{3}{2}\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \frac{1}{4}\sqrt{\frac{a}{b}}\right]|(n+1)b^n| \times \frac{4}{4} \\
&\leq \frac{(n+1)b^{n+1}(\ln b - \ln a)}{8}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right].
\end{aligned}$$

The proof of (4.17) is completed.

And

$$\begin{aligned}
&\left|\frac{a^{n+1}+b^{n+1}}{2} - \frac{b^{n+1}-a^{n+1}}{(n+1)(\ln b - \ln a)}\right| \\
&\leq \frac{b(\ln b - \ln a)}{4}(n+1)b^n \\
&\leq \frac{(n+1)b^{n+1}(\ln b - lna)}{4}.
\end{aligned}$$

The proof of (4.18) is completed.

Proposition 4.1.3. *[12] Let $a, b \in \mathbb{R}^+(a < b), a^{-1} > b^{-1}$.For $n \in \mathbb{Z}, |n| \geq 2$,we have*

(i)

$$|H^{-1}(b,a) - L(b^{-1},a^{-1})| \leq \frac{(\ln b - \ln a)}{8a}\left[6\left(\frac{\ln b - \ln a}{2}\right) + \sqrt{\frac{a}{b}}\right],$$

(ii)

$$|H^{-1}(b,a) - L(b^{-1},a^{-1})| \leq \frac{(\ln b - \ln a)}{4a},$$

(iii)

$$\left|H^{-1}(b^{n+1},a^{n+1}) - \frac{a^{-1}-b^{-1}}{\ln b - \ln a}L_n^n(b^{-1},a^{-1})\right| \leq \frac{(n+1)(\ln b - \ln a)}{8a^{n+1}}\left[6\left(\frac{\ln b - \ln a}{2}\right) + \sqrt{\frac{a}{b}}\right],$$

(iv)

$$\left|H^{-1}(b^{n+1},a^{n+1}) - \frac{a^{-1}-b^{-1}}{\ln b - \ln a}L_n^n(b^{-1},a^{-1})\right| \leq \frac{(n+1)(\ln b - \ln a)}{8a^{n+1}}$$

Proof 32 *Making the substitutions* $a \to b^{-1}, b \to a^{-1}$ *and*

$$H^{-1}(b,a) = A(a^{-1}, b^{-1})$$

in the inequalities (4.14) ,(4.15), (4.17) and (4.18) one can obtain desired inequalities respectively.

4.2 Fractionl Integral Inequalities via Hadamard's Fractional Integral

Theorem 29 *[14][15] Let* f *be an integrable function on* $[1,\infty)$. *Assume the following .*

(hy_1) *There exist two integreable function* φ_1, φ_2 *on* $[1,\infty)$ *such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t), \forall t \in [1,\infty). \tag{4.19}$$

Then for $t > 1, \alpha, \beta > 0$, *one has*

$$H^{\beta}_{a+}\varphi_1(t)H^{\alpha}_{a+}f(t) + H^{\alpha}\varphi_2(t)H^{\beta}_{a+}f(t) \ge H^{\alpha}_{a+}\varphi_2(t)H^{\beta}_{a+}\varphi_1(t) + H^{\alpha}_{a+}f(t)H^{\beta}_{a+}f(t). \tag{4.20}$$

Proof 33 *From* (hy_1), *for all* $\tau \ge 1, \rho \ge 1$, *we have*

$$(\varphi_2(\tau) - f(\tau))\,(f(\rho) - \varphi_1(\rho)) \ge 0. \tag{4.21}$$

Therefore,

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \ge \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{4.22}$$

Multiplying both sides of (4.22) by $\left(\dfrac{\ln(\frac{t}{\tau})^{\alpha-1}}{\tau\Gamma(\alpha)}\right)$, $\tau \in (1,t)$, *we get*

$$\begin{aligned} & f(\rho)\frac{\left(\ln\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + \varphi_1(\rho)\frac{\left(\ln\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau) \\ & \ge \varphi_1(\rho)\frac{\left(\ln\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + f(\rho)\frac{\left(\ln\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau). \end{aligned} \tag{4.23}$$

Integrating both sides of (4.23)with respect to τ *on* $(1, t)$*, we obtain*

$$\begin{aligned} & f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} \\ & +\varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau} \\ & \geq \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} \\ & +f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}, \end{aligned} \tag{4.24}$$

which yields

$$f(\rho)H^{\alpha}_{a+}\varphi_2(t)+\varphi_1(\rho)H^{\alpha}_{a+}f(t)\geq\varphi_1(\rho)H^{\alpha}_{a+}\varphi_2(t)+f(\rho)H^{\alpha}_{a+}f(t). \tag{4.25}$$

Multiplying both sides of(4.25) by $\dfrac{\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\rho\Gamma(\beta)}, \rho\in(1,t)$,*we have*

$$\begin{aligned} & H^{\alpha}_{a+}\varphi_2(t)\frac{\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\rho\Gamma(\beta)}f(\rho)+H^{\alpha}_{a+}f(t)\frac{\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\rho\Gamma(\beta)}\varphi_1(\rho) \\ & \geq H^{\alpha}_{a+}\varphi_2(t)\frac{\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\rho\Gamma(\beta)}\varphi_1(\rho)+H^{\alpha}_{a+}f(t)\frac{\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\rho\Gamma(\beta)}f(\rho). \end{aligned} \tag{4.26}$$

Integrating both sides of(4.26) with respect to $\rho\in(1,t)$ *,we get*

$$\begin{aligned} & H^{\alpha}_{a+}\varphi_2\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho} \\ & +H^{\alpha}_{a+}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{\rho}\right)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho} \\ & \geq H^{\alpha}_{a+}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{\rho}\right)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho} \\ & H^{\alpha}_{a+}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho} \end{aligned} \tag{4.27}$$

Hence, we deduce inequality (4.20) as requested. This completes the proof.

Corollary 30 *[14][15] Let* f *be an integrable function on* $[1,\infty)$ *satisfying* $m\leq f(t)\leq M$ *, for all* $t\in[1,\infty)$ *and* $m, M\in\mathbb{R}$*. Then for* $t>1$ *and* $\alpha,\beta>0$ *, one has*

$$\begin{aligned} & m\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}H^{\alpha}_{a+}f(t)+M\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}H^{\beta}_{a+}f(t) \\ & \geq mM\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+H^{\alpha}f(t)H^{\beta}_{a+}f(t). \end{aligned} \tag{4.28}$$

Corollary 31 *[14][15] Let f be an integrable function on $[1,\infty)$ Assume that there exists an integrable function $\varphi(t)$ on $[1,\infty)$ and a constant $M>0$ such that*

$$\varphi(t)-M\leq f(t)\leq\varphi(t)+M, \tag{4.29}$$

for all $t\in[1,\infty)$. Then for $t>1$ and $\alpha,\beta>0$, one has

$$\begin{aligned}
&H^{\beta}_{a+}\varphi(t)H^{\alpha}_{a+}f(t)+H^{\alpha}_{a+}\varphi(t)H^{\beta}_{a+}f(t)\\
&+\frac{M\,(\ln t)^{\beta}}{\Gamma(\alpha+1)}H^{\alpha}_{a+}\varphi(t)+\frac{M\,(\ln t)^{\alpha}}{\Gamma(\alpha+1)}H^{\beta}_{a+}f(t)\\
&+\frac{M^{2}(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&\geq H^{\alpha}_{a+}\varphi(t)H^{\beta}_{a+}\varphi(t)+H^{\alpha}_{a+}f(t)H^{\beta}f(t)\\
&+\frac{M(\ln t)^{\beta}}{\Gamma(\beta+1)}H^{\alpha}_{a+}f(t)+\frac{M(\ln t)^{\alpha}}{\Gamma(\alpha+1)}H^{\beta}_{a+}\varphi(t).
\end{aligned} \tag{4.30}$$

Example 4.2. Let f be a function satisfying

$$\ln t\leq f(t)\leq 1+\ln t$$

for $t\in[1,\infty)$. Then for $t>1$ and $\alpha>0$, we have

$$\begin{aligned}
&\left(\frac{2(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\right)H^{\alpha}_{a+}f(t)\\
&\geq\left(\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\right)\left(\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)}\right)\\
&+\left(H^{\alpha}_{a+}f(t)\right)^{2}.
\end{aligned} \tag{4.31}$$

Theorem 32 *[14][15] Let $\alpha,\beta>0$. Let f be an integrable function on $[1,\infty)$ and $p,q>0$ satisfying $\frac{1}{p}+\frac{1}{q}=1$ suppose that (hy_1) holds. Then for $t>1$, Then the following inequality*

$$\begin{aligned}
&\frac{1}{p}\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}H^{\alpha}_{a+}\left((\varphi_2-f)^{p}\right)(t)\\
&+\frac{1}{q}\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}H^{\beta}_{a+}\left((f-\varphi_1)^{q}\right)(t)\\
&+H^{\alpha}_{a+}\varphi_2(t)H^{\beta}\varphi_1(t)+H^{\alpha}_{a+}f(t)H^{\beta}_{a+}f(t)\\
&\geq H^{\alpha}_{a+}\varphi_2(t)H^{\beta}_{a+}f(t)+H^{\alpha}_{a+}f(t)H^{\beta}_{a+}\varphi_1(t)
\end{aligned} \tag{4.32}$$

Proof 34 *According to the well known Young's inequality.*
Setting $x = \varphi_2(\tau) - f(\tau)$ *and* $y = f(\rho) - \varphi_1(\rho), \tau, \rho > 1$, *we have*

$$\frac{1}{p}\left(\varphi_2(\tau) - f(\tau)\right)^p + \frac{1}{q}\left(f(\rho) - \varphi_1(\rho)\right)^q \geq \left(\varphi_2 - f(\tau)\right)\left(f(\rho) - \varphi_1(\rho)\right) \tag{4.33}$$

Multiplying both sides of (4.33)by $\dfrac{\left(\ln(\dfrac{t}{\tau})\right)^{\alpha-1}\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}, \tau, \rho \in (1, t)$, *we get*

$$\begin{aligned}
&\frac{1}{p}\frac{\left(\ln(\dfrac{t}{\tau})\right)^{\alpha-1}\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}\left(\varphi_2(\tau) - f(\tau)\right)^p \\
&+ \frac{1}{q}\frac{\left(\ln(\dfrac{t}{\tau})\right)^{\alpha-1}\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}\left(f(\rho) - \varphi_1(\rho)\right)^q \\
&\geq \frac{\left(\ln(\dfrac{t}{\tau})\right)^{\alpha-1}}{\tau\Gamma(\alpha)}\left(\varphi_2(\tau) - f(\tau)\right)\frac{\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\rho\Gamma(\beta)} \\
&\times\left(f(\rho) - \varphi_1(\rho)\right)
\end{aligned} \tag{4.34}$$

Integrating the above inequality with respect to τ *and* ρ *from* 1 *to* t, *we have*

$$\begin{aligned}
&\frac{1}{p}H^{\beta}_{a+}(1)(t)H^{\alpha}_{a+}\left(\varphi_2 - f\right)^p(t) \\
&+ \frac{1}{q}H^{\alpha}_{a+}(1)(t)H^{\beta}_{a+}\left(f - \varphi_1\right)^q(t) \\
&\geq H^{\alpha}_{a+}\left(\varphi_2 - f\right)(t)H^{\beta}_{a+}\left(f - \varphi_1\right)(t)
\end{aligned} \tag{4.35}$$

wich implies (4.32).

Corollary 33 *[14][15] Let* f *be an integrable function on* $[1, \infty)$ *satisfying* $m \leq f(t) \leq M$ *, for all* $t \in [1, \infty)$ *and* $m, M \in \mathbb{R}$. *Then for* $t > 1$ *and* $\alpha, \beta > 0$, *one has*

$$\begin{aligned}
&(m + M)^2\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}H^{\alpha}_{a+}f^2(t) \\
&+ \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}H^{\beta}_{a+}f^2(t) + 2H^{\alpha}_{a+}f(t)H^{\beta}_{a+}f(t) \\
&\geq 2(m + M) \\
&\times\left(\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}H^{\alpha}_{a+}f(t) + \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}H^{\beta}_{a+}f(t)\right).
\end{aligned} \tag{4.36}$$

Example 4.3. [14][15] Let f be a function satisfying

$$\ln t \leq f(t) \leq 1 + \ln t$$

for $t \in [1, \infty)$.Then for $t > 1$ and $\alpha > 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} \\
&\times \left(\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} + \frac{2\,(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{4\,(\ln t)^{\alpha+2}}{\Gamma(\alpha+3)} + 2H^{\alpha}_{a+} f^2(t) \right) \\
&+ \left(\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)} \right) \frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)} + \left(H^{\alpha}_{a+} f(t) \right)^2 \\
&\geq 2 \left(\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+2)} \right) H^{\alpha}_{a+} f(t) \\
&+ \frac{2\,(\ln t)^{\alpha}}{\Gamma(\alpha+1)} H^{\alpha}_{a+} \left(f \ln t \right)(t)
\end{aligned}$$

Theorem 34 *[14][15] Let $\alpha, \beta > 0$, f be an integrable function on $[1, \infty)$ and $p, q > 0$ satisfying $p + q = 1$. In addition, suppose that (hy_1) holds. Then, for $t > 1$, then the following inequality*

$$\begin{aligned}
&p\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)} H^{\alpha}_{a+} \varphi_2(t) + q\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+1)} H^{\beta}_{a+} f(t) \\
&\geq H^{\alpha}(\varphi_2 - f)^p(t) H^{\beta}_{a+}(f - \varphi_1)^q(t) \\
&+ p\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)} H^{\alpha}_{a+} f(t) + q\frac{(\ln t)^{\alpha+1}}{\Gamma(\alpha+1)} H^{\beta}_{a+} \varphi_1(t)
\end{aligned} \tag{4.37}$$

Proof 35 *From the well known weighted AM-GM inequality, by setting $x = \varphi_2(\tau) - f(\tau) =$ and $y = f(\rho) - \varphi_1(\rho), \tau, \rho > 1$, we have*

$$\begin{aligned}
&p \left(\varphi_2(\tau) - f(\tau) \right) + q \left(f(\rho) - \varphi_1(\rho) \right) \\
&\geq \left(\varphi_2(\tau) - f(\tau) \right)^p \left(f(\rho) - \varphi_1(\rho) \right)^q
\end{aligned} \tag{4.38}$$

Multiplying both sides of (4.38) by $\frac{\left(\ln(\frac{t}{\tau})\right)^{\alpha-1}\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}, \tau,\rho \in (1,t)$, *we get*

$$\begin{aligned}
&p\frac{\left(\ln(\frac{t}{\tau})\right)^{\alpha-1}\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau)-f(\tau)) \\
&+q\frac{\left(\ln(\frac{t}{\tau})\right)^{\alpha-1}\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\tau\rho\Gamma(\alpha)\Gamma(\beta)}(f(\rho)-\varphi_1(\rho)) \\
&\geq \frac{\left(\ln(\frac{t}{\tau})\right)^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau)-f(\tau))^p \\
&\times \frac{\left(\ln(\frac{t}{\rho})\right)^{\beta-1}}{\rho\Gamma(\beta)}(f(\rho)-\varphi_1(\rho))^q
\end{aligned} \tag{4.39}$$

Integrating the above inequality with respect to τ *and* ρ *from* 1 *to* t, *we have*

$$\begin{aligned}
&pH^{\beta}_{a+}(1)(t)H^{\alpha}_{a+}(\varphi_2-f)(t) \\
&+qH^{\alpha}_{a+}(1)(t)H^{\beta}_{a+}(f-\varphi_1)(t) \\
&\geq H^{\alpha}_{a+}(\varphi_2-f)^p(t)H^{\beta}_{a+}(f-\varphi_1)^q(t).
\end{aligned} \tag{4.40}$$

Corollary 35 *[14][15] Let* $\alpha,\beta>0$, f *be an integrable function on* $[1,\infty)$ *satisfying* $m \leq f(t) \leq M$, *for all* $t\in[1,\infty)$ *and* $m,M\in\mathbb{R}$.
Then for $t>1$ *,then the following inequality*

$$\begin{aligned}
&M\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}H^{\beta}_{a+}f(t) \\
&\geq m\frac{(\ln t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\ln t)^{\beta}}{\Gamma(\beta+1)}H^{\alpha}_{a+}f(t) \\
&+2H^{\alpha}(M-f)^{\frac{1}{2}}(t)H^{\beta}_{a+}(f-m)^{\frac{1}{2}}(t).
\end{aligned} \tag{4.41}$$

Example 4.4. [14][15] Let f be a function satisfying

$$\ln t \leq f(t) \leq 1+\ln t$$

for $t\in[1,\infty)$.

Then for $t>1$ and $\alpha>0$, we have

$$\begin{aligned}
&\frac{(\ln t)^{2\alpha}}{\Gamma^2(\alpha+1)} \\
&\geq 2H^{\alpha}_{a+}\left(\sqrt{1+\ln t-f}\right)(t)H^{\alpha}_{a+}\left(\sqrt{f-\ln t}\right)(t).
\end{aligned} \tag{4.42}$$

Conclusion

The purpose of this work is to generalize integral inequalities of Hermite-Hadamard type for the Hadamard fractional integral.

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