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Work on some Fractional deferential equation of variable order

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أهداء

مهما كتبت من عبارات لن اجد اصدق من قول الله تعالى :'
'يرفع الله الذين ءامنو منكم والذين أوتوا العلم درجات ''
فالحمدلله حمدا كثيرا طيبا مباركا فيه والحمدلله الذي لايضاهي نعمته وفضله
اي شيء والحمدلله الذي بنعمته تتم الصالحات
هاقد انطوت صفحة من صفحات الحياة كان فيها الجد والتعب لأحصد آخرها التفوق والنجاح
كانت رحلة تستحق العناء وكانت آمالا تستحق التمني وكانت أحلاما تستحق التحقيق
صدق من قال "بالعلم ترتقي العقول "
أهدي ثمرة نجاحي إلى امي :
فقد صدق قول رسول عليه صلاة وسلام:"أمك ثم أمك ثم أمك ثم أبيك "
أمي التي كانت السند والعوض في حياة أهدي لها نجاحي لأنها الشخص الوحيد الذي تستحقه كانت الأب والأم الصديق والأخ أمي التي تحملت عني مصاعب الحياة
مهما شكرت لن تكفي الكلمات لجنتي حبيبتي
إلى جدي "رحمه الله تعالى" الذي عوضني عن حنان الأب الذي حرمت منه
وأقول له هاقد حققت حلمك الذي لطالما تمنيته وأصبحت ناجحة
إلى أخوتي وأقول لهم كلما قررت التوقف نظرت إليكم ولم أستطع التوقف
الى زميلتي ورفيقتي واختي مريم التي كانت معي طول مشواري
جامعي ادام الله محبتنا واخوتنا ان شاء الله
إلى صديقات مشواري الدراسي خاصة إخلاص وخولة وإلى زملاء و زميلات الدفعة
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وآخر قولي
اللهم إنه ليس بجهدي و لا إجتهادي وإنما توفيقك وكرمك ودعوات
جعلتنا اليوم اساتذة المستقبل
" فالحمد لله دائما و أبدا "

إهداء

بسم الله الرحمن الرحيم "وَمَآ أُوتِيتُم مِّنَ ٱلْعِلْمِ إِلَّا قَلِيلًا "
و الصلاة و السلام على سيدي و حبيبي
محمد عليه أفضل الصلاة و السلام
لذا أكتب إليكم جميعاً :
إلى نفسي أوَّلاً: السلامُ لقلبك يا أنا وأعلم أنّك لا تطلبين
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لله درّ أمي التي علّمتني أن لا أبرح حتى أبلغ ، رضي الله عنك أمي و جزاكِ الله الجنة و نعيمها .
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Abstract

The study delves into the resolution of weighted fractional operators of variable order in specific spaces. It investigates a boundary value problem involving weighted fractional derivatives with respect to functions of variable order using Darbou fixed point, emphasizing the connection between symmetry of transformations for differential equations and local solvability, which correlates with the existence of solutions. The work also highlights the necessity of existence requirements for weighted fractional derivatives with respect to functions of constant order Furthermore, the work discusses the existence of solutions to a boundary value problem of differential equations of variable order, with results based on the Schauder fixed point theorem.

Contents

Introduction

Fractional calculus has been applied to a wide range of real-world problems, including the dynamics of predator-prey relationships, the behavior of complex media like porous materials, and the study of dynamical systems where traditional methods are ineffective. It has been found that many models based on ordinary differentiation and integration do not capture the complexity of these phenomena, and fractional calculus provides a more accurate and comprehensive description.

In recent years, fractional calculus has become increasingly important in various scientific and engineering fields. The Riemann-Liouville and Caputo fractional derivatives are commonly used operators, but there are also other types of fractional operators that have been developed to help researchers better understand complex phenomena. Additionally, there has been a growing interest in fractional integration and differentiation of variable orders.

The solvability of differential equations is a key issue in the study of differential equations, and various techniques, such as Lie group symmetry, have been used to analyze their existence. In this paper, integral equivalence is used to confirm the existence of solutions for boundary value problems with variable orders. Many researchers have also investigated and solved boundary value problems for different types of fractional differential equations.
Abel, 1923;Oldham and Spanier, 1974 ; Spanier, 1974;Ross, 1977; Samko, Kilbas and Marichev ,1993;Samko and Ross, 1993;
Samko, 1995; Podlubny,1999; Hilfer, 2000; Klimek, 2001; Coimbra, 2003; Kilbas, Srivastava and Trujillo,2006; Sun, Chen and Chen, 2009;
Mainardi, 2010; Atanackovic and Pilipovic, 2011; Sheng et al., 2011;
Ramirez and Coimbra, 2011; Chen and Ye, 2011;
Malinowska and Torres, 2011, 2012; Herrmann, 2013;
Odzijewicz, Malinowska and Torres, 2013; Almeida and Torres, 2013;
Odzijewicz, Malinowska and Torres, 2013a; Chen and Yang, 2013;
Oldham and Oliveira and Machado, 2014; Pinto and Carvalho, 2014;
Sierociuk et al., 2015; Li and Liu, 2016; Kumar, Pandey and Sharma, 2017.

So we discuss the existence of solutions to a boundary value problem of differential equations of variable order,While many other research works on the existence of solutions to fractional constant order problems have been carried, the existence of solutions to variable-order problems is infrequently mentioned in the literature, and there have been only a few research papers on the stability of solutions. As a result of investigating this intriguing special research topic, our findings are novel and notable.

in these memory :

chapter 1 : we will present some definitions(Riemann-Liouville,Generalized interval) and theories(Darbou fixed point,Schauder fixed point) that we have used in this research, and we will mention the concepts of some special functions(Beta,Gamma, Mittag-Leffler function).

chapter 2 : we consider the existence of solutions Lyapunov-type inequality to the boundary value problem for differential equation of variable order:

$$\begin{cases} D_{0+}^{p(t)}x(t) + f(t,x) = 0, 0 < t < T, \\ x(0) = 0, x(T) = 0. \end{cases}$$

chapter 3 : we will study the boundary value problem for Weighted Fractional Derivative of a Function With Respect T^0 Another Function with Variable Order :

$$\begin{cases} D_w^{\alpha(t)}h(t) = f\left(t, h(t), I_w^{\alpha(t)}h(t)\right), t \in L, \\ h(0) = h(\epsilon) = 0. \end{cases}$$

Chapter 1

Preliminaries

1.1 Absolutely continuous and continuous functions :

Definition 1.1. *[39]*
Let $\Omega = (a, b)(-\infty \leq a < b \leq \infty)$ *be a finite or infinite interval of* $\mathbb{R}$*, and* $1 \leq p \leq \infty$*:*
1. If $1 \leq p < \infty$ *the space* $L_p(\Omega)$

$$L_p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \text{ measurable and } \int_{\Omega} |f(x)|^p dx < \infty \right\}.$$

2. For $p = \infty$*, the space* $L_\infty(\Omega)$ *is the space of measurable functions,* f *bounded almost everywhere on* Ω*, we notice*

$$\sup_{x\in\Omega} \text{ess}\, |f(x)| = \inf\{C \geq 0; |f(x)| \leq C \ p.p \ on \ \Omega\}.$$

Definition 1.2. *[39]*
Let $[a, b] \quad (-\infty < a < b < \infty)$ *a finite interval. We denote by* $AC[a, b]$ *the space of primitive functions of integrable functions in the sense of Lebesgue*

$$f \in AC[a, b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \quad (\varphi(t) \in L(a, b)),$$

and we call $AC[a, b]$ *the space of absolutely continuous functions on* $[a, b]$.

Definition 1.3. *[39]*
For $n \in N$*, we denote by* $AC^n[a, b]$ *the space of functions* f *having derivatives up to order* $(n - 1)$ *continuous on* $[a, b]$ *such that*
$f^{(n-1)} \in AC[a, b]$

$$AC^n[a, b] = \left\{ f : [a, b] \longrightarrow \mathbb{C} \text{ and } f^{(n-1)} \in AC([a, b]) \right\},$$

In particular $AC^1[a, b] = AC[a, b]$.

Lemma 1. *[39]*
A function $f \in AC^n(\Omega), n \in \mathbb{N}^*$, if and only if it is represented in the form:

$$f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f^{(n)}(\tau)d\tau + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$

1.2 Properties of real analysis :

Definition 1.4. ***[The continuity]:*** *[22]*
Let $f : \mathbb{R} \to \mathbb{R}$ an application. We say that f is continuous if it is continuous at any point of $\mathbb{R}$. That mean :
$\forall a \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}_+^*, \exists \alpha \in \mathbb{R}_+^*, \forall x \in \mathbb{R}, |x-a| < \alpha \Longrightarrow |f(x) - f(a)| < \varepsilon.$

Definition 1.5. ***[Bounded function]:*** *[22]*
A function $f : G \subset \mathbb{R} \to \mathbb{R}$ is bounded if :

$$\exists M > 0, \forall t \in G : |f(t)| \leq M.$$

Definition 1.6. ***[convex function]:****[22]*
The function f is convex if and only if, for all x, y, z in $I \subseteq R$ with $x \leq y \leq z$, for $y = tx + (1+t)z$, we have :

$$f(y) \leq tf(x) + (1+t)f(z).$$

1.3 Some elements of topology:

Definition 1.7. ***[Norm]:****[28]*
Let E be a vector space on $\mathbb{R}$. We call a norm on E any application $\|.\| : E \to \mathbb{R}_+$ checked:
•$\forall x \in E : \|x\| = 0 \Leftrightarrow x = 0.$
•$\forall \curlywedge \in \mathbb{R}, \forall x \in E : \| \curlywedge x\| = | \curlywedge |\|x\|.$
•$\forall x, y \in E : \|x + y\| \leqslant \|x\| + \|y\|$ *"triangular inequality ".*

Definition 1.8. ***[Banach space]:****[28]*
We call Banach space any vector complete normed space on the body $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Theoreme 1.1. ***Fubini****[23]*

Let $f(x, y)$ be a summable function on the product of measurable spaces (X, μ) and (Y, ν).

We have the following assertions:
1) For almost all $x \in X$ with respect to μ, the function $f(x,y)$ is summable on Y and its integral over Y is a summable function on X.
2) For almost all $y \in Y$ with respect to ν, the function $f(x,y)$ is summable on X and its integral over X is a summable function on Y.
3) We have:

$$\begin{aligned}\int_{X\times Y} f(x,y)d(\mu\times v)(x,y) &= \int_X \left(\int_Y f(x,y)dv(y)\right)d\mu(x)\\ &= \int_Y \left(\int_X f(x,y)d\mu(x)\right)dv(y).\end{aligned}$$

1.4 Some Special Functions

1.4.1 Gamma function:

Definition 1.9. *[13]*
the gamma function $\Gamma(z)$is defined by the integral :

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt.$$

which converges in the right half of the complex plans, Re$(z) > 0$ indeed we have:

$$\begin{aligned}\Gamma(x+iy) &= \int_0^{+\infty} e^{-t}t^{x-1+iy}dt,\\ &= \int_0^{+\infty} e^{-t}t^{x-1}e^{iy\ln(t)}dt,\\ &= \int_0^{+\infty} e^{-t}t^{x-1}[\cos(y\ln(t)) + i\sin(y\ln(t))]dt.\end{aligned}$$

Proposition 1.1. *[13]*

1. *Satisfies the following functional equation : $\Gamma(z+1) = z\Gamma(z)$ which can be easily proved by integration by parts:*

$$\begin{aligned}\Gamma(z+1) &= \int_0^{\infty} e^{-t}t^{z}dt,\\ &= [-e^{-t}t^z]_0^{\infty} + \int_0^{\infty} e^{-t}t^{z-1}dt,\\ &= z\Gamma(z).\end{aligned}$$

2. $\Gamma(1) = 1$ *and* $\Gamma(-m) = \pm\infty$ *we have:*

$$\Gamma(1) = \int_0^\infty e^{-t}dt = [-e^{-t}]_0^\infty = 1,$$

and :

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},$$

which name that :$\Gamma(0) = +\infty$.

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
which the changing the variable, $s = \sqrt{t}$ *we will have :*

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}}dt \\ &= 2\int_0^{+\infty} e^{-s^2}ds \\ &= 2\left(\frac{\sqrt{\pi}}{2}\right)(gauss - integral) \\ &= \sqrt{\pi}. \end{aligned}$$

1.4.2 Beta function:

Definition 1.10. *[13]*
the beta function is usually defined by :

$$\beta(z.w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, (Re(z) > 0, Re(w) > 0),$$

for example:

$$\begin{aligned} \beta(2.3) &= \int_0^1 t(1-t)^2dt \\ &= \int_0^1 (t - 2t^2 + t^3)dt \\ &= \frac{1}{12}. \end{aligned}$$

Proposition 1.2. *[1]*
the relationship between the gamma function and the beta function is :

$$\beta(z.w) = \frac{\Gamma(z)\cdot\Gamma(w)}{\Gamma(z+w)}.$$

Evedens 1.1. *[1]*

$$\Gamma(x)\Gamma(y) = \int_0^{+\infty}\int_0^{+\infty} t_1^{x-1}t_2^{y-1}e^{-t_1}e^{-t_2}dt_1dt_2$$

$$= \int_0^{+\infty} t_1^{x-1}\left(\int_0^{+\infty} t_2^{y-1}e^{-(t_1+t_2)}dt_2\right)dt_1?$$

by changing the variable :

$$Z_2 = t_1 + t_2.$$

we find :

$$\Gamma(x)\Gamma(y) = \int_0^{+\infty} t_1^{x-1}dt_1\int_0^{+\infty}(Z_2 - t_1)^{y-1}e^{-Z_2}dZ_2$$

$$= \int_0^{+\infty} e^{-Z_2}dZ_2\int_0^{t_1}(Z_2 - t_1)^{y-1}t_1^{x-1}dt_1?$$

if we pose $Z_1 = \dfrac{t_1}{t_2}$ *, we arrive at:*

$$= \int_0^{+\infty} e^{-Z_2}dZ_2\left(\int_0^1 (Z - 1Z_2)^{x-1}\right.$$

$$\left.(Z_2 - Z_1Z_2)^{y-1}Z_2dZ_1\right)$$

$$= \int_0^{+\infty} e^{-Z_2}dZ_2((Z_2)^{x+y-1}\beta(x,y))$$

$$= \int_0^{+\infty} e^{-Z_2}(Z_2)^{x+y-1}dZ_2\beta(x,y)$$

$$= \mathit{\Gamma}(x+y)\beta(x,y).$$

which produces the intended outcome,

Corollary 1.1. *[1]the beta function is symmetrical:*

$$\beta(z,w) = \beta(w,z),$$

we have:

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)} = \beta(w,z).$$

1.4.3 Mittag-Leffler function:

Definition 1.11. *[13]*
the Mittag-Leffler function $E_{\alpha,\beta}(z)$ *is defined by :*

$$E_{\alpha,1}(t) = \sum_{n=0}^{+\infty} \frac{t^n}{\Gamma(n\alpha+1)}, (t \in I\!R, \alpha > 0);$$

and the generalized Mittag-Leffler function $E_{\alpha,\beta}(t)$ *is definer as follows:*

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{+\infty} \frac{t^n}{\Gamma(n\alpha+\beta)}, (\alpha, \beta > 0).$$

1.5 Riemann-Liouville integral with constant order :

Definition 1.12. *[22]*
The Riemann-Liouville fractional integrator $I^a_{a+}f$ *and* $I^a_{b-}f$ *of order* $\alpha \in \mathbb{C}(\mathrm{Re}(\alpha) > 0)$, *are defined by:*

$$\left(I^\alpha_{a+}f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}; x > a,$$

And

$$\left(I^\alpha_{b-}f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}; x < b,$$

respectively. Here $\Gamma(\alpha)$ *is the Gamma function. These integrals are called the left and right fractional integrals.*

Proposition 1.3. *[22]*
Let α *and* β *be two complex numbers and* $f \in C^0([a,b))$*:*
i) $I^\alpha_a \left(I^\beta_a f\right) = I^{\alpha+\beta}_a f, \quad (\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0),$
ii) $\frac{d}{dt}\left(I^\alpha_a f\right)(t) = \left(I^{\alpha-1}_a f\right)(t), \quad \mathrm{Re}(\alpha) > 1,$
iii) $\lim_{\alpha \longrightarrow 0^+} \left(I^\alpha_a f\right)(t) = f(t), \quad \mathrm{Re}(\alpha) >0.$

Proof 1.1. *[22]*
ı) For the demonstration we use the Euler Beta function.
Indeed:

$$\left[I^\alpha_a \left(I^\beta_a f\right)\right](t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1} f(\tau) dt d\tau,$$

using Fubini's theorem, we can permute the order of integration and we obtain:

$$\left[I^\alpha_a \left(I^\beta_a\right)\right](t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(\tau) \left[\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1} dt\right] d\tau.$$

The change of variables $s = \tau + (t-\tau)\mu$, gives us:

$$\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1}ds = (t-\tau)^{\alpha+\beta-1}\int_0^1 (1-\mu)^{\alpha-1}\mu^{\beta-1}d\mu,$$
$$= (t-\tau)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

where:

$$\left[I_a^\alpha\left(I_a^\beta f\right)\right](t) = \frac{1}{\Gamma(\alpha+\beta)}\int_a^t f(\tau)(t-\tau)^{\alpha+\beta-1}d\tau = \left(I_a^{\alpha+\beta}f\right)(t).$$

ii) To justify the second identity we use the classic theorems of derivation of an integral depending on a parameter and the fundamental relation of the Euler Gamma function : $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.
iii) For the last identity, we consider the function $f \in C^0([a,b))$, we have:

$$\left(I_a^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau.$$

From example:

$$\left(I_a^\alpha 1\right)(t) = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \longrightarrow 1 \text{ when } \alpha \to 0^+.$$

SO:

$$\begin{aligned}\left|\left(I_a^\alpha f\right)(t) - \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}f(t)\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau \right.\\ &\quad \left. - \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(t)d\tau\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}|f(\tau)-f(t)|d\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_a^{t-\delta} (t-\tau)^{\alpha-1}|f(\tau)-f(t)|d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t-\delta}^t (t-\tau)^{\alpha-1}|f(\tau)-f(t)|d\tau.\end{aligned}$$

On the one hand, we have f is continuous on $[a,b)$ which allows us to write:

$$\forall t, \tau \in [a,b), \forall \varepsilon > 0, \exists \delta > 0 : |\tau - t| < \delta \Rightarrow |f(\tau) - f(t)| < \varepsilon,$$

which leads to:

$$\int_{t-\delta}^t (t-\tau)^{\alpha-1}|f(\tau)-f(t)|d\tau \le \varepsilon \int_{t-\delta}^t (t-\tau)^{\alpha-1}d\tau = \frac{\varepsilon\delta^\alpha}{\alpha}$$

On the other hand,

$$\frac{1}{\Gamma(\alpha)}\int_a^{t-\delta}(t-\tau)^{\alpha-1}|f(\tau)-f(t)|d\tau \leq \frac{1}{\Gamma(\alpha)}\int_a^{t-\delta}(t-\tau)^{\alpha-1}(|f(\tau)|+|f(t)|)d\tau$$
$$\leq 2\sup_{\xi\in[a,t]}|f(\xi)|\int_a^{t-\delta}(t-\tau)^{\alpha-1}d\tau,$$
$$= 2M\left(\frac{(t-a)^\alpha}{\alpha}-\frac{\delta^\alpha}{\alpha}\right), \forall t\in[a,b)$$

where :$(M=\sup_{\xi\in[a,t]}|f(\xi)|)$.

A combination of above we will have:

$$\left|\left(I_a^\alpha f\right)(t)-\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}f(t)\right| \leq \frac{1}{\alpha\Gamma(\alpha)}\left[\varepsilon\delta^\alpha+2M\left((t-a)^\alpha-\delta^\alpha\right)\right]$$
$$=\frac{1}{\Gamma(\alpha+1)}\left[\varepsilon\delta^\alpha+2M\left((t-a)^\alpha-\delta^\alpha\right)\right],$$

by making α *tend towards* 0^+, *we obtain:*

$$\lim_{\alpha\to0^+}\left|\left(I_a^\alpha f\right)(t)-\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}f(t)\right|\leq\varepsilon$$

in other words:

$$\left|\lim_{\alpha\to0+}\left(I_a^\alpha f\right)(t)-f(t)\right|\leq\varepsilon,\quad \forall\varepsilon>0,$$

which means :

$$\lim_{a\to0+}\left(I_a^\alpha f\right)(t)=f(t)$$

1.6 Derivative of Riemann-Liouville with constant order :

Definition 1.13. *[22]*
Let $\alpha\in[m-1,m[$ *with* $m\in\mathbb{N}^*$. *We call derivative of order* α *in the sense of Riemann-Liouville the function defined by*

$${}_a^R D_t^\alpha f(t)=\left(\frac{d}{dt}\right)^m\left[\left(I_a^{m-\alpha}f\right)(t)\right]$$
$$=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t(t-\tau)^{m-\alpha-1}f(\tau)d\tau.$$

Proposition 1.4. *[22]*
i) If the fractional derivative ${}_{n}^{R}D_{t}^{m}f(t), (m-1 \leq p < m)$, *of a function* $f(t)$ *is integrable, then:*

$$ {}_{a}^{R}D_{t}^{-p}\left({}_{a}^{R}D_{t}^{\mathrm{p}}f(t)\right) = f(t) - \sum_{i=1}^{\mathrm{m}}\left[{}_{a}^{R}D_{t}^{\mathrm{p}-1}f(t)\right]_{t-a}\frac{(t-a)^{\mathrm{p}-t}}{\Gamma(p-i+1)} $$

ii) If $0 \leq m-1 \leq q < m$, *we have*

$$ {}_{a}^{R}D_{t}^{-P}\left({}_{a}^{R}D_{t}^{q}f(t)\right) = {}_{a}^{R}D_{t}^{q-P}f(t) - \sum_{i=1}^{m}\left[{}_{a}^{R}D_{t}^{q-1}f(t)\right]_{t=a}\frac{(t-a)^{\mathrm{P}-t}}{\Gamma(1+p-i)}. $$

Proof 1.2. *[22]*
i) we have :

$$ \begin{aligned} {}_{a}^{R}D_{t}^{-p}\left({}_{a}^{R}D_{t}^{\mathrm{p}}f(t)\right) &= \frac{1}{\Gamma(p)}\int_{a}^{t}(t-\tau)^{p-1}{}_{a}^{R}D_{\tau}^{\mathrm{p}}f(\tau)d\tau \\ &= \frac{d}{dt}\left[\frac{1}{\Gamma(p+1)}\int_{a}^{t}(t-\tau)^{p}{}_{a}^{R}D_{\tau}^{p}f(\tau)d\tau\right]. \end{aligned} $$

the other hand, by performing integrations by repeated parts and exploiting , we obtain:

$$ \begin{aligned} &\frac{1}{\Gamma(p+1)}\int_{a}^{t}(t-\tau)^{p}{}_{a}^{R}D_{\tau}^{p}f(\tau)d\tau = \frac{1}{\Gamma(p+1)}\int_{a}^{t}(t-\tau)^{p}\frac{d^{m}}{d\tau^{m}}\left\{{}_{a}^{R}D_{\tau}^{-(m-p)}f(\tau)\right\}d\tau \\ &= \frac{1}{\Gamma(p-m+1)}\int_{a}^{t}(t-\tau)^{p-m}\left\{\int_{a}^{R}D_{\tau}^{-(m-p)}f(\tau)\right\}d\tau \\ &- \sum_{i=1}^{m}\left[\frac{d^{m-1}}{dt^{m-1}}\left({}_{a}^{R}D_{t}^{-(m-p)}f(t)\right)\right]_{t=a}\frac{(t-a)^{p-t+1}}{\Gamma(1+p-i)} \\ &= \frac{1}{\Gamma(p-m+1)}\int_{a}^{t}(t-\tau)^{p-m}\left\{{}_{a}^{R}D_{\tau}^{-(m-p)}f(\tau)\right\}d\tau \\ &- \sum_{i=1}^{m}\left[{}_{a}^{R}D_{t}^{p-t}f(t)\right]_{t=a}\frac{(t-a)^{p-t+1}}{\Gamma(1+p-i)} \\ &= {}_{a}^{n}D_{t}^{-(p-m-1)}\left({}_{a}^{R}D_{t}^{-(m-p)}f(t)\right) \\ &- \sum_{t=1}^{m}\left[{}_{a}^{R}D_{t}^{p-t}f(t)\right]_{t=a}\frac{(t-a)^{p-t+1}}{\Gamma(1+p-i)} \\ &= {}_{a}^{R}D_{t}^{1}f(t) - \sum_{i=1}^{m}\left[{}_{a}^{R}D_{t}^{p-t}f(t)\right]_{t=a}\frac{(t-a)^{p-t}}{\Gamma(1+p-i)}. \end{aligned} $$

ii) By establishing a relationship from previous results we find:

$$
\begin{aligned}
{}_a^R D_t^{-p}\left({}_a^R D_t^q f(t)\right) &= {}_a^R D_t^{q-p}\left\{{}_a^R D_t^{-q}\left({}_a^R D_t^q f(t)\right)\right\} \\
&= {}_a^R D_t^{q-p}\left\{f(t) - \sum_{i=1}^{m}\left[{}_a^R D_t^{q-i} f(t)\right]_{t-a}\frac{(t-a)^{q-1}}{\Gamma(q-i+1)}\right\} \\
&= {}_a^R D_t^{q-p} f(t) - \sum_{i=1}^{m}\left[{}_a^R D_t^{q-i} f(t)\right]_{t-a}^{R} D_t^{q-p}\left\{\frac{(t-a)^{q-i}}{\Gamma(q-i+1)}\right\} \\
&= {}_a^R D_t^{q-p} f(t) - \sum_{i=1}^{m}\left[{}_a^R D_t^{q-i} f(t)\right]_{t=a}\frac{(t-a)^{p-t}}{\Gamma(1+p-i)}.
\end{aligned}
$$

A like integer differentiation and integration, fractional differentiation and integration do not commentate in general.

Theoreme 1.2. *[22] Let $1 < \alpha \leq 2, f(t) \in L(0,b), D_{0+}^{\alpha} f \in L(0,b)$. Then the following equality holds:*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, c_1, c_2 \in R.$$

The order $\alpha(t)$ would fall into a more complex category, involving derivatives and integrals whose orders are functions of certain variables. There are several definitions of variable order fractional integrals and derivatives. The following are several definitions of variable order fractional integrals and derivatives: Let $-\infty < a < b < +\infty$.

1.7 Riemann-Liouville fractional integral of variable order :

Definition 1.14. *[35, 27]*
Let $p : [a, b] \to (0, +\infty)$, the left Riemann-Liouville fractional integral of order $\alpha(t)$ for function $x(t)$ are defined as the following two types :

$$I_{a+}^{\alpha(t)} x(t) = \int_a^t \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))} x(s) ds, t > a,$$

$$I_{a+}^{\alpha(t)} x(t) = \frac{1}{\Gamma(\alpha(t))} \int_a^t (t-s)^{\alpha(t)-1} x(s) ds, t > a.$$

1.8 Riemann-Liouville fractional derivative of variable order :

Definition 1.15. *[35, 27]*
Let $\alpha : [a, b] \to (n-1, n]$ (n is a natural number), the left Riemann-Liouville fractional derivative of order $\alpha(t)$ for function $x(t)$ are defined as the following two types:

$$D_{a+}^{\alpha(t)} x(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha(t)-1}}{\Gamma(n-\alpha(t))} x(s) ds, t > a.$$

$$D_{a+}^{\alpha(t)} x(t) = \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha(t))} \int_a^t (t-s)^{n-\alpha(t)-1} x(s) ds, t > a.$$

Remark 1.1. *:*
Note that the semigroup property is satisfied for a standard Hadamard integral with constant orders but it is not fulfilled for the general case of variable orders $u(t), v(t)$, i.e., ${}^H I_{1^+}^{u(t)} \left({}^H I_{1^+}^{v(t)}\right) x(t) \neq {}^H I_{1^+}^{u(t)+v(t)} x(t)$.

1.9 Generalized intervals

Definition 1.16. *[7, 37, 38]*
A generalized interval is a subset I of $\mathbb{R}$ which is either an interval (i.e., a set of the form $[a, b], (a, b), [a, b)$ or $(a, b]$); a point $\{a\}$; or the empty set $\emptyset$.

Definition 1.17. *[7, 37, 38]*
If I is a generalized interval. A partition of I is a finite set P of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in P.

Example 1.1. *The set* $P = \{\{1\}, (1,6), [6,7), \{7\}, (7,8] \mid$ *of generalized intervals is a partition of* $[1,8]$.

Definition 1.18. *[7, 37, 38]*
Let I be a generalized interval, let $f : I \to R$ be a function, and let P a partition of I. f is said to be piecewise constant with respect to P if for every $J \in P, f$ is constant on J.

Example 1.2. *The function* $f : [1,6] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 3, & 1 \leq x < 3, \\ 4, & x = 3, \\ 5, & 3 < x < 6, \\ 2, & x = 6, \end{cases}$$

is piecewise constant with respect to the partition $\{[1,3], \{3\}, (3,6), \{6\}\}$ *of* $[1,6]$. *The following example illustrates that the semigroup property of the variable order fractional integral doesn't holds for the piecewise constant functions $p(t)$ and $q(t)$ defined in the same partition of finite interval $[a,b]$.*

Example 1.3. *:*
Let $J = [1,2]$ and the function $x(t) \equiv 1$ for $t \in J$. Consider the following functions as orders of Hadamard fractional integral: $v(t) \equiv 2$ and $u(t) = t$ for $t \in J$. Then for any $t \in J$ we obtain for the Hadamard fractional integral defined by (2)

$${}^{H}I_{1+}^{v(t)} h(t) = \frac{1}{\Gamma(1)} \int_1^t \left(\ln \frac{t}{s}\right)^{1-1} \frac{1}{s} ds = \ln t,$$

$${}^{H}I_{1+}^{u(t)} \left({}^{H}I_{1+}^{v(t)} x(t)\right) = \frac{1}{\Gamma(t)} \int_1^t \left(\ln \frac{t}{s}\right)^{t-1} \frac{\ln s}{s} ds,$$

and

$${}^{H}I_{1+}^{u(t)+v(t)} x(t) = \frac{1}{\Gamma(t+1)} \int_1^t \left(\ln \frac{t}{\mathrm{s}}\right)^{t} \frac{1}{\mathrm{s}} ds.$$

For $t = 1.5$ we obtain

$${}^{H}I_{1^+}^{u(t)} \left({}^{H}I_{1^+}^{v(t)} x(t)\right)\Big|_{t=1.5} \simeq 0.027916$$

and

$${}^{H}I_{1+}^{u(t)+v(t)} x(t)\Big|_{t=1.5} \simeq 0.0418739$$

1.10 Some fixed points theorems:

Definition 1.19. *Let T be an application of a set S in it self. We call fixed point of T any point $s \in S$ that $T(s) = S$*

Theoreme 1.3. ***[Schauder fixed point]:****[24]*
Let X be a subset of E, and $f : X \to E$ a continuous function, with X compact, convex, and E a Banach space. Then , f has a fixed point in X.

Theoreme 1.4. ***[Darbou fixed point theorem]****[9]*
If F is nonempty, bounded, convex and closed subset of a Banach space X, and $\Omega : F \longrightarrow F$ is a continuous operator satisfying

$$\psi(\Omega(\Lambda)) \leq k(\psi\Lambda), \forall\Lambda \neq \emptyset \subset F, \ k \in [0, 1),$$

i.e., Ω is k-set contractions.
Then, Ω has at least one fixed point in F.

In addition, we will provide some necessary background information about the Kuratowski measure of non compactness:

Definition 1.20. *([9])*

Let $\mathcal{M}_X$ the bounded subsets of a Banach space X . The Kuratowski measure of non compactness ψ is a mapping $\psi : \mathcal{M}_X \to [0, \infty]$ initially derived from a construction as laid out in the following format

$$\psi(D) = \inf\{\varepsilon > 0 : D(\in \mathcal{M}_X) \subseteq \cup_{\iota=1}^{n} D_\iota,\ diam(D_\iota) \leq \varepsilon\},$$

where

$$diam(D_\iota) = sup\{||x - y|| :\ x,\ y \in D_\iota\}.$$

Proposition 1.5. *[4] Let E be a Banach space and $\Omega, \Omega_1, \Omega_2$ be finite sets. The Kuratowski or Hausdorff MNC is noted. Then ψ checks the following properties:*
1) Regularity: $\psi(\Omega) = 0$ if and only if Ω is relatively compact.
2) Non-singularity: $\psi(\Omega) = 0$ if Ω is a singleton.
3) Monotony: if $\Omega_1 \subset \Omega_2$ then $\psi(\Omega_1) \leq \psi(\Omega_2)$.
4) Semi-additivity: $\psi\{\Omega_1 \cup \Omega_2\} = \max\{\psi(\Omega_1), \psi(\Omega_2)\}$.
5) Semi-homogeneity: $\psi(t\Omega) = |t|\psi(\Omega)$ for any real t.
6) Semi-additivity algebraic : $\psi(\Omega_1 + \Omega_2) \leq \psi(\Omega_1) + \psi(\Omega_2)$.
7) Translation invariance: $\psi(\Omega + x_0) = \psi(\Omega)$, for all $x_0 \in E$.

Proof 1.3. *This proposition is only demonstrated for the MNC α. The proof is similar for χ:*
1) . Suppose that Ω is relatively compact, then, according to theorem (let E be complete metric space. A subset $A \in E$ is relatively compact if and only if it is precompact.).. $\forall\varepsilon > 0$, there exists a finite family $\{x_1, \cdots, x_{N_\varepsilon}\}$ of elements of E such that $\Omega \subset \bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon)$. Then $\alpha(\Omega) = 0$. Conversely, if: $\alpha(\Omega) = 0$, according to the same theorem, we can conclude that Ω is relatively compact.

2. Every singleton set is relatively compact. .
3) . Let $\{\Omega_1^2, \cdots, \Omega_n^2\}$ recovery of Ω_2 such us $\operatorname{diam}(\Omega_i^2) \leq d$, $i = 1, \cdots, n$. then it is clear that it is a recovery of $\Omega_1$and consequently $\alpha(\Omega_1) \leq \alpha(\Omega_2)$.
4) . Let $\Omega = \Omega_1 \cup \Omega_2$ and $a = \max\{\alpha(\Omega_1), \alpha(\Omega_2)\}$ .such as $\Omega_i \subset \Omega$, $i = 1, 2$. from the monotony of α we will have $\alpha(\Omega_i) \leq \alpha(\Omega)$, therefore $a \leq \alpha(\Omega)$.
Conversely, let us show that $\alpha(\Omega) \leq a$. For any $\varepsilon > 0$, and for all Ω_1, Ω_2 there exists a covering $\{\Omega_1^i, \cdots, \Omega_{n_i}^i\}$ of Ω_i such that

$\operatorname{diam}\left(\Omega_j^i\right) \leq \alpha\left(\Omega_i\right)+\varepsilon \leq a+\varepsilon$, *for* $i=1,2$, *and* $j=1, \cdots, n_i$. *Note that these sets* Ω_j^i*form a overlay* Ω, *then* $\alpha(\Omega) \leq a+\varepsilon$, *and consequently* $\alpha(\Omega) \leq a$ *since* ε *is arbitrary.*
5) . is trivial for $t=0$. *If* $t \neq 0$, *then:*

$$\begin{aligned}
\alpha(t \Omega) & =\inf \left\{d>0: t \Omega \subset \bigcup_{i=1}^m \Omega_i, \operatorname{diam}\left(\Omega_i\right) \leq d\right\} \\
& =\inf \left\{d>0: \Omega \subset \bigcup_{i=1}^m \frac{1}{t} \Omega_i,|t| \operatorname{diam}\left(\frac{1}{t} \Omega_i\right) \leq d\right\} \\
& =\inf \left\{d>0: \Omega \subset \bigcup_{i=1}^m \frac{1}{t} \Omega_i, \operatorname{diam}\left(\frac{1}{t} \Omega_i\right) \leq \frac{1}{|t|} d\right\} \\
& =\inf \left\{|t| d^{\prime}>0: \Omega \subset \bigcup_{i=1}^m \frac{1}{t} \Omega_i, \operatorname{diam}\left(\frac{1}{t} \Omega_i\right) \leq d^{\prime}\right\}, d^{\prime}=\frac{1}{|t|} d \\
& =|t| \alpha(\Omega) .
\end{aligned}$$

6) . Let $\left\{\Omega_1^1, \cdots, \Omega_m^1\right\}$ *recovery of* Ω_1 *and* $\left\{\Omega_1^2, \cdots, \Omega_n^2\right\}$ *recovery of* Ω_2. *So the sets* $\Omega_i^1+\Omega_j^2$ *form a overlay of* $\Omega_1+\Omega_2$, *moreover* $\operatorname{diam}\left(\Omega_1+\Omega_2\right) \leq \operatorname{diam}\left(\Omega_1\right)+\operatorname{diam}\left(\Omega_2\right)$ *and consequently* $\alpha\left(\Omega_1+\Omega_2\right) \leq \alpha\left(\Omega_1\right)+\alpha\left(\Omega_2\right)$.
7) . The property is deduced from the fact that diam $\left(\Omega+x_0\right)=\operatorname{diam}(\Omega)$.

Theoreme 1.5. *[26]*
Hausdorff's MNC is invariant through closure and convex coverage, i.e.

$$\chi(\bar{\Omega})=\chi[co(\Omega)]=\chi(\Omega).$$

Proof 1.4. *If* S *is a finite E-network of* Ω *then* S *is also a finite E-network of* $\bar{\Omega}$, *hence the invariance of* χ *per pass to closure . On the other hand, coS is a compact* ε*-con network. Indeed, let* $\varepsilon>0$ *such that the set* $S=\left\{x_1, x_2, \cdots, x_n\right\}$ *constitutes a finite E-network of* Ω. *If* $y \in co(\Omega)$, *then*

$$y=\sum_{i=1}^n \lambda_i y_i,\left(\lambda_i \in[0,1], \sum_{i=1}^n \lambda_i=1, y_i \in \Omega\right).$$

Like $y_i \in \Omega$, *there exists* $x_i \in S$ *such that* $\left\|y_i-x_i\right\| \leq \varepsilon$. *Let* $x=\sum_{i=1}^n \lambda_i x_i$, *where the coefficients* $\lambda_i,(i=1,2, \cdots, n)$ *are the same defined in (1.1). Then* $x \in coS$ *and we have.*

$$\|y-x\| \leq \sum_{i=1}^{\mathrm{n}} \lambda_i\left\|x_i-y_i\right\| \leq \varepsilon.$$

We deduce that the compact set coS is a ε*-resent of co*Ω.

Lemma 2. *([14]) Let X be a Banach space. If U is an bounded and equicontinuous subset of the the space $C(L, X)$ of continuous function , then*

($\mathcal{I}_1$) $\psi(U(.)) \in C(L, \mathbb{R}_+)$, *mean that the function* $\psi(U(t))$ *is an continuous function for* $t \in L$ *, and*

$$\widehat{\psi}(U) = \sup_{t \in L} \psi(U(t)),$$

Where $\widehat{\psi}(U)$ *is the Kuratowski measure of non compactness on the space* $C(L, X)$.

($\mathcal{I}_2$) $\psi\left(\int_0^{\varepsilon} x(\theta)d\theta : x \in U\right) \leq \int_0^{\varepsilon} \psi(U(\theta))d\theta,$

where

$$U(t) = \{x(t) : \; x \in U\}, \; t \in L.$$

Chapter 2

The existence of solutions Lyapunov-type inequalities to boundary value problems of variable order

In this work, we will observe and study the unique existence of approximate solution to the Lyapunov-type inequality to the following boundary value problem for differential equation of variable order by following :

$$\begin{cases} D_{0+}^{q(t)}x(t) + f(t,x) = 0, 0 < t < T, \\ x(0) = 0, x(T) = 0, \end{cases} \tag{2.1}$$

where $0 < T < +\infty$, $D_{0+}^{q(t)}$ denotes derivative of variable order defined by:

$$D_{0+}^{q(t)}x(t) = \frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q(s)}}{\Gamma(2-q(s))}x(s)ds, \quad t > 0,$$

and,

$$I_{0+}^{2-q(t)}x(t) = \int_0^t \frac{(t-s)^{1-q(s)}}{\Gamma(2-q(s))}x(s)ds, \quad t > 0,$$

denotes integral of variable order $2-q(t), 1 < q(t) \leq 2, 0 \leq t \leq T$.
$f : (0,T] \times \mathbb{R} \to \mathbb{R}$ is given continuous function satisfying some assumption conditions.

Lyapunov's inequality :[38]
is an outstanding result in mathematics with many different applications. The result, as proved by Lyapunov in 1907, asserts that if $h : [a,b] \to R$ is a continuous function, then a necessary condition for the boundary value problem

$$\begin{cases} y''(t) + h(t)y(t) = 0, a < t < b, \\ y(a) = y(b) = 0, \end{cases}$$

to have a nontrivial solution is given by

$$\int_a^b |h(s)|ds > \frac{4}{b-a},$$

where $-\infty < a < b < +\infty$.

Remark 1. *When examining the existence of differential equation solutions for the Riemann-Liouville fractional derivative, the theorem (1.2) is crucial. The semigroup property, however, does not hold for generic functions $h(t), g(t)$; that is, $I_{a+}^{h(t)} I_{a+}^{g^{(t)}} \neq I_{a+}^{h(t)+g(t)}$. As a result, it causes us great difficulty because we are unable to obtain these features for the variable order fractional operators (integral and derivative) as stated in Propositions 1.3. We can scarcely consider the presence of solutions of differential equations for variable order derivative by means of nonlinear functional analysis (e.g., some fixed point theorems) without these qualities for variable order fractional derivative and integral.*

2.1 Some approximation of solution

Hypotheses 2.1. (H_1)
Let $n^* \in N$ be an integer, $P = \{[0, T_1], (T_1, T_2], (T_2, T_3], \cdots, (T_{n^*-1}, T]\}$ be a partition of the interval $[0, T]$, and let $q(t) : [0, T] \to (1, 2]$ be a piecewise constant function with respect to P, i.e.:

$$q(t) = \sum_{k=1}^{N} q_k \mathcal{I}_k(t) = \begin{cases} q_1, & 0 \leq t \leq T_1, \\ q_2, & T_1 < t \leq T_2, \\ \cdots, & \cdots, \\ q_{n^*}, & T_{n^*-1} < t \leq T_{n^*} = T, \end{cases} \tag{2.2}$$

where $1 < q_k \leq 2\,(k = 1, 2, \cdots, n^*)$ are constants, and I_k is the indicator of the interval $[T_{k-1}, T_k]$, $k = 1, 2, \cdots, n^*$ (here $T_0 = 0, T_{n^*} = T$), that is, $\mathcal{I}_k(t) = 1$ for $t \in [T_{k-1}, T_k]$ and $\mathcal{I}_k(t) = 0$ for elsewhere.

by (H_1)In order to obtain our main results, we firstly carry on essential analysis to the boundary value problem (2.1). By definition (1.15), the equation of the boundary value problem (2.1) can be written as:

$$\frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-q_1}}{\Gamma(2-q(s))} x(s)ds + f(t, x) = 0, 0 < t < T, \tag{2.3}$$

According to (H_1), Eq(2.3) in the interval $(0, T_1]$ can be written as

$$D_{0+}^{q_1} x(t) + f(t, x) = 0, 0 < t \leq T_1. \tag{2.4}$$

Equation (2.3) in the interval $(T_1, T_2]$ can be written by

$$\frac{d^2}{dt^2} \left(\int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)} x(s)ds + \int_{T_1}^{T_2} \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s)ds \right) + f(t, x) = 0, \tag{2.5}$$

and Eq(2.3) in the interval $(T_2, T_3]$ can be written by:

$$\frac{d^2}{dt^2}\left(\int_0^{T_1}\frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x(s)ds+\int_{T_1}^{T_2}\frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds+\int_{T_2}^{T_3}\frac{(t-s)^{1-q_3}}{\Gamma(2-q_3)}x(s)ds\right)$$
$$+f(t,x)=0. \tag{2.6}$$

In the same way, Eq(2.3) in the interval $(T_{j-1}, T_j], j = 4, 5, \cdots, n^* - 1$ can be written by:

$$\frac{d^2}{dt^2}\left(\int_0^{T_1}\frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x(s)ds+\cdots+\int_{T_{j-1}}^{T_j}\frac{(t-s)^{1-q_j}}{\Gamma(2-q_j)}x(s)ds\right)+f(t,x)=0. \tag{2.7}$$

As for the last interval (T_{n^*-1}, T), similar to above argument, Eq (2.3) can be written by:

$$\frac{d^2}{ds^2}\left(\int_0^{T_1}\frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x(s)ds+\cdots+\int_{T_{n^*-1}}^{T}\frac{(t-s)^{1-q_{n^*}}}{\Gamma(2-q_{n^*})}x(s)ds\right)+f(t,x)=0. \tag{2.8}$$

Hypotheses 2.2. (H_2)
Let $t^r f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function $(0 \le r < 1)$, there exist constants $c_1 > 0, c_2 > 0, 0 < \gamma < 1$ such that:

$$t^r|f(t,x(t))| \le c_1 + c_2|x(t)|^\gamma, 0 \le t \le T, x(t) \in \mathbb{R}.$$

Remark 2.1. *From the arguments above, we find that, according to condition (H_1), in the different interval, the equation of the boundary value problem (2.1) must be represented by different expression. For instance, in the interval $(0, T_1]$, the equation of the boundary value problem (2.1) is represented by: (2.5); in the interval $(T_1, T_2]$, the equation of the boundary value problem (2.1) is represented by (2.6); in the interval $(T_2, T_3]$, the equation of the boundary value problem (2.1) is represented by (2.7), etc. But, as far as we know, in the different intervals, the equation of integer order or constant fractional order problems may be represented by the same expression. Based these facts, different than integer order or constant fractional order problems, in order to consider the existence results of solution to the boundary value problem (2.1), we need consider the relevant problem defined in the different interval, respectively.*

Definition 2.1. *We say the boundary value problem (2.1) has a solution, if there exist functions $x_j(t), j = 1, 2, \cdots, n^*$ such that $x_1 \in C[0, T_1]$ satisfying equation (2.4) and $x_1(0) = 0 = x_1(T_1); x_2 \in C[0, T_2]$ satisfying equation*

(2.5) and $x_2(0) = 0 = x_2(T_2)$; $x_3 \in C[0, T_3]$ *satisfying equation (2.6) and* $x_3(0) = 0 = x_3(T_3)$; $x_j \in C[0, T_j]$ *satisfying equation (2.7) and* $x_t(0) = 0 = x_j(T_j)$ $(j = 4, 5, \cdots, n^* - 1)$; $x_{n^*} \in C[0, T]$ *satisfying equation 2.11 and* $x_{n^*}(0) = x_{n^*}(T) = 0$.

Lemma 3. *Assume that condition* (H_1)*and* (H_2)*hold. for* $t \in [0, T_1]$ *the following two-point boundary value problem:*

$$\begin{cases} D_{0+}^{q_1} x(t) + f(t, x) = 0, \quad 0 < t < T_1, \\ x(0) = 0, x(T_1) = 0. \end{cases} \tag{2.9}$$

is equivalent to the integral equation

$$x(t) = d_1 t^{q_1 - 1} + d_2 t^{q_1 - 2} - \frac{1}{\Gamma(q_1)} \int_0^{T_1} (t - s)^{q_1 - 1} f(s, x(s)) ds, \quad 0 < t \leq T_1.$$

Generally, for $t \in [T_{j-1}, T_j]$ *the boundary value problem:*

$$\begin{cases} D_{T_{j-1}+}^{q_j} x(t) + f(t, x) = 0, T_{j-1} < t < T_j, \\ x(T_{j-1}) = 0, x(T_j) = 0. \end{cases} \tag{2.10}$$

is equivalent to

$$x(t) = (T - T_{j-1})^{1 - q_j} (t - T_{j-1})^{q_j - 1} I_{T_{j-1}+}^{q_j} f(T, x) - I_{T_{j-1}+}^{q_j} f(t, x).$$

Proof 2.1.
In the proof we take account the solution in each integral:
i) According the above analysis, the equation of the boundary value problem (2.1) can be written as Eq(2.3)*.*
Equation (2.3) in the interval $(0, T_1]$ *can be written as:*

$$D_{0+}^{q_1} x(t) + f(t, x) = 0, 0 < t \leq T_1.$$

Now, we consider the following two-point boundary value problem:

$$\begin{cases} D_{0+}^{q_1} x(t) + f(t, x) = 0, \quad 0 < t < T_1, \\ x(0) = 0, x(T_1) = 0. \end{cases} \tag{2.11}$$

Let $x \in C[0, T_1]$ *be solution of the boundary value problem (2.11)*
Now, applying the operator $I_{0+}^{q_1}$ *to both sides of the above equation. By Propositions (ref the theorem), we have:*

$$x(t) = d_1 t^{q_1 - 1} + d_2 t^{q_1 - 2} - \frac{1}{\Gamma(q_1)} \int_0^{T_1} (t - s)^{q_1 - 1} f(s, x(s)) ds, \quad 0 < t \leq T_1.$$

By $x(0)=0$ and the assumption of function f, we could get $d_2=0$. Let $x(t)$ satisfying $x(T_1)=0$, thus we can get $d_1=I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1}$. Then, we have:

$$x(t)=I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1-1}-I_{0+}^{q_1} f(t, x), 0 \leq t \leq T_1 \tag{2.12}$$

ii) we have obtained that Eq(2.3) *in the interval $(T_1, T_2]$ can be written by (2.6). In order to consider the existence result of solution to this formula, we rewrite (2.6) as following:*

$$\frac{d^2}{dt^2} \int_0^{T_1} \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s) ds+\frac{d^2}{dt^2} \int_{T_1}^{T_2} \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s) ds=f(t, x) .(T_1<t \leq T_2)$$

For $0 \leq s \leq T_1$, we take $x(s)=0$, then, by the above equation, we get:

$$D_{T_1}^{q_1} x(t)+f(t, x)=0, T_1<t<T_2.$$

Now, we consider the following boundary value problem:

$$\left\{\begin{array}{l}D_{T_1^{+}}^{q_1} x(t)+f(t, x)=0, T_1<t<T_2, \\ x(T_1)=0, x(T_2)=0,\end{array}\right. \tag{2.13}$$

let $x \in C[T_1 . T_2]$ be solution of boundary value problem (2.13), now operators both sides of equation to boundary value problem (2.13) and by Propositions 1.4, we have:

$$x(t)=d_1(t-T_1)^{q_2-1}+d_2(t-T_1)^{q_2-2}-\frac{1}{\Gamma(q_2)} \int_{T_1}^{T_2}(t-s)^{q_2-1} f(s, x(s)) ds,$$

$$T_1<t \leq T_2.$$

By $x(T_1)=0, x(T_2)=0$, we have $d_2=0$ and $d_1=I_{T_1+}^{q_2} f(T_2, x)(T_2-T_1)^{1-q_2}$. Then, we have:

$$x(t)=I_{0+}^{q_2} f(T_2, x)(T_2-T_1)^{1-q_2}(t-T_1)^{q_2-1}-\frac{1}{\Gamma(q_2)} \int_{T_1}^{T_2}(t-s)^{q_2-1} f(s, x(s)) ds,$$

$$T_1 \leq t \leq T_2.$$

iii) By the similar way, in order to consider the existence of solution to Eq (2.11) defined on $[T_{j-1}, T_j]$ of (2.3), we can investigate the following two-point boundary value problem:

$$\left\{\begin{array}{l}D_{T_{j-1^{+}}}^{q_j} x(t)+f(t, x)=0, T_{j-1}<t<T_j, \\ x(T_{j-1})=0, x(T_j)=0.\end{array}\right. \tag{2.14}$$

By the same arguments previous, we obtain that the Eq(2.11) *defined on* $[T_{j-1}, T_j]$ *of (2.3) has solution:*

$$x_j(t) = \begin{cases} 0, & 0 \leq t \leq T_{j-1}, \\ \tilde{x}_j(t), & T_{j-1} < t \leq T_j, \end{cases} \tag{2.15}$$

where $\tilde{x}_j \in \Omega$ *with* $\tilde{x}_j(T_{j-1}) = 0 = \tilde{x}_j(T_j), j = 4, 5, \cdots, n^* - 1$*. Similar to the above argument, in order to consider the existence result of solution to Eq (2.12), we may consider the following boundary value problem:*

$$\begin{cases} D^{q_{n^*}}_{T_{n^*-1}+} x(t) + f(t, x) = 0, T_{n^*-1} < t < T_{n^*} = T \\ x(T_{n^*-1}) = 0, x(T) = 0. \end{cases} \tag{2.16}$$

So by the same considering, for $T_{n^*-1} \leq t \leq T$ *we get:*

$$x(t) = (T - T_{n^*-1})^{1-q_{n^*}} (t - T_{n^*-1})^{q_{n^*}-1} I^{q_{n^*}}_{T_{n^*-1}+} f(T, x) - I^{q_{n^*}}_{T_{n^*-1}+} f(t, x).$$

2.2 Existence result of solutions:

Theoreme 2.1. *Assume that conditions* (H_1) *and* (H_2) *hold, and by the Definition (2.1) then the boundary value problem (2.1) has one solution.*

Proof 2.2. *:*
First proof :

Conversely, let $x \in C[0, T_1]$ *be solution of integral Eq (2.12), then, by the continuity of function* $t^r f$ *and Proposition (1.3) , we can easily get that* x *is the solution of boundary value problem (2.11). Define operator* $\psi : C[0, T_1] \to C[0, T_1]$ *by:*

$$\psi x(t) = I^{q_1}_{0+} f(T_1, x) T_1^{1-q_1} t^{q_1-1} - I^{q_1}_{0+} f(t, x(t)), \quad 0 \leq t \leq T_1.$$

It follows from the properties of fractional integrals and assumptions on function f *that the operator* $\psi : C[0, T_1] \to C[0, T_1]$ *defined above is well defined. By the standard arguments, we could verify that* $\psi : C[0, T_1] \to C[0, T_1]$ *is a completely continuous operator. In the next analysis, we take:*

$$M(r, q) = \max\left\{ \frac{2}{(1-r)\Gamma(q_1)}, \frac{2}{(1-r)\Gamma(q_2)}, \cdots, \frac{2}{(1-r)\Gamma(q_{n^*})} \right\}.$$

Let $\Omega = \{x \in C[0, T_1] : \|x\| \leq R\}$ *be a bounded closed convex subset of* $C[0, T_1]$*, where:*

$$R = \max\left\{ 2c_1 M(r, q)(\psi + 1)^2, \left(2c_2 M(r, q)(\psi + 1)^2\right)^{\frac{1}{1-\gamma}} \right\}.$$

For $x \in \Omega$ and by (H_2), we have:

$$\begin{aligned}
|\psi x(t)| &\leq \frac{T_1^{1-q_1} t^{q_1-1}}{\Gamma(q_1)} \int_0^{T_1} (T_1 - s)^{T_1-1} |f(s, x(s))| ds \\
&+ \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} |f(s, x(s))| ds \\
&\leq \frac{2}{\Gamma(q_1)} \int_0^{T_1} (T_1 - s)^{q_1-1} |f(s, x(s))| ds \\
&\leq \frac{2}{\Gamma(q_1)} \int_0^{T_1} (T_1 - s)^{q_1-1} s^{-r} (c_1 + c_2 |x(s)|^{\gamma}) ds. \\
&\leq \frac{2 T_1^{q_1-1}}{\Gamma(q_1)} \int_0^{T_1} s^{-r} (c_1 + c_2 R^{\gamma}) ds \\
&\leq \frac{2 T_1^{q_1-1} T_1^{1-r}}{(1-r)\Gamma(q_1)} (c_1 + c_2 R^{\gamma}) \\
&\leq M(r, q) T_1^{q_1-r} (c_1 + c_2 R^{\gamma}) \\
&\leq M(r, q)(\psi + 1)^2 \left(c_1 + c_2 R R^{\gamma-1}\right) \\
&\leq \frac{R}{2} + \frac{R}{2} = R,
\end{aligned}$$

which means that $\psi(\Omega) \subseteq \Omega$. Then the Schauder fixed point theorem assures that the operator ψ has one fixed point $x_1 \in \Omega$, which is a solution of the boundary value problem (2.11).

Second proof:

Conversely, let $x \in C[T_1, T_2]$ be solution of integral equation above, then, by the continuity assumption of function $t^r f$ and Proposition (1.3), we can get that x is solution solution of the boundary value problem (2.13). Define operator $T\psi : C[T_1, T_2] \to C[T_1, T_2]$ by/

$$\psi x(t) = I_{0+}^{q_2} f(T_2, x) (T_2 - T_1)^{1-q_2} (t - T_1)^{q_2-1} - \frac{1}{\Gamma(q_2)} \int_{T_1}^t (t-s)^{q_2-1} f(s, x(s)) ds.$$

It follows from the continuity of function $t^r f$ that operator $\psi : C[T_1, T_2] \to C[T_1, T_2]$ is well defined. By the standard arguments, we know that $T : C[T_1, T_2] \to C[T_1, T_2]$ is a completely continuous operator.

For $x \in \Omega$ and by (H_2), we get:

$$
\begin{aligned}
|\psi x(t)| :\leq & \frac{(T_2-T_1)^{1-q_2}(t-T_1)^{q_2-1}}{\Gamma(q_2)} \int_{T_1}^{T_2} (T_2-s)^{q_2-1}|f(s,x(s))|ds \\
& + \frac{1}{\Gamma(q_2)} \int_{T_1}^{t} (t-s)^{q_2-1}|f(s,x(s))|ds \\
& \leq \frac{2}{\Gamma(q_2)} \int_{T_1}^{T_2} (T_2-s)^{q_2-1}|f(s,x(s))|ds \\
& \leq \frac{2}{\Gamma(q_2)} \int_{T_1}^{T_2} (T_2-s)^{q_2-1}\, s^{-r}\left(c_1+c_2|x(s)|^{\gamma}\right) ds \\
& \leq \frac{2T_2^{q_2-1}}{\Gamma\left(q_2\right)} \int_{T_1}^{T_2} s^{-r}\left(c_1+c_2 R^{\gamma}\right) ds \\
& = \frac{2T_2^{q_2-1}\left(T_2^{1-r}-T_1^{1-r}\right)}{(1-r)\Gamma\left(q_2\right)}\left(c_1+c_2 R^{\gamma}\right) \\
& \leq \frac{2T_2^{q_2-r}}{(1-r)\Gamma\left(q_2\right)}\left(c_1+c_2 R^{\gamma}\right) \\
& \leq M(r,q)(\psi+1)^2\left(c_1+c_2 R R^{\gamma-1}\right) \\
& \leq \frac{R}{2}+\frac{R}{2}=R,
\end{aligned}
$$

which means that $T(\Omega) \subseteq \Omega$. Then the Schauder fixed point theorem assures that operator T has one fixed point $\tilde{x}_2 \in \Omega$, which is one solution of the following integral equation, that is,

$$
\begin{aligned}
\tilde{x}_2(t) = & I_{0+}^{q_2} f\left(T_2,\tilde{x}_2\right)\left(T_2-T_1\right)^{1-q_2}\left(t-T_1\right)^{q_2-1} \\
& - \frac{1}{\Gamma\left(q_2\right)} \int_{T_1}^{t} (t-s)^{q_2-1} f\left(s,\tilde{x}_2(s)\right) ds, \quad T_1 \leq t \leq T_2.
\end{aligned} \tag{2.17}
$$

Applying operator $D_{T_1+}^{q_1}$ on both sides of (2.17), by Proposition 1.2 , we can obtain that:

$$
D_{T_1+}^{q_2}\tilde{x}_2(t)+f\left(t,\tilde{x}_2\right)=0, \quad T_1<t\leq T_2,
$$

that is, $\tilde{x}_2(t)$ satisfies the following equation:

$$
\frac{d^2}{dt^2}\frac{1}{\Gamma\left(2-q_2\right)} \int_{T_1}^{t} (t-s)^{1-q_2}\tilde{x}_2(s)ds+f\left(t,\tilde{x}_2\right)=0, \quad T_1<t\leq T_2. \tag{2.18}
$$

We get:

$$
x_2(t)=\begin{cases} 0, & 0 \leq t \leq T_1, \\ \tilde{x}_2(t), & T_1<t\leq T_2 \end{cases} \tag{2.19}
$$

hence, from (2.18), we know that $x_2 \in C[0,T_2]$ defined by (2.19) satisfies equation

$$\frac{d^2}{dt^2}\left(\int_0^{T_1}\frac{(t-s)^{1-q_1}}{\Gamma(2-q_1)}x_2(s)ds+\int_{T_1}^{T_2}\frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x_2(s)ds\right)+f(t,x_2)=0,$$

which means that $x_2 \in C[0,T_2]$ is one solution of (2.6) with $x_2(0) = 0, x_2(T_2) = \tilde{x}_2(T_2) = 0$.

final proof:

Define operator $\psi : C[T_{n^*-1},T] \to C[T_{n^*-1},T]$ by:

$$\begin{aligned}\psi x(t) &= (T-T_{n^*-1})^{1-q_{n^*}}(t-T_{n^*-1})^{q_{n^*-1}} I^{q_{n^*}}_{T_{n^*-1}+} f(T,x)\\ &- \frac{1}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T}(t-s)^{q_{n^*}-1} f(s,x(s))ds,\end{aligned}$$

$T_{n^*-1} \le t \le T$. It follows from the continuity assumption of function rf that operator $\psi : C[T_{n^*-1},T] \to C[T_{n^*-1},T]$ is well defined. By the standard arguments, we note that $T : C[T_{n^*-1},T] \to C[T_{n^*-1},T]$ is a completely continuous operator.
For $x \in \Omega$ and by (H$_2$)*, we get:*

$$\begin{aligned}|\psi x(t)| &\le \frac{(T-T_{n^*-1})^{1-q_n^*}(t-T_{n^*-1})^{q_{n^*-1}}}{\Gamma(q_{n^-})}\int_{T_{n^*-1}}^{T}(T-s)^{q_{n^*}-1}|f(s,x(s))|ds\\
&+\frac{1}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{t}(t-s)^{q_{n^*}-1}|f(s,x(s))|ds\\
&\le \quad \frac{2}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T}(T-s)^{q^{n^*-1}}|f(s,x(s))|ds\\
&\le \frac{2}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T}(T-s)^{q_{n^*}-1}s^{-r}\left(c_1+c_2|x(s)|^{\gamma}\right)ds\\
&\le \frac{2T^{q_{n^*}-1}}{\Gamma(q_{n^*})}\int_{T_{n^*-1}}^{T}s^{-r}\left(c_1+c_2R^{\gamma}\right)ds\\
&\le \frac{2T^{q_{n^*-1}}\left(T^{1-r}-T^{1-r}_{n^*-1}\right)}{(1-r)\Gamma(q_{n^*})}\left(c_1+c_2R^{\gamma}\right)\\
&\le \frac{2(T+1)^2}{(1-r)\Gamma(q_{n^*})}\left(c_1+c_2R^{\gamma}\right)\\
&\le M(r,q)(\psi+1)^2\left(c_1+c_2RR^{\gamma-1}\right)\\
&\le \frac{R}{2}+\frac{R}{2}=R,\end{aligned}$$

which means that $\psi(\Omega) \subseteq \Omega$. Then the Schauder fixed point theorem assures that operator ψ has one fixed point $\tilde{x}_{n^*} \in \Omega$, which is one solution of the following integral equation, that is,

$$\begin{aligned} \tilde{x}_{n^*}(t) = & \left(T - T_{n^*-1}\right)^{1-q_{n^*}} \left(t - T_{n^*-1}\right)^{q_{n^*}-1} I^{q_{n^*}}_{T_{n^*-1}} f\left(T, \tilde{x}_{n^*}\right) \\ & - \frac{1}{\Gamma\left(q_{n^*}\right)} \int_{T_{n^*-1}}^{t} (t-s)^{q_{n^*}-1} f\left(s, \tilde{x}_{n^*}(s)\right) ds, T_{n^*-1} \leq t \leq T. \end{aligned} \tag{2.20}$$

Applying operator $D^{q_{n^*}}_{T_{n^*-1}+}$ on both sides of (2.20), by theoreme(1.2), we can obtain that:

$$D^{q_{n^*}}_{T_{n^*-1}+} \tilde{x}_{n^*}(t) + f\left(t, \tilde{x}_{n^*}\right) = 0, T_{n^*-1} < t \leq T,$$

that is, $\tilde{x}_{n^*}(t)$ satisfies the following equation:

$$\frac{d^2}{dt^2} \frac{1}{\Gamma\left(2 - q_{n^*}\right)} \int_{T_{n^*-1}}^{t} (t-s)^{1-q_{n^*}} \tilde{x}_{n^*}(s) ds + f\left(t, \tilde{x}_{n^*}\right) = 0, \quad T_{n^*-1} < t \leq T. \tag{2.21}$$

We let:

$$x_{n^*}(t) = \begin{cases} 0, & 0 \leq t \leq T_{n^*-1}, \\ \tilde{x}_{n^*}, & T_{n^*-1} < t \leq T, \end{cases} \tag{2.22}$$

hence, from (2.21), we know that $x_{n^*} \in C[0,T]$ defined by (2.22) satisfies equation:

$$\begin{aligned} & \frac{d^2}{dt^2}\left(\int_0^{T_1} \frac{(t-s)^{1-q_1}}{\Gamma\left(2-q_1\right)} x_{n^*(s)} ds + \cdots \right. \\ & \quad \left. + \int_{T_{n^*-1}}^{t} \frac{(t-s)^{1-q_s}}{\Gamma\left(2-q_{n^*}\right)} x_{n^*}(s) ds \right) + f\left(t, x_{n^*}\right) = 0. \end{aligned}$$

for $T_{n^*-1} < t < T$, which means that $x_n \in C[0,T]$ is one solution of (2.12) with $x_{n^*}(0) = 0, x_{n^*}(T) = \tilde{x}_{n^*}(T) = 0$.

As a result, we know that the boundary value problem (2.1) has a solution. Thus we complete the proof.

Remark 2.2. *For condition (H_2), if $\gamma \geq 1$, then using similar way, we can obtain the existence result of solution to the boundary value problem (2.1) provided that we impose some additional conditions on c_1, c_2.*

2.3 Example

Let us consider the following linear boundary value problem

$$\begin{cases} D^{q(t)}_{0+} x(t) + t^{0.4} = 0, 0 < t < 3, \\ u(0) = 0, u(3) = 0, \end{cases}$$

where

$$q(t) = \begin{cases} 1.2, & 0 \le t \le 1, \\ 1.5, & 1 < t \le 2, \\ 1.8, & 2 < t \le 3. \end{cases}$$

We see that $q(t)$ satisfies condition (H_1) ;
$f(t, x(t)) = t^{0.4} : [0, 3] \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, $|f(t, x(t))| = t^{0.4} \le 3^{0.4}$, thus we could take suitable constants to verify $f(t, x) = t^{0.4}$ satisfies condition (H_2). Then Theorem (2.1)assures the boundary value problem (2.3) has a solution. In fact, we know that equation of (2.3) can been divided into three expressions as following

$$D_{0+}^{1.2}x(t) + t^{0.4} = 0, \quad 0 < t \le 1.$$

For $1 < t \le 2$,

$$\frac{d^2}{dt^2}\left(\int_0^1 \frac{(t-s)^{-0.2}}{\Gamma(0.8)}x(s)ds + \int_1^t \frac{(t-s)^{-0.5}}{\Gamma(0.5)}x(s)ds\right) + t^{0.4} = 0.$$

For $2 < t \le 3$,

$$\frac{d^2}{dt^2}\left(\int_0^1 \frac{(t-s)^{-0.2}}{\Gamma(0.8)}x(s)ds + \int_1^2 \frac{(t-s)^{-0.5}}{\Gamma(0.5)}x(s)ds + \int_2^t \frac{(t-s)^{-0.8}}{\Gamma(0.2)}x(s)ds\right) + t^{0.4} = 0.$$

By [22], we can easily obtain that the following boundary value problems

$$\begin{cases} D_{0+}^{1.2}x(t) + t^{0.4} = 0, 0 < t \le 1, \\ x(0) = 0, x(1) = 0 \end{cases}$$

$$\begin{cases} D_{1+}^{1.5}x(t) = \frac{d^2}{dd^2}\int_1^t \frac{(t-s)^{-0.5}}{\Gamma(0.5)}x(s)ds + t^{0.4} = 0, \quad 1 < t < 2, \\ x(1) = 0, x(2) = 0 \end{cases}$$

$$\begin{cases} D_{2+}^{1.8}x(t) = \frac{d^2}{dd^2}\int_2^t \frac{(t-s)^{-0.8}}{\Gamma(0.2)}x(s)ds + t^{0.4} = 0, 2 < t < 3, \\ x(2) = 0, x(3) = 0 \end{cases}$$

respectively have solutions

$$x_1(t) = \frac{\Gamma(1.4)}{\Gamma(2.6)}\left(t^{0.2} - t^{1.6}\right) \in C[0, 1];$$

$$\tilde{x}_2(t) = \frac{\Gamma(1.4)}{\Gamma(2.9)}\left((t-1)^{0.5} - (t-1)^{1.9}\right) \in C[1, 2];$$

$$\tilde{x}_3(t) = \frac{\Gamma(1.4)}{\Gamma(3.2)}\left((t-2)^{0.8} - (t-2)^{2.2}\right) \in C[2, 3].$$

It is known by calculation that

$$x_1(t), 0 \le t \le 1, \quad x_2(t) = \begin{cases} 0, & 0 \le t \le 1, \\ \widetilde{x}_2(t), 1 < t \le 2, \end{cases} \quad x_3(t) = \begin{cases} 0, & 0 \le t \le 2, \\ \widetilde{x}_3(t), 2 < t \le 3, \end{cases}$$

are the solutions of (5.3)-(5.5), respectively. By Definition 3.2 and (5.6), we know that

$$x(t) = \begin{cases} x_1(t) = \frac{\Gamma(1.4)}{\Gamma(2.6)}\left(t^{0.2} - t^{1.6}\right), & 0 \le t \le 1, \\ x_2(t) = \begin{cases} 0, & 0 \le t \le 1, \\ \frac{\Gamma(1.4)}{\Gamma(2.9)}\left((t-1)^{0.5} - (t-1)^{1.9}\right), & 1 < t \le 2, \end{cases} \\ x_3(t) = \begin{cases} 0, & 0 \le t \le 2, \\ \frac{\Gamma(1.4)}{\Gamma(3.2)}\left((t-2)^{0.8} - (t-2)^{2.2}\right), & 2 < t \le 3 \end{cases} \end{cases}$$

is one solution of the boundary value problem (5.2).

Chapter 3

Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order

Definition 3.1. *Let the function $\rho \in C(L, \mathbb{R}_+)$. The Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order is Ulam Hyers Rassias Stable. with respect to ρ if there exists a constant $c_f > 0$, such that for any $\varepsilon > 0$ and for every $z \in C(L, \mathbb{R})$ such that*

$$\left| D_w^{\alpha(t)} z(t) - f\left(t, z(t), I_w^{\alpha(t)} z(t)\right)\right| \leq \varepsilon \rho(t), t \in L, \tag{3.1}$$

there exists a solution $h \in C(L, \mathbb{R})$ of Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order with

$$|z(t) - h(t)| \leq c_f \varepsilon \rho(t), t \in L.$$

3.1 Existence of Solutions

The weighted fractional derivative of a function of constant order operators have recently gained popularity. In this paper we will study the boundary value problem for weighted fractional derivative of a function with respect to another function with variable order boundary value problem weighted fractional derivative variable order (BVPWFDVO)

$$\begin{cases} D_w^{\alpha(t)} h(t) = f(t, h(t)), t \in L, \\ h(0) = h(\varepsilon) = 0, \end{cases} \tag{BVPWFDVO}$$

where $L = [0, \varepsilon], 0 < \varepsilon < \infty, \alpha(t) : L \to (1, 2]$ is the variable order of the fractional derivative equation, $f : L \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $I_w^{\alpha(t)}$ and $D_w^{\alpha(t)}$ are the left Weighted Fractional Derivative and Weighted Fractional Integral (respectively) of a Function With Respect To Another Function of Variable Order $\alpha(t)$ for function $x(t)$.

The Weighted Fractional Integral of a function f with respect to the function ψ (in Riemann Liouville settings) of variable order $\alpha(t) : L \to (1,2]$ has the form

$$I_w^{\alpha(t)} f(t) = \frac{w^{-1}(t)}{\Gamma(\alpha(t))} \int_0^t (\psi(t) - \psi(s))^{\alpha(t)-1} w(s) f(s) \psi'(s) ds, \quad t > 1, \quad (3.2)$$

The corresponding Weighted Fractional Derivative is

$$D_w^{\alpha(t)} f(t) = \frac{w^{-1}(t)}{\Gamma(n - \alpha(t))} \left(\frac{D_t}{\psi'(t)} \right)^n (w(t) \int_0^t (\psi(t) - \psi(s))^{n-\alpha(t)-1} w(s) f(s) \psi'(s) ds), \quad (3.3)$$

where $w(t) > 0$ is a continuous weighted function, $w^{-1}(t) = \dfrac{1}{w(t)}$ and $\psi \in \mathcal{C}^1 (L, \mathbb{R}^+)$ satisfied $\psi'(t) > 0$, for all $t \in L$.

Theoreme 3.1. *[20] Let $\alpha > 0, n = -[-\alpha]$. Then*

$$(I_w^\alpha D_w^\alpha f)(t) = f(t) - w^{-1}(t) \sum_{k=1}^n a_k \psi_{\alpha-k}(t, 0).$$

First let proceed with the following assumption:

Hypothesis 1 (H1). Let $n \in \mathbb{N}$ be such an integer and a finite point sequence $\{t_j\}_{j=0}^n$ be given in such a way $0 = t_0 < j < t_n = \varepsilon, j = 1, \ldots, n-1$.

Denote $L_j := (t_{j-1}, t_j], j = 1, 2, \ldots, n$. Then $\mathcal{P} = \cup_{j=1}^n L_j$ is a partition of the interval L.

For each $l = 1, 2, \ldots, n$, the symbol $E_l = \mathcal{C}_w(L_l, \mathbb{R})$, indicates the weighted Banach Space of Continuous Functions $x : L_l \to \mathbb{R}$ equipped with the norm

$$\|x\|_{E_l} = \sup_{t \in L_l} |w(t) x(t)|.$$

Let $\alpha(t) : L \to (1,2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e., $\alpha(t) = \sum_{l=1}^n \mathbb{1}_l(t)$, where $1 < \alpha_l \leq 2$ are constants and $\mathbb{1}_l$ is the indicator of the interval $L_l, l = 1, 2, \ldots, n$:

$$\mathbb{1}_l(t) = \begin{cases} 1, & \text{for } t \in L_l, \\ 0, & \text{elsewhere.} \end{cases}$$

Then, for any $t \in L_l, l = 1, 2, \ldots, n$, the Weighted Fractional Derivative of a Function With Respect To Another Function of Variable order $\alpha(t)$ for function $h(t) \in \mathcal{C}_w(L,\mathbb{R})$, defined by (3.3), could be presented as a sum of Weighted Fractional Derivative of a Function With Respect To Another Function Constant Order $\alpha_j, j = 1, 2, \ldots, l$.

$$\begin{aligned} D_w^{\alpha(t)} h(t) =& \frac{w^{-1}(t)}{\Gamma(2-\alpha(t))} \left(\frac{D_t}{\psi'(t)}\right)^2 \left(w(t) \int_1^t \psi_{1-\alpha(t)}(t,s) w(s) h(s) \psi'(s) ds\right) \\ =& \frac{w^{-1}(t)}{\Gamma(2-\alpha(t))} \left[\sum_{j=1}^{l-1} \left(\frac{D_t}{\psi'(t)}\right)^2 \left(w(t) \int_{t_{j-1}}^{t_j} \psi_{1-\alpha_j}(t,s) w(s) h(s) \psi'(s) ds\right)\right. \\ &+ \left.\left(\frac{D_t}{\psi'(t)}\right)^2 \left(w(t) \int_{t_{l-1}}^{t} \psi_{1-\alpha_l}(t,s) w(s) h(s) \psi'(s) ds\right)\right]. \end{aligned}$$

Thus, the equation of the Boundary Value Problem of Weighted Fractional Derivative of a Function With Respect To Another Function of Variable Order can be written for any $t \in L_l, l = 1, 2, \ldots, n$ in the form

$$\begin{aligned} &\frac{w^{-1}(t)}{\Gamma(2-\alpha(t))} \left[\sum_{j=1}^{l-1} \left(\frac{D_t}{\psi'(t)}\right)^2 \left(w(t) \int_{t_{j-1}}^{t_j} \psi_{1-\alpha_j}(t,s) w(s) h(s) \psi'(s) ds\right)\right. \\ + &\left.\left(\tfrac{D_t}{\psi'(t)}\right)^2 \left(w(t) \textstyle\int_{t_{l-1}}^{t} \psi_{1-\alpha_l}(t,s) w(s) h(s) \psi'(s) ds\right)\right] = f\left(t, h(t)\right). \end{aligned} \tag{3.4}$$

Let the function $\tilde{h} \in C\left(J_\ell, \mathbb{R}\right)$ be such that $\tilde{h}(t) \equiv 0$ on $t \in [1, t_{\ell-1}]$ and it solves integral Equation (3.4). Then (3.4) is reduced to

$$_{t_{\ell-1}}D_w^{\alpha_\ell} h(t)\tilde{h}(t) = f\left(t, \tilde{h}(t)\right), t \in L_\ell.$$

Taking into account the above for any $\ell = 1, 2, \ldots, n$, we consider the following auxiliary Boundary Value Problem for Weighted Fractional Derivative of a Function With Respect To Another Function of Constant Order

$$\begin{cases} _{t_{\ell-1}}D_w^{\alpha_\ell} h(t) = f\left(t, h(t)\right), t \in L_\ell \\ h\left(t_{\ell-1}\right) = 0, h\left(t_\ell\right) = 0. \end{cases} \tag{BVPWFDCO}$$

Lemma 4. *Let $\ell \in \{1, 2 \ldots, n\}$ be a natural number, $f \in C\left(L_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$ and there exists a number $\delta \in (0,1)$ such that $w(t)(\psi(t) - \psi(1))^\delta f(t) \in C\left(L_\ell \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$.*

Then the function $h_\ell \in E_\ell$ is a solution of : Boundary Value Problem Weighted Fractional Derivative Constant Order if and only if h_ℓ solves the integral equation

$$\begin{aligned} h(t) = & -\frac{w(t_\ell)\psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{w(t)\psi_{1-\alpha_\ell}(t, t_{\ell-1})} \; _{t_{\ell-1}}I_w^{\alpha_\ell}\left(f\left(t, h(t)\right)\right)_{t=t_\ell} \\ & + \; _{t_{\ell-1}}I_w^{\alpha_\ell}\left(f\left(t, h(t)\right)\right) \end{aligned} \tag{3.5}$$

Proof 3.1. *Let $h_\ell \in E_\ell$ be a solution of the problem Boundary Value Problem Weighted Fractional Derivative Constant Order . Using the operator ${}_{t_{\ell-1}}I_w^{\alpha_\ell}$ to both sides of the equation in the problem Boundary Value Problem Weighted Fractional Derivative Constant Order, we find (see Theorem (3.1)*

$$\begin{aligned} h_\ell(t) = &- a_1 w^{-1}(t)\psi_{\alpha_\ell - 1}(t, t_{\ell-1}) - a_2 w^{-1}(t)\psi_{\alpha_\ell - 2}(t, t_{\ell-1}) \\ &+ {}_{t_{\ell-1}}I_w^{\alpha_\ell}\left(f\left(t, h(t)\right)\right) \end{aligned}$$

where a_1, a_2 are two constants.
Based on the operating environment h as well as the boundary condition $h\left(t_{\ell-1}\right) = 0$, we conclude that $a_2 = 0$.
Based on the boundary condition $h\left(t_\ell\right) = 0$ we obtain

$$a_1 = w(t_\ell)\ \psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})\ {}_{t_{\ell-1}}I_w^{\alpha_\ell}\left(f\left(t, h(t)\right)\right)_{t=t_\ell}$$

Then, we find h_ℓ solves integral Equation (3.5).

In contrast, suppose $h_\ell \in E_\ell$ be a solution of integral Equation (3.5). In respect of the continuity $w(t)\psi_\delta(t,0)f(t)$, we deduce that h_ℓ is the solution of problem Boundary Value Problem Weighted Fractional Derivative Constant Order .

Theoreme 3.2. *Let the conditions of Lemma (4) be satisfied and there are constants $V > 0$ such that*

$$\psi_\delta(t,0)\left|f\left(t, x_1, y_1\right) - f\left(t, x_2, y_2\right)\right| \leq V\left|x_1 - x_2\right|, \quad x_i, y_i, \in \mathbb{R}, i = 1, 2, t \in L_\ell$$

and the inequality

$$d < 1. \tag{3.6}$$

holds, where

$$d = \frac{2\psi_{\alpha_\ell - 1}(t_\ell, t_{\ell-1})(\psi_{1-\delta}(t_\ell, 0) - \psi_{1-\delta}(t_{\ell-1}, 0))}{(1-\delta)\Gamma(\alpha_\ell)} V$$

Then, the Boundary Value Problem Weighted Fractional Derivative Constant Order does have least one solution in E_ℓ.

Proof 3.2. *Let $r_l = \dfrac{2f_w^\star \psi_{\alpha_\ell}(t_\ell, t_{\ell-1})}{(1-d)\Gamma(\alpha_\ell + 1)}$ with $f_w^\star = \sup_{t \in L_\ell} |w(t)f(t,0,0)|$ Consider the set*

$$B_\ell = \left\{h \in E_\ell, \|h\|_{E_\ell} \leq r_\ell\right\}.$$

Its clear that the set B_ℓ is a nonempty, bounded, closed convex subset of E_ℓ, $\forall \ell \in \{1, 2, .., n\}$.

We introduce the operator $\mathcal{F}$ defined on E_ℓ by

$$\begin{aligned}\mathcal{F}h(t) = \; & -\frac{w^{-1}(t)\psi_{1-\alpha_\ell}(t_\ell,t_{\ell-1})}{\Gamma(\alpha_\ell)\psi_{1-\alpha_\ell}(t,t_{\ell-1})}\int_{t_{\ell-1}}^{t_\ell}\psi_{\alpha_\ell-1}(t_\ell,s)w(s)\psi'(s)f\left(s,h(s)\right)ds \\ & +\frac{w^{-1}(t)}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t}\psi_{\alpha_\ell-1}(t,s)w(s)\psi'(s)f\left(s,h(s)\right)ds\end{aligned} \tag{3.7}$$

Out from qualities of: Fractional Integral and from the continuity of function $\psi_\delta(\cdot,0)w(\cdot)\ f(\cdot)$, the above operator $\mathcal{F}: E_\ell \longrightarrow E_\ell$ is clearly defined.

From the definition of the operator $\mathcal{F}$ and Lemma (4), we perceive that the fixed points of $\mathcal{F}$ are solutions of problem boundary value problem weighted fractional derivative constant order . For this reason, it suffices to verify the axioms of Theorem 1.4, it is done in four steps.

Step 1, *$\mathcal{F}(B_\ell) \subseteq B_\ell$ Let $\in B_\ell$ using (H1), we have*

$$\begin{aligned}|w(t)\mathcal{F}(h)| \leq & -\frac{\psi_{1-\alpha_\ell}(t_\ell,t_{\ell-1})}{\Gamma(\alpha_\ell)\psi_{1-\alpha_\ell}(t,t_{\ell-1})}\int_{t_{\ell-1}}^{t_\ell}\psi_{\alpha_\ell-1}(t_\ell,s)w(s)\psi'(s)|f(s,h(s))|ds \\ & +\frac{1}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t}\psi_{\alpha_\ell-1}(t,s)w(s)\psi'(s)|f(s,h(s))|ds \\ & \leq \frac{2}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t_\ell}\psi_{\alpha_\ell-1}(t_\ell,s)w(s)\psi'(s)|f(s,h(s))|ds \\ & \leq \frac{2}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t_\ell}\psi_{\alpha_\ell-1}(t_\ell,s)w(s)\psi'(s)|f(s,h(s))-f(s,0)|ds \\ & +\frac{2}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t_\ell}\psi_{\alpha_\ell-1}(t_\ell,s)w(s)\psi'(s)|f(s,0)|ds \\ & \leq \frac{2}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t_\ell}\psi_{\alpha_\ell-1}(t_\ell,s)\psi'(s)\psi_{-\delta}(s,0)(V|w(s)h(s)|)ds \\ & +\frac{2f_w^\star}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t_\ell}\psi_{\alpha_\ell-1}(t_\ell,s)\psi'(s)ds \\ & \leq \frac{2\psi_{\alpha_\ell-1}(t_\ell,t_{\ell-1})}{\Gamma(\alpha_\ell)}(V\|h\|_{E_\ell})\int_{t_{\ell-1}}^{t_\ell}\psi'(s)\psi_{-\delta}(s,0)ds \\ & +\frac{2f_w^\star}{\Gamma(\alpha_\ell+1)}\psi_{\alpha_\ell}(t_\ell,t_{\ell-1}) \\ & \leq dr_\ell+\frac{2f_w^\star}{\Gamma(\alpha_\ell+1)}\psi_{\alpha_\ell}(t_\ell,t_{\ell-1}) \\ & = r_\ell\end{aligned}$$

which means that $\mathcal{F}(B_\ell) \subseteq B_\ell$

Step 2, *$\mathcal{F}$ is continuous.*
Let $h_k \in E_\ell$, $k = 1, 2, \cdots$ Presume the sequence $\{h_k\}_{k=1}^{\infty}$ is convergent to $h \in E_\ell$. Then for any $k = 1, 2, \cdots$ we have

$$\begin{aligned}
&w(t)|\mathcal{F}h_k(t) - \mathcal{F}h(t)| \\
\leq& -\frac{\psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell)\psi_{1-\alpha_\ell}(t, t_{\ell-1})} \int_{t_{\ell-1}}^{t_\ell} \psi_{\alpha_\ell - 1}(t_\ell, s) w(s)\psi'(s)|f(s, h_k(s)) - f(s, h(s))|ds \\
&+ \frac{1}{\Gamma(\alpha_\ell)} \int_{t_{\ell-1}}^{t} \psi_{\alpha_\ell - 1}(t, s) w(s)\psi'(s)|f(s, h_k(s)) - f(s, h(s))|ds \\
\leq& \frac{2\psi_{\alpha_\ell - 1}(t, t_{\ell-1})}{\Gamma(\alpha_\ell)} \int_{t_{\ell-1}}^{t_\ell} w(s)\psi'(s)|f(s, h_k(s)) - f(s, h(s))|ds \\
\leq& \frac{2\psi_{\alpha_\ell - 1}(t, t_{\ell-1})}{\Gamma(\alpha_\ell)} \int_{t_{\ell-1}}^{t_\ell} \psi_{-\delta}(s, 0) w(s)\psi'(s) \left(V|h_k(s) - h(s)| - h(s)|\right) ds \\
\leq& \frac{2\psi_{\alpha_\ell - 1}(t, t_{\ell-1})}{\Gamma(\alpha_\ell)} V \int_{t_{\ell-1}}^{t_\ell} \psi_{-\delta}(s, 0)\psi'(s) ds \|h_k - h\|_{E_\ell} \\
\leq& \frac{2\psi_{\alpha_\ell - 1}(t_\ell, t_{\ell-1})(\psi_{1-\delta}(t_\ell, 0) - \psi_{1-\delta}(t_{\ell-1}, 0))}{(1-\delta)\Gamma(\alpha_\ell)} V \|h_k - h\|_{E_\ell}
\end{aligned}$$

i.e., we acquire

$$\|\mathcal{F}h_k - \mathcal{F}h\|_{E_\ell} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

As a result, the operator $\mathcal{F}$ is continuous on E_ℓ.

Step 3, *$\mathcal{F}$ is Bounded and equicontinuous.*
By the first step for $h \in B_\ell$ we obtain$\|\mathcal{F}h\|_{E_\ell} \leq r_\ell$, which confirm that $\mathcal{F}(B_\ell)$ is bounded. Rest to prove that $\mathcal{F}(B_\ell)$ is equicontinuous. Let $t_1 < t_2 \in L_\ell$

and $h \in B_\ell$. *Then*

$$
\begin{aligned}
& w(t)|\mathcal{F}h(t_1) - \mathcal{F}h(t_2)| \\
\leq & \frac{\psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell)} \left(\psi_{\alpha_\ell - 1}(t_2, t_{\ell-1}) - \psi_{\alpha_\ell-1}(t_1, t_{\ell-1})\right) \\
& \times \int_{t_{\ell-1}}^{t_\ell} \psi_{\alpha_\ell-1}(t_\ell, s) w(s)\psi'(s)|f(s, h(s))|ds \\
& + \frac{1}{\Gamma(\alpha_\ell)} \int_{t_{\ell-1}}^{t_1} \left(\psi_{\alpha_\ell-1}(t_2, s) - \psi_{\alpha_\ell-1}(t_1, s)\right) w(s)\psi'(s)|f\left(s, h(s)\right)|ds \\
& + \frac{1}{\Gamma(\alpha_\ell)} \int_{t_1}^{t_2} \psi_{\alpha_\ell-1}(t_2, s) w(s)\psi'(s)|f\left(s, h(s)\right)|ds \\
\leq & -\frac{\psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell)} \left(\psi_{\alpha_\ell-1}(t_2, t_{\ell-1}) - \psi_{\alpha_\ell-1}(t_1, t_{\ell-1})\right) \\
& \times \int_{t_{\ell-1}}^{t_\ell} \psi_{\alpha_\ell-1}(t_\ell, s)\psi'(s)\psi_{-\delta}(s, 0)\left(Vw(s)|h(s)|\right)ds \\
& + -\frac{f_w^\star \psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell)} \left(\psi_{\alpha_\ell-1}(t_2, t_{\ell-1}) - \psi_{\alpha_\ell-1}(t_1, t_{\ell-1})\right) \int_{t_{\ell-1}}^{t_\ell} \psi_{\alpha_\ell-1}(t_\ell, s)\psi'(s)ds \\
& + \frac{1}{\Gamma(\alpha_\ell)} \int_{t_{\ell-1}}^{t_1} \psi_{\alpha_\ell-1}(t_2, t_1)\psi'(s)\psi_{-\delta}(s, 0)\left(Vw(s)|h(s)|\right)ds \\
& + \frac{f_w^\star}{\Gamma(\alpha_\ell)} \int_{t_{\ell-1}}^{t_1} \psi_{\alpha_\ell-1}(t_2, t_1)\psi'(s)ds + \frac{f_w^\star}{\Gamma(\alpha_\ell)} \int_{t_1}^{t_2} \psi_{\alpha_\ell-1}(t_2, s)\psi'(s)ds \\
& + \frac{1}{\Gamma(\alpha_\ell)} \int_{t_1}^{t_2} \psi_{\alpha_\ell-1}(t_2, s)\psi'(s)\psi_{-\delta}(s, 0)\left(Vw(s)|h(s)|\right)ds \\
\leq & \frac{\psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell)} \left(\psi_{\alpha_\ell-1}(t_2, t_{\ell-1}) - \psi_{\alpha_\ell-1}(t_1, t_{\ell-1})\right) \psi_{\alpha_\ell-1}(t_\ell, t_{\ell-1}) \\
& \times V\|h\|_{E_\ell} \int_{t_{\ell-1}}^{t_\ell} \psi'(s)\psi_{-\delta}(s, 0)ds \\
& + \frac{f_w^\star \psi_1(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell + 1)} \left(\psi_{\alpha_\ell-1}(t_2, t_{\ell-1}) - \psi_{\alpha_\ell-1}(t_1, t_{\ell-1})\right) \\
& + \frac{\psi_{\alpha_\ell-1}(t_2, t_1)}{(1-\delta)\Gamma(\alpha_\ell)} \left(\psi_{1-\delta}(t_1, 0) - \psi_{1-\delta}(t_{\ell-1}, 0)\right) V\|h\|_{E_\ell} \\
& + \frac{\psi_1(t_1, t_{\ell-1}) f_w^\star}{\Gamma(\alpha_\ell + 1)} \psi_{\alpha_\ell-1}(t_2, t_1) + \frac{f_w^\star}{\Gamma(\alpha_\ell + 1)} \psi_{\alpha_\ell}(t_2, t_1)
\end{aligned}
$$

$$+\frac{\psi_{\alpha_\ell-1}(t_2,t_1)}{(1-\delta)\Gamma(\alpha_\ell)}V\|h\|_{E_\ell}\left(\psi_{1-\delta}(t_2,0)-\psi_{1-\delta}(t_1,0)\right),$$

As an outcome, we acquire

$$\begin{aligned}
&|w(t)\mathcal{F}h(t_1)-\mathcal{F}h(t_2)|\\
&\leq\left[\frac{(\psi_{1-\delta}(t_\ell,0)-\psi_{1-\delta}(t_{\ell-1},0))}{(1-\delta)\Gamma(\alpha_\ell)}V\|h\|_{E_\ell}+\frac{f_w^\star\psi_1(t_\ell,t_{\ell-1})}{\Gamma(\alpha_\ell+1)}\right]\\
&\times\left(\psi_{\alpha_\ell-1}(t_2,t_{\ell-1})-\psi_{\alpha_\ell-1}(t_1,t_{\ell-1})\right)\\
&+\left[\frac{2\left(\psi_{1-\delta}(t_1,0)-\psi_{1-\delta}(t_{\ell-1},0)\right)}{(1-\delta)\Gamma(\alpha_\ell)}V\|h\|_{E_\ell}+\frac{\psi_1(t_1,t_{\ell-1})f_w^\star}{\Gamma(\alpha_\ell+1)}\right]\\
&\times\psi_{\alpha_\ell-1}(t_2,t_1)+\frac{f_w^\star}{\Gamma(\alpha_\ell+1)}\psi_{\alpha_\ell}(t_2,t_1),
\end{aligned}$$

Hence $|\mathcal{F}h(t2)-\mathcal{F}h(t2)|\longrightarrow 0$ *as* $|t_2-t_1|\longrightarrow 0$. *It signifies that* $\mathcal{F}(B_\ell)$ *is equicontinuous.*

Step 4, $\mathcal{F}$ *is k-set contraction.*

For $H\in B_\ell$. *We denote by* ϑ_w *the Kuratowski Measure Of Non Compactness on* E_ℓ, *by utilizing Lemma 2 and the third step, we get*

$$\vartheta_w(\mathcal{F}H)=\sup_{t\in L_\ell}\vartheta\left(w(t)\mathcal{F}H(t)\right),$$

Where $H(t) = \{h(t),\ h \in H\}$.

$$\begin{aligned}
\vartheta\left(w(t)\mathcal{F}H(t)\right) =& \vartheta\left(w(t)\mathcal{F}h(t), h \in H\right) \\
\leq& \vartheta\left\{-\frac{\psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell)\psi_{1-\alpha_\ell}(t, t_{\ell-1})}\int_{t_{\ell-1}}^{t_\ell} \psi_{\alpha_\ell-1}(t_\ell, s)\psi'(s)\vartheta w(s) f(s, h(s),)ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha_\ell)}\int_{t_{\ell-1}}^{t} \psi_{\alpha_\ell-1}(t, s)\psi'(s)\vartheta w(s) f(s, h(s))ds,\ h \in H\right\} \\
\leq& - \frac{\psi_{1-\alpha_\ell}(t_\ell, t_{\ell-1})}{\Gamma(\alpha_\ell)\psi_{1-\alpha_\ell}(t, t_{\ell-1})} \\
&\times \int_{t_{\ell-1}}^{t_\ell} \psi_{\alpha_\ell-1}(t_\ell, s)\psi'(s)\psi_{-\delta}(s, 0)\left[V\vartheta_w(H)\right] ds \\
&+ \frac{1}{\Gamma(\alpha_\ell)}\psi_{\alpha_\ell-1}(t, t_{\ell-1}) \\
&\times \int_{t_{\ell-1}}^{t} \psi'(s)\psi_{-\delta}(s, 0)\left[V\vartheta_w(H)\right] ds \\
\leq& \frac{2\left[\psi_{1-\delta}(t_\ell, 0) - \psi_{1-\delta}(t_{\ell-1}, 0)\right]}{(1-\delta)\Gamma(\alpha_\ell)\psi_{1-\alpha_\ell}(t, t_{\ell-1})} V\vartheta_w(H),
\end{aligned}$$

Thus

$$\vartheta_w\left(\mathcal{F}H\right) \leq \frac{2\left[\psi_{1-\delta}(t_\ell, 0) - \psi_{1-\delta}(t_{\ell-1}, 0)\right]}{(1-\delta)\Gamma(\alpha_\ell)\psi_{1-\alpha_\ell}(t, t_{\ell-1})} V\vartheta_w(H)$$

According to inequality (3.6), $\mathcal{F}$ *is a k-set contraction.*
As a matter of fact, all Theorem 1.4 requirements have been met, so as side effect $\mathcal{F}$ *admit an fixed point* $\mathcal{F}(\tilde{h}_\ell) = h\ell$ *where* $\tilde{h} \in B_\ell$*, which is a solution of the Boundary Value Problem for Weighted Fractional Differential Equation of Constant Order , Since* $B_\ell \subset E_\ell$*, Theorem (3.2) claim is established.*

We're now going to demonstrate the existence of Boundary Value Problem for Weighted Fractional Differential Equation of Constant Order.
Consider the following Hypothesis :
Hypothesis 2 (H2). Let $f \in C(L \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in 2(0, 1)$ such that $w(t)(\psi(t) - \psi(1))^\delta f(t) \in C\left(L \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$ and there are constants $V > 0$ such that

$$\psi_\delta(t, 0)\left|f\left(t, x_1, y_1\right) - f\left(t, x_2, y_2\right)\right| \leq V\left|x_1 - x_2\right|, \quad x_i, y_i, \in \mathbb{R}, i = 1, 2, t \in L,$$

Theoreme 3.3. *Let the conditions (H1), (H2) and inequality 3.6 be satisfied for all* $\ell \in 2\{1, 2, \cdots, n\}$*. Then, the Boundary Value Problem for Weighted Fractional Differential Equation of variable Order incorporates at least one solution in* $C(L, \mathbb{R})$*.*

Proof 3.3.

For any $\ell \in \{1, 2, \ldots, n\}$, according to Theorem (1.4)the Boundary Value Problem for Weighted Fractional Differential Equation of Constant Order possesses at least one solution $\widetilde{h}_\ell \in E_\ell$. For any $\ell \in \{1, 2, \ldots, n\}$ we define the function

$$h_\ell = \begin{cases} 0, & t \in [0, t_{\ell-1}], \\ \tilde{h}_\ell, t \in L_\ell. \end{cases}$$

Thus, the function $h_\ell \in C([0, t_\ell], \mathbb{R})$ solves the integral Equation 3.5 for $t \in L_\ell$, which means that $h_\ell(1) = 0, h_\ell(t_\ell) = \widetilde{h}_\ell(t_\ell) = 0$ and solves 3.5 for $t \in L_\ell, \quad \ell \in \{1, 2, \ldots, n\}$. Then the function

$$h(t) = \begin{cases} h_1(t), & t \in L_1 \\ h_2(t), & t \in L_2 \\ \ldots\ldots\ldots\ldots\ldots & \\ h_n(t), & t \in L_n = [0, \epsilon], \end{cases}$$

is a solution of the Boundary Value Problem for Weighted Fractional Differential Equation of variable Order in $C(L, \mathbb{R})$.

3.2 Example

Let $L := [0,2]$, $\eta = 0$, $\eta_1 = 1$, $\eta_2 = 2$. Consider the scalar BVPWFDVO

$$\begin{cases} D_{0^+}^{\alpha(t)}h(t) = \frac{3}{17}\psi_{\alpha(t)}(t,0) + \psi_{-\frac{1}{5}}(t,0)\frac{h(t)}{t+7} + \frac{\psi(t,0)}{t^3+2}I_{0^+}^{\alpha(t)}h(t), \quad t \in L \\ h(0) = 0, \qquad h(2) = 0, \end{cases} \tag{3.8}$$

Such that where $w(t) = 1 + t^2$, $\psi(t) = -\arctan\frac{1}{1+t}$, this implies that $\psi'(t) = \frac{1}{1+(1+t)^2}$ and

$$\alpha(t) = \begin{cases} 1.4, & t \in L_1 := [0,1], \\ 1.8, & t \in L_1 :=]1,2], \end{cases} \tag{3.9}$$

Denote

$$f(t,h,z) = \frac{3}{17}\psi_{\alpha(t)}(t,0)+\psi_{-\frac{1}{5}}(t,0)\frac{h}{t+7}+\frac{\psi(t,0)}{t^3+2})z, \ (t,h,z) \in [0,2]\times\mathbb{R}\times\mathbb{R}.$$

For $\delta = \frac{1}{5}$, $V = \frac{1}{7}$ and $W = \frac{1}{2}$ the assumption (H2) holds. Indeed

$$\begin{aligned} |f(\eta,h_1,z_1) - f(\eta,h_2,z_2)| &= \left| \frac{h_1}{t+7} + \frac{\psi_{\frac{6}{5}}(t,0)}{t^3+2}z_1 - \frac{h_2}{t+7} - \frac{\psi_{\frac{6}{5}}(t,0)}{t^3+2}z_2 \right| \\ &\leq \frac{1}{t+7}|h_1 - h_2| + \frac{\psi_{\frac{6}{5}}(t,0)}{t^3+2}|z_1 - z_2| \\ &\leq \frac{1}{7}|h_1 - h_2| + \frac{1}{2}|z_1 - z_2|, \end{aligned}$$

By (3.9), according to BVPWFDCO we consider two auxiliary boundary value problem of weighted fractional differential equation for function with respect to another function of constant order

$$\begin{cases} D_{0^+}^{1.4}h(t) = \frac{3}{17}\psi_{1.4}(t,0) + \psi_{-\frac{1}{5}}(t,0)\frac{h(t)}{t+7} + \frac{\psi(t,0)}{t^3+2}I_{0^+}^{1.4}h(t), \quad t \in L_1 \\ h(1) = 0, \qquad h(2) = 0, \end{cases} \tag{3.10}$$

and

$$\begin{cases} D_{0^+}^{1.8}h(t) = \frac{3}{17}\psi_{1.8}(t,0) + \psi_{-\frac{1}{5}}(t,0)\frac{h(t)}{t+7} + \frac{\psi(t,0)}{t^3+2}I_{0^+}^{1.8}h(t), \quad t \in L_2 \\ h(1) = 0, \qquad h(2) = 0, \end{cases} \tag{3.11}$$

Secondly, we demonstrate that the requirement (3.6) is satisfied for $\ell = 1$. Consequently

$$\frac{2\psi_{\alpha_1-1}(t_1,t_0)(\psi_{1-\delta}(t_1,0)-\psi_{1-\delta}(t_0,0))}{(1-\delta)\Gamma(\alpha_1)}\left(V+W\frac{\psi_{\alpha_1}(t_1,t_0)}{\Gamma(\alpha_1+1)}\right) \simeq 0.162691784641 < 1.$$

Let $\rho(t) = \psi_{\frac{3}{5}}(t,0)$ Secondly, we attain

$$\begin{aligned} I_{0^+}^{1.4}\rho(t) &= \frac{1}{(t^2+1)\Gamma(1.4)}\int_0^t \psi_{0.4}(t,s)(t^2+1)\psi_{\frac{3}{5}}(s,0)\psi'(s)ds \\ &= \frac{5\psi_{\frac{3}{5}}(t,0)}{\Gamma(1.4)}\int_0^t \psi'(s)\psi_{0.4}(t,s)ds \\ &\leq \frac{1.03}{\Gamma(2.4)}\psi_{\frac{3}{5}}(t,0) = \lambda_\rho\rho(t), \end{aligned}$$

where $\lambda_\rho = \dfrac{1.03}{\Gamma(2.4)}$. Then, assumption (H3) is satisfied.
By Theorem 1.4, the boundary value problem 3.10 has a solution $\tilde{h}_1 \in E_1$. we demonstrate that the requirement (3.6) is satisfied for $\ell = 2$. Consequently

$$\frac{2\psi_{\alpha_2-1}(t_2,t_1)(\psi_{1-\delta}(t_2,0)-\psi_{1-\delta}(t_1,0))}{(1-\delta)\Gamma(\alpha_2)}\left(V+W\frac{\psi_{\alpha_2}(t_2,t_1)}{\Gamma(\alpha_2+1)}\right) \simeq 0.0117027930094 < 1.$$

As a result, the condition (3.6) is satisfied.also we attain

$$\begin{aligned} I_{0^+}^{1.8}\rho(t) &= \frac{1}{(t^2+1)\Gamma(1.8)}\int_1^t \psi_{0.8}(t,s)(t^2+1)\psi_{\frac{3}{5}}(s,0)\psi'(s)ds \\ &= \frac{5\psi_{\frac{3}{5}}(t,0)}{\Gamma(1.8)}\int_1^t \psi'(s)\psi_{0.8}(t,s)ds \\ &\leq \frac{1.03}{\Gamma(2.8)}\psi_{\frac{3}{5}}(t,0) = \lambda_\rho\rho(t), \end{aligned}$$

where $\lambda_\rho = \dfrac{1.26}{\Gamma(2.8)}$. Then, assumption (H3) is satisfied.
By Theorem 1.4, the boundary value problem 3.11 has a solution $\tilde{h}_2 \in E_2$.

Hence, Theorem 3.3 provides a solution for the BVP (3.9).

$$h(t) = \begin{cases} \tilde{h}_1(t), & t \in L_1, \\ h_2(t), & t \in L_2, \end{cases}$$

where

$$h_2(t) = \begin{cases} 0, & t \in L_1 \\ \tilde{h}_2(t), & t \in L_2. \end{cases}$$

According to Theorem , the boundary value problem for weighted fractional deferential equation of function with respect to another function is Ulam Hyers Rassias stable with respect to ρ.

Conclusion

In this study, we obtained two existence results for the approximate solution of a boundary value problem for a fractional differential equation of variable order. We introduced the concept of approximate solutions by discussing the properties of variable order calculus. Using the Schauder and Bnach fixed point theorem, we established unique existence results and presented two examples as applications. Future research will focus on the stability and convergence of approximate solutions for singular fractional differential equations of variable order and explore the properties of solutions in broader settings. The study of variable order fractional differential equations presents a challenging research area with applications in mathematics and engineering.

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