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Abstract:

In this paper, we present the classical Ostrowski's inequality and some applications for some special means. Also, we prove some inequalities for the class of s-convex functions and their analogous fractional.

Key words: Ostrowski inequality ,convex function.

خلاصة:

في هذه الورقة، نقدم عدم المساواة الكلاسيكية لأوستروفسكي وبعض التطبيقات لبعض الوسائل الخاصة. أيضًا، نثبت بعض التفاوتات لفئة الوظائف المحدبة وجزئتها المماثلة.

كلمات مفتاحية: عدم المساواة استروفسكي ، الدوال المحدبة

Dedication:

In the Name of Allah, the Most Gracious and the Most Merciful

All the Praise is due to Allah alone the Sustainer of all the worlds.

My deepest gratitude and sincere thanks go to my parents mother **"Daher Khiera"** thank you for you love, support and prayers, father **"Miloud"** thank you for your hardwork, encouragement.

To my lovely brothers **Mohamed, Mustapha** and **Abdallah**

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To all my best friends: **Manel, Sarra, Chaima, Katia, Wiam, Asma, Zohra, Souad**

IKRAM

Dedication:

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Contents

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Introduction

Inequalities play an important role in various branches of modern such as Hilbert's theory of spaces, the theory of probability and statistics,analysis, numerical analysis, qualitative theory of differentials equations and differences equations, etc $\cdots$.
The mathematical basis of this theory was partly established during the 18^{th} and 19^{th} century by eminent mathematicians such as: Gauss, Cauchy,Chebychev in the following years the subject attracted many researchers : Poincaré, Lyapunov,Gronwall, Hölder, Hadamard, Pólya, Bellman and Ostrowski. Literary in this context is vast and varied among the works of which a very good description can be found of the historical evolution of inequalities it is possible to consult, Mitrinovic, Pecaric and Fink[[30];[31];[32]].
This theory continues to evolve in many directions and in many different ways. New inequalities have been established, generalizations and extensions as well as variants on several unidimensional, multidimensional, fractional and discreet.
The objective of this work is to present inequality of Ostrowski and some type inequalities in classical and fractional calculus. The brief consists of four chapters divided as follows:
In the first chapter are defined some classes of function as integrable functions ,continuous and absolutely continuous functions,then we report some type of classic convexity,a sketch concerning fractional integrations,and some inequalities of Hôlder,Hermite-Hadamard...etc
The second chapter is devoted to Osrowski's inequality and some applications to

some Special Means.
The third chapter affected inegalities of type Ostrowski for s-convex functions.
In the fourth chapter we consider some integral fractional inequalities of Riemann-Liouville called in the literature Ostrowski type inequalities.
We conclude this modest work by a general conclusion and an interesting bibiography.

Chapter 1

PRELIMINARIES

1.1 Some Function Spaces

1.1.1 Spaces of integrable functions

Definition 1.1.1. Let $(a, b)(\infty \leq a < b \leq +\infty)$ be finite or infinite interval in $\mathbb{R}, 1 \leq p \leq \infty$..

1.For $1 \leq p < \infty$,, the space $L^p((a, b))$ is the space of real function f on (a, b) such that f is mesurable function and

$$\int_a^b |f(x)|^p dx < \infty.$$

2.For $p = \infty$ the spaces $L^\infty$is the space of classes of mesurable functions f bounded almost every where on (a, b)

Theorem 1 *Let (a, b) be a finite or infinite interval of $\mathbb{R}$*

1.For $1 \leq p < +\infty$, the space $L^p((a, b)$ is Banach spaces endowed with the norm

$$\|f\|_p = (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}$$

2.the spaces$L^\infty(\Omega)$ is a Banach space endowed with the norm:

$$\|f\|_\infty = \{\inf M \geq 0 : |f(x)| \leq M \;\; p.p \; on \;\; (a, b)\}$$

1.1.2 Continuous and absolutely continuous functions space

Definition 1.1.2. [33] let $\Omega = [a, b](-\infty \leq a \leq b \leq \infty)$ and $n \in \mathbb{N} = 0, 1, ...$ is referred by $\mathcal{C}^n(\Omega)$ the space of functions f which have their derivatives of lower order or equal to $(n-1)$ continuous on equipped with the standard norm:

$$\|f\|_{c^n} = \sum_{k=0}^{n} \|f^{(k)}(x)\| = \sum_{k=0}^{n} \max_{x\in\Omega} |f^{(k)}(x)|, n \in \mathbb{N}$$

In particular if $n = 0, \mathcal{C}^0(\Omega)$ the space of continuous functions f in Ω equipped of the standard norm:

$$\|f\|_c = \max_{x\in\Omega} |f(x)|.$$

1.2 Some Concepts In Fractional Calculus

In this section, we recall some fondamental concepts in fractional calculus such that: Euler Gamma,Beta functions, considered as special functions. It extends the factorial function to all complex numbers exception of whole negatives. Also we recall the definition of Riemann-Liouville fractional integrals.

1.2.1 Some special functions

Definition 1.2.1. [10](Gamma function) For any complex z number such as $\mathcal{R}e(z) > 0$, the next function, called Gamma function as follows

$$\Gamma(z) = \int_0^{\infty} \exp^{-t} t^{z-1} dt \tag{1.1}$$

Remark 2 *for* $z \in \mathbb{N}$,*we have* $\Gamma(z) = (z-1)! = 1 \times 2 \times 3 \cdots \times (z-1)$.

Definition 1.2.2. [34](Beta function) Euler's beta function is defned for all complex numbers x and y real parts strictly positive by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt. \tag{1.2}$$

Remark 3 *The relationship between Gamma function and Beta function is given as follows:*

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.3}$$

1.2.2 Riemann-Liouville fractional integrals

The history of the non-integer order derivative stretches from the end of the 17th century until today. Specialists agree to go back to the end of the year 1695 when The Hospital raised a question in Leibniz asking about the significance de $\frac{d^n y}{dx^n}$ when $n = \frac{1}{2}$. Leibniz, in his reply, wanted to make a reflection on a possible theory of non whole derivation, and responded to L Hospital:" $\cdots$ thiswould lead to a paradox $\cdots$ " It was not until the 1990s that the first useful consequences. The first serious attempt to give a logic for the fractional derivative is due to Liouville who published nine papers in this subject between 1832 and 1837. Independently, Riemann proposed an approach that this is mainly the case of Liouville, and since then it has carried the non Approach of Riemann-Liouville . Later, other theories made their appearances as that of Grünwald-Leitnikov, Weyl and Caputo etc. This theory has not ceased to attract the scope of its application in image processing,biology, mechanical civil engineering.

Definition 1.2.3. [11] Let $f \in L^1([a,b])$, Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f(x)$ of order $\alpha > 0$, where $\alpha > 0$ is defined by

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, x > a, \quad (left), \tag{1.4}$$

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, x < b, \quad (right) \tag{1.5}$$

Remark 4 *By laying agreement* $I^{0}_{a^+}f(x) = f(x)$.

1.3 Some Inequalities

We recall the famous inequality called Hermite-Hadamard for convex functions then we will state its generalization for s-convex functions . In all that follows we refer to $I = [a,b] \subset \mathbb{R}$.

Definition 1.3.1. [1](on convexity) A set$I \subseteq \mathbb{R}^n$ is said to be convex if for all $x, y \in I$ and for all $t \in [0,1]$, we have

$$tx + (1-t)y \in I \tag{1.6}$$

Definition 1.3.2. [?] A function $f : I \to \mathbb{R}$ is called convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.7}$$

is satisfied for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 1.3.3. [4] A positive function $f : I \subset [0, \infty[\to \mathbb{R}$ is called s-convex at second sense for a number $s \in]0.1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.8}$$

is satisfied for all $x, y \in I$ and $t \in [0.1]$.

Definition 1.3.4. [7] A positive function$f : I \subset [0, \infty[\to \mathbb{R}$is said to be extended s- convex for a certain number $s \in]-1, 1[$,if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.9}$$

is satisfied for all $x, y \in I$ and $t \in]0, 1[$.

Lemma 1.3.1 *[12](Hermite-Hadamard Inequality) Let $f : [a, b] \to \mathbb{R}$, a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.10}$$

Lemma 1.3.2 *[13](Hôlder Inequality) Let $p > 1$ as $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are two real functions defined on $[a, b]$ and if $|f|^p$and $|g|^q$ are integrable functions on $[a, b]$, then*

$$\int_a^b f(x)g(x)dx \leq (\int_a^b f^p(x)dx)^{\frac{1}{p}}((\int_a^b g^q(x)dx)^{\frac{1}{q}}). \tag{1.11}$$

Chapter 2

INEQUALITIES OF OSTROWSKI TYPE

2.1 Ostrowski's Inequalities

Ostrowski Inequality.In 1938,the following celebrated inequality was estblished by Ostrowski.This type of inequality provides estimates of net errors in the approximation the value of a function relative to its full average. They apply to obtaining approximations a priori and calculating error limits for different quadrature rules.

Lemma 2.1.1 *Let* $a, b \in \mathbb{R}$*,and* $x \in [a, b]$

$$\left(\frac{1}{4} + \frac{\left(x - \dfrac{a+b}{2} \right)^2}{(b-a)^2} \right) .(b-a) = \frac{(x-a)^2 + (b-x)^2}{2(b-a)}. \qquad (2.1)$$

Proof 1

$$\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right).(b-a)=\frac{(b-a)}{4}+\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)}\right)$$

$$=\frac{b-a}{4}+\frac{x^2+\left(\frac{a+b}{2}\right)^2-2x\left(\frac{a+b}{2}\right)}{(b-a)}$$

$$=\frac{b-a}{4}+\frac{x^2+\left(\frac{a^2+b^2+2ab}{4}\right)-2x\left(\frac{a+b}{2}\right)}{(b-a)}$$

$$=\frac{\frac{(b-a)^2}{4}+\frac{4x^2+(a^2+b^2+2ab)}{4}-\frac{4x(a+b)}{4}}{(b-a)}$$

$$=\frac{\frac{b^2+a^2-2ab+4x^2+a^2+b^2+2ab-4x(a+b)}{4}}{(b-a)}$$

$$=\frac{\frac{2b^2+2a^2+4x^2-4x(a+b)}{2}}{2(b-a)}$$

$$=\frac{b^2+a^2+2x^2-2ax-2bx}{2(b-a)}$$

$$=\frac{x^2+b^2-2bx+x^2+a^2-2ax}{2(b-a)}$$

$$=\frac{(b-x)^2+(x-a)^2}{2(b-a)}.$$

Theorem 5 *[14] Let I be an interval in $\mathbb{R}$, I^o the interior of I and $a,b\in I^o, a<b$, and $f:I\to\mathbb{R}$ such that $f\in\mathcal{C}^1([a,b]), x\in[a,b]$. If $|f'(t)|\leq M$,for all $t\in[a,b]$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M \tag{2.2}$$

for $x\in[a,b]$.

Inequality (2.2)is sharp since the function cannot be replaced by a smaller.

Proof 2 *We consider the Montgomery's identity (see theorem 5),for all* $x \in [a,b]$

$$f(x) - \frac{1}{b-a}.\int_a^b f(t)dt = \frac{1}{b-a}.\int_a^b p(x,t)f'(t)dt \tag{2.3}$$

where

$$p(x,t) = \begin{cases} t-a, if & t \in [a,x] \\ t-b, if & t \in [x,b]. \end{cases}$$

Thus ,we have

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}.\int f(t)dt \right| &\leq \frac{1}{b-a}\int_a^b |p(x,t)||f'(t)|dt \\ &\leq \frac{M}{b-a}\int_a^b |p(x,t)|dt. \end{aligned}$$

We have

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}.\int f(t)dt \right| &\leq \int_a^x (t-a)dt + \int_x^b (t-b)dt \\ &= \left[\frac{1}{2}(t-a)^2\right]_a^x + \left[\frac{1}{2}(t-b)^2\right]_x^b \\ &= \frac{1}{2}(x-a)^2 + \frac{1}{2}(b-x)^2 \\ &= \frac{(x-a)^2 + (b-x)^2}{2} \end{aligned}$$

from there

$$\begin{aligned} &= \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right] \\ &= \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].(b-a)M. \end{aligned}$$

Next,we give a different proof to(2.2) from that of Ostrowski's initial proof given in 1938 (see [14]).

Theorem 6 *The constant* $\frac{1}{4}$ *in theorem (5) is optimale (sharp). Inequqlity (2.3) is sharp,namely the optimal function is*

$$f^*(y) := |y-x|^\alpha.(b-a)\alpha > 1 \tag{2.4}$$

Proof 3 *We have*

$$\begin{aligned}
\left|\frac{1}{b-a}.\int_a^b f(y)dy - f(x)\right| &= \frac{1}{b-a}.\left|\int_a^b f(y) - f(x)dy\right| \\
&\leq \frac{1}{b-a}\int_a^b |f(y) - f(x)|\, dy \\
&\leq \frac{1}{b-a}\int_a^b k|y-x|.dy \\
&\leq \frac{1}{b-a}\int_a^b f'(y)|y-x|dy \\
&\leq \frac{1}{b-a}\int_a^b |f'(y)||y-x|dy \\
&\leq \frac{1}{b-a}\|f'\|_\infty \int_a^b |y-x|dy
\end{aligned}$$

We compute $\int_a^b |y-x|.dy$

$$\begin{aligned}
\int_a^b |y-x| &= \int_a^x (x-y)dy + \int_x^b (y-x)dy \\
&= \int_a^x xdy - \int_a^x ydy + \int_x^b ydy - \int_x^b xdy \\
&= x^2 - ax - \frac{1}{2}x^2 + \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}x^2 - bx + x^2 \\
&= (x-a)^2 + (b-x)^2
\end{aligned}$$

Finally, we obtain

$$\left|\frac{1}{b-a}.\int_a^b f(y)dy - f(x)\right| \leq \frac{1}{b-a}\|f'\|_\infty((x-a)^2 + (b-x)^2)$$

So,we have established inquality (2.2).

Note that for

$$f^*(y) = \alpha.|y-x|^{\alpha-1}.sign(y-x)(b-a)$$

$$sign(x) = \begin{cases} 1, \text{if} & y > x \\ 0, \text{if} & y = x \\ -1, \text{if} & x > y \end{cases}$$

thus

$f^{'*}(y) = \alpha.|y-x|^{\alpha-1}.(b-a)$

$$\begin{aligned}\|f^{'*}\|_\infty &= sup(|f'*(x)|)\\ &= sup(\alpha.|y-x|^{\alpha-1}.(b-a))\\ &= \alpha.(b-a).(max(b-x,x-a))^{\alpha-1}\end{aligned}$$

Also we notice that $f^*(x) = 0$

Therefore we have for f^* that $f^*(x) = 0$

$$\begin{aligned}L.H.S &= \left(\frac{1}{b-a}.\int_a^b f^*(y)dy - f^*(x)\right)\\ &= \frac{1}{b-a}.\int_a^b |y-x|^\alpha.(b-a)dy\\ &= \int_a^b |y-x|^\alpha dy\end{aligned}$$

From there:

$$\begin{aligned}\int_a^b |y-x|^\alpha dy &= \frac{1}{\alpha+1}|y-x|^{\alpha+1}\\ &= \frac{((x-a)+(b-x))^{\alpha+1}}{\alpha+1}\\ &= \frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\end{aligned}$$

and

$$\begin{aligned}\lim_{\alpha\to1} L.H.S = \lim_{\alpha\to1} &\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\\ = &\frac{(x-a)^2+(b-x)^2}{2}\end{aligned} \tag{2.5}$$

Also we observe that:

$$R.H.S = \left(\frac{(x-a)^2+(b-x)^2}{2.(b-a)}\right).\|f'\|_\infty$$

We compensate $\|f'\|_\infty$ in R.H.S

$$\begin{aligned}&= \left(\frac{(x-a)2(b-x)^2}{2.(b-a)}\right).\alpha.(b-a).max((b-x,x-a))^{\alpha-1}\\ &= \left(\frac{(x-a)^2+(b-x)^2}{2}\right).\alpha.(max(b-x,x-a))^{\alpha-1}\end{aligned}$$

$$\lim_{\alpha\to 1} R.H.S = \lim_{\alpha\to 1}\left(\frac{(x-a)^2(b-x)^2}{2}\right).\alpha.(max(b-x,x-a))^{\alpha-1}$$
$$= \frac{(x-a)^2+(b-x)^2}{2}$$

$$\lim_{\alpha\to 1} L.H.S = \frac{(x-a)^2(b-x)^2}{2}$$
$$= \lim_{\alpha\to 1} R.H.S$$

Note that when $x = a$ or $x = b$, inequality (2.2) can be attaiend by

$$f_a(y) = f_b(y) = (y-a).(b-a)$$

respectively then both equal to $(b-a)^2$.

2.2 Ostrowski's Inequality For Higher Derivation

The following material has been greatly motivated by the important work in [35].Let $f \in C^{n+1}([a,b]), n \in \mathbb{N}, x \in [a,b]$, be fixed.Then by Taylor's theorem we get

$$f(y) - f(x) = \sum_{k=1}^{n} \frac{f^k(x)}{k!}.(y-x)^k + R_n(x,y) \tag{2.6}$$

Where

$$R_n(x,y) := \int_x^y (f^{(n)}(t) - f^{(n)}(x)).\frac{(y-t)^{n-1}}{(n-1)!}dt \tag{2.7}$$

Here y can be $\geq x$ or $\leq x$

$$|R_n(x,y)| \leq \int_x^y \frac{1}{(n-1)!}(y-t)^{n-1}|f^n(t) - f^n(x).dt$$
$$\leq \|f^{(n+1)}\|_\infty \int_x^y |t-x|.\frac{|y-t|^{n-1}}{(n-1)!}.dt$$

We compute

$$\int_x^y |t-x|.\frac{|y-t|^{n-1}}{(n-1)!} = \frac{1}{(n-1)!}.\int_x^y |t-x|.|y-t|^{n-1}dt$$
$$= \frac{1}{(n-1)!}.\int_x^y \left|(t-x)(y-t)^{n-1}\right| dt.$$

For $y \geq x$

We compute $\int_x^y |(t-x)(y-t)^{n-1}|dt$. By integration by parts with $u = t-x, \quad dv = (y-t)^{n-1}$

$$\begin{aligned}
\int_x^y |(t-x)(y-t)^{n-1}|dt &= \left[(t-x)\left(\frac{-1}{n}\right)(y-t)^n\right]_x^y - \int_x^y (\frac{-1}{n})(y-t)^n dt \\
&= \frac{1}{n}\frac{1}{(n-1)!}\left[\frac{1}{(n+1)}(y-t)^{n+1}\right]_x^y \\
&= \frac{1}{n!}.\frac{1}{n+1}.(y-t)^{n+1} \\
&= \frac{1}{(n+1)!}.(y-t)^{n+1}
\end{aligned}$$

Thus

$$|R_n(x,y)| \leq \frac{\|f^{n+1}\|_\infty}{(n+1)!}(x-y)^{n+1}\|. \tag{2.8}$$

For $y \leq x$

$$\begin{aligned}
|R_n(x,y)| &= \left|\int_y^x (f^n(t) - f^n(x)).\frac{(y-t)^{n-1}}{(n-1)!}.dt\right| \\
&\leq \int_y^x |f^n(t) - f^n(x)|.\frac{|y-t|^{n-1}}{(n-1)!}.dt \\
&\leq \frac{\|f^{n+1}\|_\infty}{(n-1)!}.\int_y^x (x-t)(t-y)^{n-1}dt.
\end{aligned}$$

We compute $\int_y^x (x-t)(t-y)^{n-1}dt$. By integration by parts with: $u(t) = x-t, \; dv = (t-y)^{n-1}$

$$\begin{aligned}
\int_y^x (x-t)(t-y)^{n-1}dt &= \left[(x-t).\frac{1}{n}(t-y)^n\right]_y^x + \int_y^x \frac{1}{n}(t-y)^n dt \\
&= \frac{1}{(n-1)!}\frac{1}{n}\left[\frac{1}{(n+1)}(t-y)^{n+1}\right]_y^x \\
&= \frac{1}{(n+1)!}(x-y)^{n+1}.
\end{aligned}$$

Thus

$$|R_n(x,y)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}.|y-x|^{n+1} \tag{2.9}$$

From (2.8) and (2.9)

$$R_n(x,y) \leq \frac{\|f^{n+1}\|_\infty}{(n+1)!}.|y-x|^{n+1} \;\; \forall\, x,y \in [a,b] \tag{2.10}$$

Next,we treat:

$$\left|\frac{1}{b-a}.\int_a^b f(y)-f(x)\right| = \frac{1}{b-a}.\left|\int_a^b (f(y)-f(x)).dy\right|$$
$$= \frac{1}{b-a}.\left|\int_a^b \left[\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}.(y-x)^k + R_n(x,y)\right]dy\right|.$$

we have $\int_a^b (y-x)^k = \dfrac{(b-x)^{k+1}+(x-a)^{k+1}}{k+1}$, hence

$$\left|\frac{1}{b-a}.\int_a^b f(y)-f(x)\right| = \frac{1}{b-a}\left|\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}.\frac{(b-x)^{k+1}+(x-a)^{k+1}}{k+1}+\int_a^b R_n(x,y).dy\right|$$
$$= \frac{1}{b-a}.\left|\sum_{k=1}^{n}\frac{f^{k}(x)}{k!}.\frac{1}{(k+1)}.\left[(b-x)^{k+1}-(a-x)^{k+1}\right]+\int_a^b R_n(x,y).dy\right|$$

(by2.10)

$$\leq \frac{1}{b-a}.\left[\sum_{k=1}^{n}\frac{f^{k}(x)}{(k+1)!}|(b-x)^{k+1}-(a-x)^{k+1}| + \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}.\int_a^b |y-x|^{n+1}dy\right]$$

i,e, we proved that

$$\int_a^b |y-x|^{n+1} = \frac{(x-a)^{n+2}+(b-x)^{n+2}}{(n+2)}$$
$$= \left|\frac{1}{b-a}.\int_a^b f(y)-f(x)\right|$$
$$\leq \frac{1}{b-a}.\left[\sum_{k=1}^{n}\frac{f^{K}(x)}{(k+1)!}.|(b-x)^{k+1}-(a-x)^{k+1}|\right]$$
$$+\frac{1}{b-a}\left[\frac{\|f^(n+1)\|_\infty}{(n+1)!}\frac{1}{(n+2)}((x-a)^{n+2}+(b-x)^{n+2})\right]$$
$$\leq \frac{1}{b-a}\left[\sum_{k=1}^{n}\frac{f^{(k)}(x)}{(k+1)!}|(b-x)^{k+1}-(a-x)^{k+1}+\frac{\|f^{(n+1)}\|_\infty}{(n+2)!}((x-a)^{n+2}(b-x)^{n+2})\right] \tag{2.11}$$

Where $f\in\mathcal{C}^{n+1}([a,b]), n\in\mathbb{N}, x\in[a,b]$ is fixed
if we choose $x=\dfrac{a+b}{2}$ thus $b-x=x-a=\dfrac{b-a}{2}$, then

$$\left|\frac{1}{b-a}\int_a^b f(y)dy - f(\frac{a+b}{2})\right| \leq \frac{1}{b-a}$$
$$\sum_{1\leq k even\leq n}\frac{\left|f^{(k)}\left(\frac{a+b}{2}\right)\right|}{(k+1)!}.\left(\frac{(b-a)^{k+1}}{2^{k+1}}+\frac{(b-a)^{k+1}}{2^{k+1}}\right)+\frac{\|f^{(n+1)}\|_\infty}{(n+2)!}.\left(\left(\frac{b-a}{2}\right)^{n+2}+\left(\frac{b-a}{2}\right)^{n+2}\right)$$

$$\leq \frac{1}{b-a}$$

$$\left[\sum_{1\leq k\ even\ \leq n} \frac{f^{(k)}\left(\dfrac{a+b}{2}\right)}{(k+1)!}.\frac{2.(b-a)^{k+1}}{2^1 * 2^k} + \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}.\frac{2(b-a)^{n+2}}{2^1 * 2^k}\right]$$

$$\leq \frac{1}{b-a}.\left[\sum_{1\leq k\ even\ \leq n} \frac{f^{(k)}\left(\dfrac{a+b}{2}\right)}{(k+1)!}.\frac{(b-a)}{2^k} + \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}.\frac{(b-a)^{n+2}}{2^{n+1}}\right] \tag{2.12}$$

Theorem 7 *Let* $f \in \mathcal{C}^{n+1}([a,b]).n \in \mathbb{N}$ *and* $x \in [a,b]$ *be fixed ,such that* $f^{(k)}(x) = 0, k = 1, \cdots, n,$ *then*

$$\left|\frac{1}{b-a}.\int_a^b f(y)dy - f(x)\right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}.\left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a}\right) \tag{2.13}$$

Inequality (2.13) comes immediately from (2.11) next we prove the sharpness of inquality (2.11). When n is odd :Notice that $f^{*(k)}(x) = 0, k = 0, 1, 2, \cdots, n$ and

$$f^{*(k)}(y) = (n+1)!(b-a)$$
$$f^*(y) = (y-x)^{n-1}(b-a)$$
$$f^{*'}(y) = (n-1)(b-a)(y-x)^{n-2}$$
$$f^{(2)*}(y) = (n-1)(n-2)(b-a).(y-x)^{n-3}$$
$$....f^{(n)*}(y) = (n-1)(n-2)\cdots(n+1)(y-x)^n$$

When $\|f\|_\infty = sup(|f|)$

$$f^{*(n+1)} = (n+1)!.(b-a)$$

$$\|f^{*(n+1)}\|_\infty = (n+1)!.(b-a)$$

plugging f^* into (2.13), we get

$$\begin{aligned}
L.H.S &= \left| \frac{1}{b-a}. \int_a^b f^*(y)dy - f^*(x) \right| \\
&= \left| \frac{1}{b-a} \int_a^b f^*(y)dy \right| \\
&= \frac{1}{b-a} \int_a^b |y-x|^{n+1}.(b-a)dy \\
&= \left[\frac{1}{n+2}(y-x)^{n+2} \right]_a^b \\
&= \frac{1}{n+2} |(b-x)^{n+2} - (x-a)^{n+2}| \\
&= \frac{(b-a)^{n+2} - (x-a)^{n+2}}{n+2} \qquad (2.14)
\end{aligned}$$

$\|f^{(n+1)}\|_\infty = (n+1)!(b-a)$.

Also

$$\begin{aligned}
R.H.S &= \frac{\|f_\infty^{(n+1)}\|}{(n+2)!} \frac{((x-a)^{n+2} + (b-x)^{n+2})}{b-a} \\
&= \frac{(n+1)!(b-a)}{(n+2)!} \frac{((x-a)^{n+2} + (b-x)^{n+2}}{b-a} \\
&= \frac{(n+1)!}{(n+1)!(n+2)} ((x-a)^{n+2} + (b-x)^{n+2}) \\
&= \frac{(x-a)^{n+2} + (b-x)^{n+2}}{n+2} \qquad (2.15)
\end{aligned}$$

from (2.14) and (2.15),when n is odd inequality (2.15) was proved to be sharp, in partcular attained by f^*.

When n is even :Notice that

$$
\begin{aligned}
f^{(k)}(x) &= 0, k = 0, 1, 2, ..., n \\
f(y) &= |y-x|^{n+\alpha}.(b-a) \\
f'(y) &= (n+\alpha).(b-a).|y-x|^{(n+\alpha-1)} \\
f^{(2)}(y) &= (n+\alpha).(n+\alpha-1).(b-a).|y-x|^{n+\alpha-2} \\
f^{(n)}(y) &= (n+\alpha).(n+\alpha-1)\cdots(\alpha+1).(b-a)(y-x)^{\alpha} \\
f^{n+1}(y) &= (n+\alpha).(n+\alpha-1)\cdots(\alpha+1).\alpha.(b-a)(y-x)^{\alpha-1} \\
f^{n+1}(y) &= (\prod_{j=0}^{n}(n+\alpha-j)).|y-x|^{\alpha-1}.(b-a) \\
&= (\prod_{j=0}^{n}(n+\alpha-j)).((b-x, x-a))^{\alpha-1}.(b-a)
\end{aligned}
$$

and

$$
\begin{aligned}
&= \sup(\prod_{j=0}^{n}(n+\alpha-j)).((b-x, x-a))^{\alpha-1}.(b-a) \\
\|f^{(n+1)}\|_{\infty} &= (\prod_{j=0}^{n}(n+\alpha-j))\max((b-x, x-a))^{\alpha-1}.(b-a)
\end{aligned}
$$

Consequently we have:

$$
\begin{aligned}
R.H.S &= \frac{\|f^{n+1}\|_{\infty}}{(n+2)!}\,\frac{(x-a)^{n+2}+(b-x)^{n+2}}{b-a} \\
R.H.S &= \frac{(\prod_{j=1}^{n}(n+\alpha-j))\max((b-x, x-a))^{\alpha-1}.(b-a)}{(n+2)!}\,\frac{(x-a)^{n+2}+(b-x)^{n+2}}{(b-a)} \\
\lim_{\alpha\to 1} R.H.S &= \lim_{\alpha\to 1}\frac{(\prod_{j=1}^{n}(n+\alpha-j))\max((b-x, x-a))^{\alpha-1}.(b-a)}{(n+2)!}.\frac{(x-a)^{n+2}+(b-x)^{n+2}}{(b-a)} \\
&= \frac{(n+1)!}{(n+1)!(n+2)}.((x-a)^{n+2}+(b-x)^{n+2}) \\
&= \frac{(x-a)^{n+2}+(b-x)^{n+2}}{n+2} \qquad (2.16)
\end{aligned}
$$

and

$$\begin{aligned}
L.H.S &= \left| \frac{1}{b-a} . \int_a^b f(y)dy - f(x) \right| \\
&= \frac{1}{b-a} . \int_a^b f(y)dy \\
&= \frac{1}{b-a} . \int_a^b |y-x|^{n+\alpha} .(b-a)dy \\
&= \int_a^b |y-x|^{n+\alpha} dy \\
&= \left[\frac{1}{n+\alpha+1} .(y-x)^{n+\alpha+1} \right]_a^b \\
&= \frac{1}{n+\alpha+1} .((b-x)^{n+\alpha+1} - (a-x)^{n+\alpha+1}) \\
&= \frac{(b-x)^{n+\alpha+1} + (x-a)^{n+\alpha+1}}{n+\alpha+1} \\
\lim_{\alpha \to 1} L.H.S &= \lim_{\alpha \to 1} \frac{(b-x)^{n+\alpha+1} + (x-a)^{n+\alpha+1}}{n+\alpha+1} \\
\lim_{\alpha \to 1} L.H.S &= \frac{(x-a)^{n+2} + (b-x)^{n+2}}{n+2} \qquad (2.17)
\end{aligned}$$

from (2.16)and (2.17) we get that (2.13) is sharp also when n is even

Note that when $x = a$ or $x = b$ and n is even,inequality (2.13)can be attained by

$$f_a(y) := (y-a)^{n+1}.(b-a)$$

,

$$f_b(y) := (y-b)^{n+1}(b-a)$$

respectively (then both sides of (2.12) are equal to
$\frac{(b-a)^{n+2}}{n+2}$.

When $x = \frac{a+b}{2}$, we have a case of special interest which is described next.

Theorem 8 *Let* $f \in \mathcal{C}^{n+1}([a,b]).n \in \mathbb{N}$ *such that*
$f^{(k)}\left(\frac{a+b}{2}\right) = 0$ *forall* k *even* $\in \mathbb{N}$ *and* $\in \{1, ..., n\}$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(y)dy - f\left(\frac{a+b}{2}\right) \right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!} . \frac{(b-a)^{n+1}}{2^{n+1}} \qquad (2.18)$$

Inequality (2.18) is sharp .Namely ,when n *is odd it is attained by*

$$f^*(y) := \left(y - \frac{a+b}{2}\right)^{n+1} .(b-a).$$

While when is even the optimal function is

$$f(y) := \left| y - \frac{a+b}{2} \right|^{n+\alpha} .(b-a), \quad \alpha > 1.$$

Corollary 9 *Let* $f \in C^2([a,b])$ *such that* $f''\left(\frac{a+b}{2}\right) = 0$. *Then*

$$\left| \frac{1}{b-a} . \int_a^b f(y)dy - f\left(\frac{a+b}{2}\right) \right| \leq \|f''\|_\infty . \frac{(b-a)^2}{24}$$

Proof 4 . *For* $n = 1$ *in (2.18)*

$$\begin{aligned} \left| \frac{1}{b-a} . \int_a^b f(y)dy - f\left(\frac{a+b}{2}\right) \right| &\leq \frac{\|f^{(1+1)}\|_\infty}{(1+2)!} . \frac{(b-a)^{(1+1)}}{2^{(1+1)}} \\ &\leq \frac{\|f^{(2)}\|_\infty (b-a)^2}{24} \end{aligned} \tag{2.19}$$

While the assumption $f^{(k)}(\frac{a+b}{2}) = 0$ *forall* k *even* $\in [1, ..., n]$

$$\begin{aligned} . \left| \frac{1}{b-a} \int_a^b f(y)dy - f\left(\frac{a+b}{2}\right) \right| &\leq \frac{1}{b-a} . \frac{\|f^{(n+1)}\|_\infty}{(n+2)!} . \frac{(b-a)^{n+2}}{2^{n+1}} \\ &\leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!} \frac{(b-a)^{n+1}}{2^{n+1}} \end{aligned}$$

Next we prove the sharpness of inequality (2.18) when n *is odd ,we notice that* $f^{*(k)}\left(\frac{a+b}{2}\right) = 0$, *for* $k = 0$ *and all* k *even in* $\{1, ..., n\}$. *And for there more*

$$\begin{aligned} f^*(y) &= \left(y - \frac{a+b}{2}\right)^{n+1} .(b-a) \\ f'^*(y) &= (n+1)\left(y - \frac{a+b}{2}\right)^{n} .(b-a) \\ f^{*(2)}(y) &= n(n+1)\left(y - \frac{a+b}{2}\right)^{n-1} .(b-a) \\ f^{*(n+1)} &= (n+1)!.(b-a) \\ \|f^{*(n+1)}\|_\infty &= \sup[(n+1)!(b-a)] \end{aligned}$$

thus $\|f^{*(n+1)}\|_\infty = (n+1)!(b-a)$

$$\begin{aligned}
R.H.S &= \frac{\|f^{*(n+1)}\|_\infty}{(n+2)!}\frac{(b-a)^{n+1}}{2^{n+1}} \\
&= \frac{(n+1)!(b-a)}{(n+2)!}\frac{(b-a)^{n+1}}{2^{n+1}} \\
&= \frac{(n+1)!(b-a)^{n+2}}{(n+1)!(n+2).2^{n+1}} \\
&= \frac{(b-a)^{n+2}}{(n+2)2^{n+1}} \qquad (2.20)
\end{aligned}$$

Also we find

$$\begin{aligned}
L.H.S &= \left| \frac{1}{b-a}.\int_a^b f^*(y)dy - f^*\left(\frac{a+b}{2}\right)\right| \\
&= \left| \frac{1}{b-a}\int_a^b f^*(y)dy\right| \\
&= \left| \frac{1}{b-a}.\int_a^b \left(y-\frac{a+b}{2}\right)^{n+1}.(b-a)dy\right| \\
&= \int_a^b \left(y-\frac{a+b}{2}\right)^{n+1} dy \\
&= \left[\frac{1}{n+2}.\left(y-\frac{a+b}{2}\right)^{n+2}\right]_a^b \\
&= \frac{1}{n+2}\left[\left(b-\frac{a+b}{2}\right)-\left(a-\frac{a+b}{2}\right)\right]^{n+2} \\
&= \frac{1}{n+2}.\left[\frac{2b-a+b-2a+a-b}{2}\right]^{n+2} \\
&= \frac{2.(b-a)^{n+2}}{(n+2).2^{n+2}} \\
&= \frac{(b-a)^{n+2}}{(n+2)2^{n+1}} \qquad (2.21)
\end{aligned}$$

From (2.20) and (2.21), we get that (2.18) is attained by f^**, therefore (2.18) has been proved as sharp when* n *is odd.*

When n *is even :we notice that furthemore*

$$f^{(k)}\left(\frac{a+b}{2}\right),$$

for $k = 0$ *and all* k *even* $\in [1, ..., n]$,

$$f^{(n+1)}(y) = \prod_{j=0}^{n}(n+\alpha-j).\left|y - \frac{a+b}{2}\right|.sign\left(y - \frac{a+b}{2}.(b-a)\right)$$

and

$$\|f^{n+1}\|_{\infty} = sup f^{(n+1)}(y)$$
$$\|f^{(n+1)}\|_{\infty} = \left(\prod_{j=0}^{n}(n+\alpha-J)\right).\left(\frac{b-a}{2}\right)^{\alpha-1}(b-a)$$

thus

$$R.H.S = \frac{\|f^{(n+1)}\|_{\infty}}{(n+2)!}.\frac{(b-a)^{(n+1)}}{2^{(n+1)}}$$
$$= \frac{\left(\prod_{j=1}^{n}(n+\alpha-j)\right)\left(\dfrac{b-a}{2}\right)^{\alpha-1}(b-a)}{(n+2)!}\frac{(b-a)^{n+1}}{2^{n+1}}$$
$$\lim_{\alpha\to 1} R.H.S = \frac{\prod_{j=0}^{n}(n+\alpha-j)\left(\dfrac{b-a}{2}\right)^{1-1}.(b-a)^{(n+2)}}{(n+2)!\,2^{(n+1)}}$$
$$\lim_{\alpha\to 1} R.H.S = \frac{(b-a)^{n+2}}{(n+2)\,2^{n+1}} \tag{2.22}$$

Also we find

$$L.H.S = \left|\frac{1}{b-a}\int_a^b f(y)dy - f\left(\frac{a+b}{2}\right)\right|dy$$
$$= \frac{1}{b-a}.\int_a^b |y - \frac{a+b}{2}|^{n+\alpha}(b-a)dy$$
$$= \int_a^b \left|y - \frac{a+b}{2}\right|^{n+\alpha} dy$$
$$= \int_a^{\frac{a+b}{2}}\left(y - \frac{a+b}{2}\right)^{n+\alpha} dy + \int_{\frac{a+b}{2}}^{b}\left|\frac{a+b}{2} - y\right|^{n+\alpha}$$
$$= \frac{2\left(\frac{b-a}{2}\right)^{n+\alpha+1}}{n+\alpha+1}$$
$$\lim_{\alpha\to 1} L.H.S = \frac{(b-a)^{n+2}}{2^{n+1}(n+2)} \tag{2.23}$$

Also we find from (2.22) and (2.23) we have established that inequality (2.18) is sharp again when n is even

2.3 Applications of Ostrowski's Inequality

In this section, we give somme estimations of error bounds for some sepecial means and for some numerical quadrature rules for theorem (5)

2.3.1 Applications to Some Special Means

we first discuss the application of (2.2) to lower and upper bounds estimation of some important relationships between the following means.

(a)the arthmetic mean:

$$A = A(a,b) := \left(\frac{a+b}{2}\right)$$

(b)the geometric mean:

$$G = G(a,b) := \sqrt{ab}, a, b \geq 0$$

. (c)the harmonic mean:

$$H = H(a,b) := \frac{2}{(\frac{1}{a}) + (\frac{1}{b})}, \quad a, b > 0$$

(d)The logarithmic mean:

$$L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, \text{if} & a \neq b \\ a, \text{if} & a = b \end{cases} a, b > 0$$

(e) the identric mean:

$$I = I(a,b) \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\left(\frac{1}{b-a}\right)}, \text{if} & a \neq b, \\ a, \text{if} & a = b \end{cases} a, b > 0$$

(f) The $p-$ logarithmic mean:

$$L_p = L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \text{if} & a \neq b, p \in \mathbb{R}/[-1,0], \\ a, \text{if} & a = b, \end{cases} a, b > 0$$

These means are often used in numerical approximation and in other areas. However ,only the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in R$ with $L_0 = I$ and $L-1 = L$ We now derive some sophisticated bounds for some differences and ratios of the above means . These bounds are very useful in application .our discussion is based on the following three mappings **Case 1.**$f(x) = x^p$ with $p \in \mathbb{R}/[-1, 0]$.substituting this f into (2.2),we have ,for all $x \in [a, b]$, $f(x) = x^p$

$$|x^p - L_p^p| \leq \left[\frac{1}{4} + \frac{(x-A)^2}{(b-a)^2}\right](b-a)\gamma_p(a, b) \tag{2.24}$$

$$A(a, b) = \frac{a+b}{2}$$

with

$$\gamma_p(a, b) = \begin{cases} pb^{p-1}, \text{if} & p \geq 1, \\ |p|a^{p-1}, \text{if} p & \in (-\infty, 0) \cup (0, 1)/[-1], \end{cases}$$

if we further choose $x = A$ in 2.24,then we have

$$|A^p - L_p^p| \leq \left[\frac{1}{4} + 0\right](b-a)\gamma_p(a, b)$$

$$|A^p - L_p^p| \leq \frac{b-a}{4}\gamma_p(a, b)$$

when $p \geq 1$,the above inequality reduces to

$$0 \leq L_p^p - A^p \leq \frac{p(b-a)b^{p-1}}{4}$$

when $p \geq 1$

$$\gamma_p(a, b) = Pb^{-p-1}$$

$$0 \leq L_p^p - A^p \leq \frac{b-a}{4}(pb^{p-1})$$

$$0 \leq L_p^p - A^p \leq \frac{p(b-a)b^{p-1}}{4}$$

and when $p \in (-\infty, 0)/[-1]$

$$0 \leq L_p^p - A^p \leq \frac{(a-b)pa^{p-1}}{a}$$

$$0 \leq L_p^p - A^p \leq \frac{b-a}{4}(|p|a^{p-1})$$

$$|p|a^{p-1} = \begin{cases} -p, \text{if} & (-\infty, 0) \\ p, \text{if} & (0, +\infty) \end{cases}$$

$$0 \leq L_p^p - A^p \leq \frac{-(b-a)Pa^{p-1}}{4}$$

$$0 \leq L_p^p - A^p \leq \frac{p(a-b)a^{p-1}}{4}$$

Further more if $p \in (0,1)$then we have,

$$0 \leq A^p - L_p^p \leq \frac{(b-a)|p|a^{p-1}}{4}$$

when

$$|p|a^{p-1} = \begin{cases} -pa^{p-1}, \text{if} & p \in (-\infty, 0), \\ pa^{p-1}, \text{if} & p \in (0,1) \end{cases}$$

$$0 \leq A^p - L_p^p \leq \frac{(b-a)pa^{p-1}}{4}.$$

Now choosing x=I in (2.24),we get

$$I = I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \text{if} & a \neq b \\ a, \text{if} & a = b, \end{cases}$$

$$a, b > 0$$

$$|I^p - L_p^p| \leq \left[\frac{1}{4} + \frac{(I-A)^2}{(b-a)^2}\right].(b-a)\gamma_{p(a,b)}$$

Furthemore,if we choose $x = L$ and $x = G$ in (2.24),we have,respectively,

$$|L^p - L_p^p| \leq \left[\frac{1}{4} + \frac{(L-A)^2}{(b-a)^2}\right](b-a)\gamma_p(a,b)$$

and

$$|G^p - L_p^p| \leq \left[\frac{1}{4} + \frac{(G-A)^2}{(b-a)^2}\right](b-a)\gamma_p(a,b)$$

Case2. $f(x) = \dfrac{1}{x}$ substituting this f into (2.2),we obtain

$$|L - x| \leq \frac{xL(b-a)}{a^2}\left[\frac{1}{4} + \frac{(x-A)^2}{(b-a)^2}\right], \forall x \in [a,b]. \tag{2.25}$$

Now,taking $x = A, x = I, x = G$, and $x = H$, respectively, in (2.25), we have the following bounds for the differences of the means:

$$\leq A - L \leq \frac{AL(b-a)}{4a^2}, \leq I - L$$
$$\leq \frac{IL(b-a)}{a^2}\left[\frac{1}{4} + \frac{(I-A)^2}{(b-a)^2}\right],$$
$$\leq L - G \leq \frac{GL(b-a)}{a^2}\left[\frac{1}{4} + \frac{(G-A)^2}{(b-a)^2}\right],$$

and

$$0 \leq L - H \leq \frac{HL(b-a)}{a^2}\left[\frac{1}{4} + \frac{(H-A)^2}{(b-a)^2}\right]$$

Case3. $f(x) = -\ln x$ substituting this f into(2.2),we get

$$|lnI - lnx| \leq \frac{b-a}{a}\left[\frac{1}{4} + \frac{(x-A)^2}{(b-a)^2}\right], x \in [a,b] \tag{2.26}$$

Analogous to the previous cases,taking $x = A, x = I, x = G$, and $x = H$, respectively in (2.26),we obtain the estimates for the ratios of the means as follows:

$$1 \leq \frac{A}{I} \leq exp\left(\frac{1}{4(b-a)}\right)$$
$$1 \leq \frac{I}{L} \leq exp\left\{\frac{b-a}{a}\left[\frac{1}{4} + \frac{(L-A)^2}{(b-a)^2}\right]\right\}$$
$$1 \leq \frac{I}{G} \leq exp\left\{\frac{b-a}{a}\left[\frac{1}{4} + \frac{(G-A)^2}{(b-a)^2}\right]\right\}$$
$$1 \leq \frac{I}{H} \leq exp\left\{\frac{b-a}{a}\left[\frac{1}{4} + \frac{(H-A)^2}{(b-a)^2}\right]\right\}$$

Chapter 3

OSTROWSKI'S INEQUALITIES VIA S-CONVEX FUNCTIONS

3.1 Some Related Results

In all the following I means an interval of $\mathbb{R}$, of which the interior is noted by I^o , $a, b \leq I^o$ with $a < b$, $L_1([a, b])$ the space of integrable functions on $[a, b]$, $s \in (0, 1]$.

Theorem 10 *[16] Suppose that $f : [0, \infty[. \to [[0, \infty[$ is an s-convex function in the second sence where $s \in (0, 1)$ and let $a, b \in [0, \infty[$, $a < b$ if $f \in L^1([a, b])$ then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{3.1}$$

Lemma 3.1.1 *[16] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o where $a, b \in I$ with $a < b$. If $f' \in L^[1][a, b]$, then the following equality :*

$$f(x) - \frac{1}{b-a} \int_a^b f(u)du = (a-b) \int_0^1 p(t) f'(ta + (1-t)b)dt \tag{3.2}$$

holds for each $t \in [0, 1]$, where

$$p(t) = \begin{cases} t, & , \quad t \in \left[0, \dfrac{b-x}{b-a}\right] \\ t-1, & , \quad t \in \left(\dfrac{b-x}{b-a}, 1\right] \end{cases}$$

for all $x \in [a, b]$.

Theorem 11 *[16] Let $f : I \subset [0,\infty[. \to \mathbb{R}$ a differenttiable function on I^o such that $f' \in L^1([a,b])$. where $a,b \in I$ with $a<b$. If $|f'|$ is s-convex in the second sense on $[a,b]$, for some fixed $s \in]0.1]$, then the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq \frac{b-a}{(s+1)(s+2)}\left\{\left[2(s+1)\left(\frac{b-x}{b-a}\right)^{s+2}-(s+2)\left(\frac{b-x}{b-a}\right)^{s+1}+1\right]|f'(a)|\right\}$$
$$+\frac{b-a}{(s+1)(s+2)}\left\{\left[2(s+1)\left(\frac{x-a}{b-a}\right)^{s+2}-(s+2)\left(\frac{x-a}{b-a}\right)^{s+1}+1\right]|f'(b)|\right\} \quad (3.3)$$

For $x \in [a,b]$.

Proof 5 *by lemma (3.1.1) and since $|f'|$ is s-convexe on $[a,b]$, then we have:*

$$\begin{aligned}
|f(x)-\frac{1}{b-a}\int_a^b f(u)du| &\leq (b-a)\int_0^{\frac{b-x}{b-a}} t|f'(ta+(1-t)b)|dt \\
&+ (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1||f'(ta+(1-t)b)|dt \\
&\leq (b-a)\int_0^{\frac{b-x}{b-a}} t(t^s|f'(a)|+(1-t)^s|f'(b)|)dt \\
&+ (b-a)\int_{\frac{b-x}{b-a}}^1 (1-t)(t^s|f'(a)|+(1-t)^s|f'(b)|)dt \\
&= (b-a)\left\{|f'(a)|\int_0^{\frac{b-x}{b-a}} t^{s+1}dt+|f'(b)|\int_0^{\frac{b-x}{b-a}} t(1-t)^s dt\right\} \\
&+ (b-a)\left\{|f'(a)|\int_{\frac{b-x}{b-a}}^1 (t^s-t^{s+1})dt+|f'(b)|\int_{\frac{b-x}{b-a}}^1 (1-t)^{s+1}dt\right\} \\
&= \frac{b-a}{(s+1)(s+2)}\left\{\left[2(s+1)\left(\frac{b-x}{b-a}\right)^{s+2}-(s+2)\left(\frac{b-x}{b-a}\right)^{s+1}+1\right]|f'(a)|\right\} \\
&+ \frac{b-a}{(s+1)(s+2)}\left\{\left[s\left(\frac{x-a}{b-a}\right)^{s+2}-(s+2)\frac{b-x}{b-a}\left(\frac{x-a}{b-a}\right)^{s+1}+1\right]|f'(b)|\right\}
\end{aligned}$$

where we use the fact

$$\int_0^{\frac{b-x}{b-a}} t^{s+1}dt = \frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}$$

$$\int_0^{\frac{b-x}{b-a}} t(1-t)^s dt = \frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}-\frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1}+\frac{1}{(s+1)(s+2)}$$

$$\int_{\frac{b-x}{b-a}}^{1}(t^s-t^{s+1})dt=\frac{1}{(s+1)(s+2)}-\frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1}+\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}$$

$$\int_{\frac{b-x}{b-a}}^{1}(1-t)^{s+1}dt=\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}$$

Which completes the proof.

Corollary 12 *[16] In Theorem (11), if we take* $x=\dfrac{a+b}{2}$*, then we have the following midpoint inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{(s+1)(s+2)}\left(1-\frac{1}{2^{s+1}}\right)[|f'(a)+f'(b)|]. \quad (3.4)$$

Remark 13 *[16] In corollary (12),if* $s=1$*, then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$

Theorem 14 *[16] Let* $f:I\subset[0,\infty[\to\mathbb{R}$ *a function differenttiable on* I^o *such that* $f'\in L^1([a,b])$*,where* $a,b\in I$ *with* $a<b$*.If* $|f'|^q$ *is s-convex on* $[a,b]$*, such as* $q>1$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$*, then the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq(b-a)\frac{1}{(p+1)^{\frac{1}{p}}}\frac{1}{(q+1)^{\frac{1}{q}}}$$
$$\left\{\left(\frac{b-x}{b-a}\right)^{1+\frac{1}{p}}\left(\left(\frac{b-x}{b-a}\right)^{s+1}|f'(a)|^q+\left[1-\left(\frac{x-a}{b-a}\right)^{s+1}\right]|f'(b|^q\right)^{\frac{1}{q}}\right\}$$
$$+b-a)\frac{1}{(p+1)^{\frac{1}{p}}}\frac{1}{(q+1)^{\frac{1}{q}}}$$
$$\left\{\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{p}}\left(\left[1-\left(\frac{b-x}{b-a}\right)^{s+1}\right]|f'(a)|^q+\left(\frac{x-a}{b-a}\right)^{s+1}|f'(b)|^q\right)^{\frac{1}{q}}\right\}$$

holds for each $x\in[a,b]$.

Proof 6 *Suppose that* $p>1$*. From lemma (3.1.1) and using the Hôlder Inequality,* *we have:*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| &\leq (b-a)\int_0^{\frac{b-x}{b-a}} t|f'(ta+(1-t)b)|dt \\ &+(b-a)\int_{\frac{b-x}{b-a}}^1 |t-1||f'(ta+(1-t)b)|dt \\ &\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\ &+(b-a)\left(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\end{aligned} \tag{3.5}$$

Using the s-convexity of $|f'|^q$,we obtain

$$\begin{aligned}\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt &\leq \int_0^{\frac{b-x}{b-a}} [t^s|f'(a)|^q+(1-t)^s|f'(b)|^q]dt \\ &= \frac{1}{s+1}\left\{\left(\frac{b-x}{b-a}\right)^{s+1}|f'(a)|^q+\left[1-\left(\frac{b-x}{b-a}\right)^{s+1}\right]|f'(b)|^q\right\}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q dt &\leq \int_{\frac{b-x}{b-a}}^1 [t^s|f'(a)|^q+(1-t)^s|f'(b)|^q]dt \\ &= \frac{1}{s+1}\left\{\left[1-\left(\frac{b-x}{b-a}\right)^{s+1}\right]|f'(a)|^q+\left(\frac{x-a}{b-a}\right)^{s+1}|f'(b)|^q\right\}\end{aligned} \tag{3.7}$$

Further,we have

$$\int_0^{\frac{b-x}{b-a}} t^p dt = \frac{1}{(p+1)}\left(\frac{b-x}{b-a}\right)^{p+1} \tag{3.8}$$

and

$$\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt = \frac{1}{(p+1)}\left(\frac{x-a}{b-a}\right)^{p+1}. \tag{3.9}$$

A combination of (3.6)-(3.9) gives the required inequality (3.5).

Remark 15 *In Theorem(14) if we choose $x=\frac{a+b}{2}$ and $s=1$, then we have*

$$\begin{aligned}&\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(u)du\right| \\ &\leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}[(|f'(a)|^q+3|f'(b)|^q)^{\frac{1}{q}}+(3|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}}]\end{aligned}$$

Theorem 16 *[16] Let $f : I \subset [0,\infty[\to \mathbb{R}$ be a differenttiable function on I^o such that $f' \in L^1([a,b])$, where $a,b \in I$,with $a < b$. If $|f'|^q$ is s-convex on $[a;b]$ such as $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}$$
$$\left\{\left(\frac{b-x}{b-a}\right)^2\left(\frac{|f'(x)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{x-a}{b-a}\right)^2\left(\frac{|f'(a)|^q + |f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right\}$$

for each $x \in [a,b]$.

Proof 7 *Suppose that $p > 1$. From lemma(3.1.1)and using the Hôlder inequality ,we have:*

$$\begin{aligned}|f(x) - \frac{1}{b-a}\int_a^b f(u)du| &\leq (b-a)\int_0^{\frac{b-x}{b-a}} t|f'(ta+(1-t)b)|dt \\ &+ (b-a)\int_{\frac{b-x}{b-a}}^1 |t-1||f'(ta-(1-t)b)|dt \\ &\leq (b-a)\left(\int_0^{\frac{b-x}{b-a}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\ &+ (b-a)(\int_{\frac{b-x}{b-a}}^1 (1-t)^p dt)^{\frac{1}{p}}(\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b|^q dt)^{\frac{1}{q}}\end{aligned} \tag{3.10}$$

Since $|f'|^q$is s-convex,by(3.1), we have:

$$\int_0^{\frac{b-x}{b-a}} |f'(ta+(1-t)b)|^q dt \leq \frac{b-x}{b-a}\left(\frac{|f'(x)|^q + |f'(b)|^q}{s+1}\right) \tag{3.11}$$

$$\int_{\frac{b-x}{b-a}}^1 |f'(ta+(1-t)b)|^q dt \leq \frac{x-a}{b-a}\left(\frac{|f'(a)|^q + |f'(x)|^q}{s+1}\right) \tag{3.12}$$

Therefore:

$$|f(x) - \frac{1}{b-a}\int_a^b f(u)du| \leq \frac{1}{b-a}\frac{1}{(p+1)^{\frac{1}{p}}}$$
$$\left\{(b-x)^2\left(\frac{|f'(x)|^q + |f'(b)|^q}{s+1}\right)^{\frac{1}{q}} + (x-a)^2\left(\frac{|f'(a)|^q + |f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right\}$$

Where $\frac{1}{p}+\frac{1}{q}=1$. *Also we note that*

$$\int_0^{\frac{b-x}{b-a}} t^p dt = \frac{1}{p+1}\left(\frac{b-x}{b-a}\right)^{p+1}$$

and

$$\int_{\frac{b-x}{b-a}}^1 t^p dt = \frac{1}{p+1}\left(\frac{x-a}{b-a}\right)^{p+1}$$

This complet the proof.

Corollary 17 *In Theorem (16), if we choose* $x=\frac{a+b}{2}$, *then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{4(p+1)^{\frac{1}{p}}}$$
$$\left\{\left(\frac{|f'\left(\frac{a+b}{2}\right)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'\left(\frac{a+b}{2}\right)|^q}{s+1}\right)^{\frac{1}{q}}\right\}$$

Also assuming $f'(a)=f'\left(\frac{a+b}{2}\right)=f'(b)=$ *and* $s=1$, *we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|+|f'(b)|}{4}\right)$$

Theorem 18 *[16] Let* $f: I\subset [0,\infty[\to \mathbb{R}$ *be a differenttiable on* I^o *such that* $f'\in L^1[a,b]$. *Where* $a,b\in I$ *with* $a<b$.*if* $|f'|^q$*is s-convex on* $[a,b]$, *for some* $s\in]0,1]$. *such as* $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *then the following inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{b-a}{(p+1)^{\frac{1}{p}}}$$
$$\left[\left(\frac{b-x}{b-a}\right)^{p+1}\left(\frac{x-a}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}+\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}} \quad (3.13)$$

for each $x\in[a,b]$.

Proof 8 *Suppose that* $p>1$. *From lemma(3.1.1)and using the Hôlder inequality ,we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le (b-a)(\int_0^1 |p(t)|^p dt)^{\frac{1}{p}}(\int_0^1 |f'(ta+(1-t)b)|^q dt)^{\frac{1}{q}} \quad (3.14)$$

Since $|f'|^q$ *is s-convex,we have:*

$$\int_0^1 |f'(ta+(1-t)b)|^q dt \leq \int_0^1 (t^s|f'(a)|^q + (1-t)^s|f'(b)|^q)dt$$
$$= \frac{|f'(a)|^q + |f'(b)|^q}{s+1} \quad (3.15)$$

and

$$\int_0^1 |p(t)|^p dt = \int_0^{\frac{b-x}{b-a}} t^p dt + \int_{\frac{b-x}{b-a}}^1 (1-t)^p dt$$
$$\frac{1}{p+1}\left[\left(\frac{b-x}{b-a}\right)^{p+1} + \left(\frac{x-a}{b-a}\right)^{p+1}\right] \quad (3.16)$$

Using (3.15)and(3.16)in (3.14),we obtain (3.13)

Corollary 19 *[16] Under the assumptions of Theorem (18), further if we suppose that* $p = q = 2$, *we get*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{b-a}{\sqrt{3}}\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]^{\frac{1}{2}} \left(\frac{|f'(a)|^2 + |f'(b)|^2}{s+1}\right)^{\frac{1}{2}}$$

More if we take $\frac{a+b}{2}$ *and* $s = 1$, *we find*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M(b-a)}{\sqrt{3}}\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]^{\frac{1}{2}}$$

Remark 20 *Under the hypotheses of Theorem (18), if moreover we suppose that*$p = q = 2, |f'| \leq M, M > 0$ *et* $s = 1$,*then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M(b-a)}{\sqrt{3}}\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

Chapter 4

OSTROWSKI TYPE INEQUALITIES VIA FRACTIONAL CALCULUS

4.1 Ostrowski Type Inequalities Via Riemann-Liouville Integrals

Fractional calculus deals with the study of integral and dierential operators of non-integral order.Many mathematicians like Liouville,Riemann and weyl made major contributions to the theory of fractional calculs.the study on the fractional calculus continued with contributions from Fourier,Abel,Lcroix,Leibniz,Grunwald and Letnikov,(for more details (see,[20],21,22,24,25,26).Riemann-Liouville fractional integral operator is the first formulation of non-integral order.In this section we present with proofs some fractional Ostrowski Type Inequalities involving the Riemann-Liouville fractional integrals.

Definition 4.1.1. [23]Let $f \in L_1[a,b]$. Then the Riemann-Liouville fractional integrals left and right of f of order $\alpha > 0$ with are defined by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

$$I_{b_-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b$$

Theorem 21 *Under the assumptions of theorem(5), we have*

$$|f(x)((b-x)^{\beta} + (x-a)^{\alpha}) - (\Gamma(\beta+1)I_{b}^{\beta} - f(x) + \Gamma(\alpha+1)I_{a+}^{\alpha})|$$
$$\leq M\left(\frac{\beta}{\beta+1}(b-x)^{\beta+1} + \frac{\alpha}{\alpha+1}(x-a)^{\alpha+1}\right), x \in [a,b] \quad (4.1)$$

Proof 9 *For* $t \in [a,x], \alpha > 0,$ *we have*

$$(x-t)^{\alpha} \leq (x-a)^{\alpha} \quad (4.2)$$

Under given condition on f' *and by (4.2),we have*

$$\int_a^x (M - f'(t))(x-t)^{\alpha} dt \leq (x-a)^{\alpha} \int_a^x (M - f'(t))dt$$

$$= \int_a^x M(x-t)^{\alpha} dt - \int_a^x f'(t)(x-t)^{\alpha} dt \leq (x-a)^{\alpha} \int_a^x (M - f'(t))dt$$

integrating and simplifying

$$\int_a^x M(x-t)^{\alpha} dt = \frac{-M}{\alpha+1}[(x-t)^{\alpha+1}]_a^x$$
$$= \frac{-M}{\alpha+1}(x-a)^{\alpha+1} \quad (4.3)$$

By integrating by parts with $u = (x-t)^{\alpha}, \quad dv = f'(t),$ *we have*

$$\int_a^x f'(t)(x-t)^{\alpha} = [(x-t)^{\alpha} f(t)]_a^x - \frac{1}{\alpha}\int_a^x f(t)(x-t)^{\alpha-1}$$
$$= -(x-a)^{\alpha} f(x) + \frac{1}{\alpha}\int_a^x (x-t)^{\alpha-1} f(t) \quad (4.4)$$

$$(x-a)^{\alpha}. \int_a^x (M - f'(t))dt = (x-a)^{\alpha}.[Mt - f(t)]_a^x$$
$$= M(x-a)^{\alpha+1} - f(x)(x-a)^{\alpha} + f(a)(x-t)^{\alpha} \quad (4.5)$$

by (4.3),(4.4),(4.5),we already

$$\frac{-M}{\alpha+1}(x-a)^{\alpha+1} + f(a)(x-a)^{\alpha} - \frac{1}{\alpha}\int_a^x f(t)(x-t)^{\alpha-1} dt \leq M(x-a)^{\alpha+1} - f(x)(x-a)^{\alpha} + f(a)(x-a)^{\alpha}$$

$$-\frac{M}{\alpha+1}(x-a)^{\alpha+1}-\alpha\Gamma(\alpha)I_{a^+}^{\alpha}f(x)+f(x)(x-a)^{\alpha}\leq M(x-a)^{\alpha+1}$$

$$f(x)(x-a)^{\alpha}-\Gamma(\alpha+1)I_{a^+}^{\alpha}f(x)\leq\frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}$$

and

$$\int_a^x(M+f'(t))(x-t)^{\alpha}dt\leq(x-a)^{\alpha}\int_a^x(M+f'(t))dt$$

$$=\int_a^x M(x-t)^{\alpha}dt+\int_a^x f'(t)(x-t)^{\alpha}dt\leq(x-a)^{\alpha}\int_a^x(M+f'(t))dt$$

integrating and simplifying

$$\begin{aligned}\int_a^x M(x-t)^{\alpha}dt&=\frac{-M}{\alpha+1}[(x-t)^{\alpha+1}]_a^x\\&=\frac{-M}{\alpha+1}(x-a)^{\alpha+1}\end{aligned}\tag{4.6}$$

$$\begin{aligned}\int_a^x f'(t)(x-t)^{\alpha}&=[(x-t)^{\alpha}f(t)]_a^x-\frac{1}{\alpha}\int_a^x f(t)(x-t)^{\alpha-1}\\&=-(x-a)^{\alpha}f(a)+\frac{1}{\alpha}\int_a^x(x-t)^{\alpha-1}f(t)\end{aligned}\tag{4.7}$$

$$\begin{aligned}(x-a)^{\alpha}.\int_a^x(M+f'(t))dt&=(x-a)^{\alpha}.[Mt+f(t)]_x^a\\&=M(x-a)^{\alpha+1}+f(x)(x-a)^{\alpha}-f(a)(x-t)^{\alpha}\end{aligned}\tag{4.8}$$

by(4.6),(4.7),(4.8), we already

$$\frac{-M}{\alpha+1}(x-a)^{\alpha+1}-f(a)(x-a)^{\alpha}+\frac{1}{\alpha}\int_a^x f(t)(x-t)^{\alpha-1}dt\leq M(x-a)^{\alpha+1}+f(x)(x-a)^{\alpha}-f(a)(x-a)^{\alpha}$$

$$-\frac{M}{\alpha+1}(x-a)^{\alpha+1}-\alpha\Gamma(\alpha)I_{a^+}^{\alpha}f(x)\leq M(x-a)^{\alpha+1}+f(x)(x-a)^{\alpha}$$

$$\Gamma(\alpha+1)I_{a^+}^{\alpha}f(x)-f(x)(x-a)^{\alpha})\leq\frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}$$

Above inequalities result we get the following inequality

$$|f(x)(x-a)^{\alpha}-\Gamma(\alpha+1)I_{a^+}^{\alpha}f(x)|\leq\frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}\tag{4.9}$$

the other hand for $t\in[x,b],\beta>0$, *we have*

$$(t-x)^{\beta}\leq(b-x)^{\beta}\tag{4.10}$$

Under given condition on f' and by (4.10),we have

$$\int_x^b (M - f'(t))(t-x)^\beta dt \leq (b-x)^\beta \int_x^b (M - f'(t))dt$$

$$= \int_x^b M(t-x)^\beta dt - \int_x^b f'(t)(t-x)^\beta dt \leq (b-x)^\beta \int_x^b (M - f'(t))dt$$

Integrating and simplfying

$$\begin{aligned} \int_x^b M(t-x)^\beta dt &= \frac{M}{\beta+1}[(t-x)^{\beta+1}]_x^b \\ &= \frac{M}{\beta+1}(b-x)^{\beta+1} \end{aligned} \tag{4.11}$$

Integrating by parts $\int_x^b f'(t)(t-x)^\beta dt =$ *with* $u = (t-x)^\beta, \quad dv = f'(t)$

$$\begin{aligned} \int_x^b f'(t)(t-x)^\beta &= [(t-x)^\beta f(t)]_x^b - \frac{1}{\beta}\int_x^b f(t)(t-x)^{\beta-1} \\ &= (b-x)^\beta f(b) - \frac{1}{\beta}\int_x^b (t-x)^{\beta-1} f(t)dt \end{aligned} \tag{4.12}$$

$$\begin{aligned} (b-x)^\beta . \int_x^b (M - f'(t))dt &= (b-x)^\beta . [Mt - f(t)]_x^b \\ &= (b-x)^\beta [M(b-x) - f(b) + f(x)] \\ &= M(b-x)^{\beta+1} - f(b)(b-x)^\beta + f(x)(b-x)^\beta \end{aligned} \tag{4.13}$$

by (4.11),(4.12),(4.13),we already

$$\frac{M}{\beta+1}(b-x)^{\beta+1} - f(b)(b-x)^\beta - \frac{1}{\beta}\int_x^b f(t)(t-x)^{\beta-1}dt \leq M(b-x)^{\beta+1} - f(b)(b-x)^\beta + f(x)(b-x)^\beta$$

$$= \frac{M}{\beta+1}(b-x)^{\beta+1} - f(b)(b-x)^\beta - \Gamma(\beta+1) I_{b^-}^\beta f(x) \leq M(b-x)^{\beta+1} - f(b)(b-x)^\beta - f(x)(b-x)^\beta$$

$$= f(x)(b-x)^\beta - \Gamma(\beta+1) I_{b^-}^{\beta+1 f(x)} \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}$$

and

$$\int_x^b (M + f'(t))(t-x)^\beta dt \leq (b-x)^\beta \int_x^b (M + f'(t))dt$$

$$= \int_x^b M(t-x)^\beta dt + \int_x^b f'(t)(t-x)^\beta dt \leq (b-x)^\beta \int_x^b (M + f'(t))dt$$

Integrating and simplfying

$$\int_x^b M(t-x)^\beta dt = \frac{M}{\beta+1}[(t-x)^{\beta+1}]_x^b$$
$$= \frac{M}{\beta+1}(b-x)^{\beta+1}. \tag{4.14}$$

We have

$$\int_x^b f'(t)(t-x)^\beta = [(t-x)^\beta f(t)]_x^b - \frac{1}{\beta}\int_x^b f(t)(t-x)^{\beta-1}dt$$
$$= (b-x)^\beta f(b) - \frac{1}{\beta}\int_x^b (t-x)^{\beta-1} f(t)dt \tag{4.15}$$

$$(b-x)^\beta . \int_x^b (M+f'(t))dt = (b-x)^\beta .[Mt+f(t)]_x^b$$
$$= (b-x)^\beta[M(b-x)+f(b)+f(x)]$$
$$= M(b-x)^{\beta+1} + f(b)(b-x)^\beta + f(x)(b-x)^\beta \tag{4.16}$$

by (4.14),(4.15),(4.16)we have

$$\frac{M}{\beta+1}(b-x)^{\beta+1} - f(b)(b-x)^\beta + \frac{1}{\beta}\int_x^b f(t)(t-x)^{\beta-1}dt \leq M(b-x)^{\beta+1} - f(b)(b-x)^\beta + f(x)(b-x)^\beta$$

$$= \frac{M}{\beta+1}(b-x)^{\beta+1} + f(b)(b-x)^\beta + \Gamma(\beta+1)I_{b-}^{\beta} f(x) \leq M(b-x)^{\beta+1} - f(b)(b-x)^\beta + f(x)(b-x)^\beta$$

$$= \Gamma(\beta+1)I_{b^-}^{\beta+1 f(x)} - f(x)(b-x)^\beta \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}$$

Above inequalities result the following inequality

$$|f(x)(b-x)^\beta - \Gamma(\beta+1)I_{b-}^{\beta} f(x)| \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1} \tag{4.17}$$

By adding (4.9)and(4.17), we get(4.1) The following more general result for a differentiable function which is bounded below as well as bounded above holds.

Theorem 22 *Let $f : I \to \mathbb{R}$where $I$is an interval in $\mathbb{R}$,be a mapping differentiable in I^o,the interior of I and $a, b \in I^o, a < b, if m < f'(t) \leq M$ for all $t, x \in [a, b]$, then we have*

$$((x-a)^\alpha - (b-x)^\beta)f(x) - (\Gamma(\alpha+1)I_{a+}^{\alpha} f(x) - \Gamma(\beta+1)I_{b-}^{\beta} f(x)) \leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1} - \frac{m\beta}{\beta+1}(b-x)^{\beta+1}.$$

and

$$((b-x)^{\alpha}-(x-a)^{\beta})f(x)-(\Gamma(\alpha+1)I_{a^+}^{\alpha}f(x)-\Gamma(\beta+1)I_{b^-}^{\beta}f(x)) \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}-\frac{m\alpha}{\alpha+1}(x-a)^{\alpha+1}$$

Where $\alpha, \beta > 0$.

Proof 10 *Proof is on the same lines just after comparing conditions on derivative of* f, *of the proof of theorem(21),let we omit it.*

In the following we have obtained a related result to fractional Ostrowski inequality (4.1)

Theorem 23 *Under the assumptions of theorem (5), we have*

$$\begin{aligned} &|((b-x)^{\beta}f(b)+(x-a)^{\alpha}f(a))-\Gamma(\beta+1)I_{x^+}^{\beta}f(b)+\Gamma(\alpha+1)I_{x^-}^{\alpha}f(a))| \\ &\qquad \leq \left(\frac{\beta}{\beta+1}(b-x)^{\beta+1}+\frac{\beta}{\alpha+1}(x-a)^{\alpha+1}\right) \end{aligned} \tag{4.18}$$

Where $\alpha, \beta > 0$.

Proof 11 *For* $t \in [a, x], \alpha > 0$, *we have*

$$(t-a)^{\alpha} \leq (x-a)^{\alpha} \tag{4.19}$$

Under given condition on f' *and by (4.19),we have*

$$\int_a^x (M-f'(t))(t-a)^{\alpha}dt \leq (x-a)^{\alpha}\int_a^x (M-f'(t))dt$$

$$= \int_a^x M(t-a)^{\alpha}dt - \int_a^x f'(t)(t-a)^{\alpha}dt \leq (x-a)^{\alpha}\int_a^x (M-f'(t))$$

Integrating and simplifying

$$\begin{aligned} \int_a^x M(t-a)^{\alpha}dt &= \left[\frac{M}{\alpha+1}(t-a)^{\alpha+1}\right]_a^x \\ &= \frac{M}{\alpha+1}(x-a)^{\alpha+1} \end{aligned} \tag{4.20}$$

We integrate by parts

$$\int_a^x f'(t)(t-a)^\alpha$$

with $u = (t-a)^\alpha, \quad dv = f'(t)$,*we get*

$$\begin{aligned}\int_a^x f'(t)(t-a)^\alpha dt &= [(t-a)^\alpha . f(t)]_a^x - \frac{1}{\alpha}\int_a^x f(t)(t-a)^{\alpha-1}.dt \\ &= (x-a)^\alpha . f(x) - \frac{1}{\alpha}\int_a^x f(t)(t-a)^{\alpha-1} \qquad (4.21)\end{aligned}$$

$$\begin{aligned}(x-a)^\alpha \int_a^x (M - f'(t))dt &= (x-a)^\alpha [Mt - f(t)]_a^x \\ &= M(x-a)^{\alpha+1} - f(x)(x-a)^\alpha + f(a)(x-a)^\alpha \qquad (4.22)\end{aligned}$$

by (4.20),(4.21),(4.22),we already

$$\frac{M}{\alpha+1}(x-a)^{\alpha+1} - (x-a)^\alpha . f(a) - \frac{1}{\alpha}\int_a^x f(t)(t-a)^{\alpha-1}dt \leq M(x-a)^{\alpha+1} - f(x)(x-a)^\alpha - f(a)(x-a)$$

$$= f(x)(x-a)^\alpha - \Gamma(\alpha+1)I_{a^+}^\alpha \leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}$$

and

$$\int_a^x (M + f'(t))(t-a)^\alpha dt \leq (x-a)^\alpha \int_a^x (M + f'(t))dt$$

$$= \int_a^x M(t-a)^\alpha dt + \int_a^x f'(t)(t-a)^\alpha dt \leq (x-a)^\alpha \int_a^x (M + f'(t))dt$$

Integrating and simplifying

$$\begin{aligned}\int_a^x M(t-a)^\alpha dt &= \left[\frac{M}{\alpha+1}(t-a)^{\alpha+1}\right]_a^x \\ &= \frac{M}{\alpha+1}(x-a)^{\alpha+1} \qquad (4.23)\end{aligned}$$

$$\begin{aligned}\int_a^x f'(t)(t-a)^\alpha dt &= [(t-a)^\alpha . f(t)]_a^x - \frac{1}{\alpha}\int_a^x f(t)(t-a)^{\alpha-1}.dt \\ &= (x-a)^\alpha . f(x) - \frac{1}{\alpha}\int_a^x f(t)(t-a)^{\alpha-1} \qquad (4.24)\end{aligned}$$

$$\begin{aligned}(x-a)^\alpha \int_a^x (M + f'(t))dt &= (x-a)^\alpha [Mt + f(t)]_a^x \\ &= M(x-a)^{\alpha+1} + f(x)(x-a)^\alpha + f(a)(x-a)^\alpha \qquad (4.25)\end{aligned}$$

by (4.23),(4.24),(4.25)

$$\frac{M}{\alpha+1}(x-a)^{\alpha+1}+(x-a)^{\alpha}.f(a)+\frac{1}{\alpha}\int_a^x f(t)(t-a)^{\alpha-1}dt \leq M(x-a)^{\alpha+1}+f(x)(x-a)^{\alpha}+f(a)(x-a)$$

$$= f(x)(x-a)^{\alpha}+\Gamma(\alpha+1)I_{a^+}^{\alpha} \leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}.$$

Above inequalities result the following inequality

$$|f(a)(x-a)^{\alpha}-\Gamma(\alpha+1)I_{x^-}^{\alpha}f(a)| \leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1} \tag{4.26}$$

Now on the other hand for $t \in [x,b], \beta > 0,$ *we have*

$$(b-t)^{\beta} \leq (b-x)^{\beta} \tag{4.27}$$

Under given condition on f' *and by 4.27,we have*

Proof 12

$$\int_x^b (M-f'(t))(b-t)^{\beta}dt \leq (b-x)^{beta}\int_x^b (M-f'(t))dt$$

$$= \int_x^b M(b-t)^{\beta}dt - \int_x^b f'(t)(b-t)^{\beta}dt \leq (b-x)^{\beta}\int_x^b (M-f'(t))dt$$

Integrating and simplifying

$$\begin{aligned}\int_x^b M(b-t)^{\beta}dt &= \left[\frac{-M}{\beta+1}(b-t)^{\beta+1}\right]_x^b \\ &= \frac{-M}{\beta+1}(b-x)^{\beta+1}\end{aligned} \tag{4.28}$$

$$\begin{aligned}\int_x^b f'(t)(b-t)^{\beta}dt &= [(b-t)^{\beta}.f(t)]_x^b + \frac{1}{\beta}\int_x^b f(t)(b-t)^{\beta-1}.dt \\ &= -(b-x)^{\beta}.f(x)+\Gamma(\beta+1)I_{x^+}^{\beta}f(b)\end{aligned} \tag{4.29}$$

$$\begin{aligned}(b-x)^{\beta}\int_x^b (M-f'(t))dt &= (b-x)^{\beta}[Mt-f(t)]_x^b \\ &= M(b-x)^{\beta+1}+f(x)(b-x)^{\beta}+f(b)(b-x)^{\beta}\end{aligned} \tag{4.30}$$

by (4.28),(4.29),(4.30),we already

$$\frac{-M}{\beta+1}(b-x)^{\beta+1}+(b-x)^{\beta}.f(x)-\Gamma(\beta+1)I_{x^+}^{\beta}f(b) \leq M(b-x)^{\beta+1}-f(b)(b-x)^{\beta}+f(x)(b-x)^{\beta}$$

$$= f(b)(b-x)^{\beta}-\Gamma(\beta+1)I_{x^+}^{\beta}f(b) \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}$$

and

$$\int_x^b (M+f'(t))(b-t)^{\beta}dt \leq (b-x)^{beta}\int_x^b (M+f'(t))dt$$

$$= \int_x^b M(b-t)^{\beta}dt + \int_x^b f'(t)(b-t)^{\beta}dt \leq (b-x)^{\beta}\int_x^b (M+f'(t))dt$$

Integrating and simplifying

$$\int_x^b M(b-t)^{\beta}dt = \left[\frac{-M}{\beta+1}(b-t)^{\beta+1}\right]_x^b$$
$$= \frac{-M}{\beta+1}(b-x)^{\beta+1} \tag{4.31}$$

so

$$\int_x^b f'(t)(b-t)^{\beta}dt = [(b-t)^{\beta}.f(t)]_x^b + \frac{1}{\beta}\int_x^b f(t)(b-t)^{\beta-1}.dt$$
$$= -(b-x)^{\beta}.f(x) + \Gamma(\beta+1)I_{x^+}^{\beta}f(b) \tag{4.32}$$

$$(b-x)^{\beta}\int_x^b (M-f'(t))dt = (b-x)^{\beta}[Mt+f(t)]_x^b$$
$$= M(b-x)^{\beta+1}+f(x)(b-x)^{\beta}-f(b)(b-x)^{\beta} \tag{4.33}$$

by (4.31),(4.32),(4.33),we already

$$\frac{-M}{\beta+1}(b-x)^{\beta+1}-(b-x)^{\beta}.f(x)+\Gamma(\beta+1)I_{x^+}^{\beta}f(b) \leq M(b-x)^{\beta+1}+f(b)(b-x)^{\beta}-f(x)(b-x)^{\beta}$$

$$= \Gamma(\beta+1)I_{x^+}^{\beta}f(b)-f(b)(b-x)^{\beta} \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}$$

Above inequalities result the following inequality

$$\left|f(b)(b-x)^{\beta}-\Gamma(\beta+1)I_{x^+}^{\beta}f(b)\right| \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1} \tag{4.34}$$

by adding (4.26) and (4.34), we get (4.18)

Some Implications. *Following implications have been observed*

Corollary 24 *If $\beta = \alpha$ in (4.1),then we leads the folowing fractional Ostrowski inequality*

$$|f(x)((b-x)^{\alpha}+(x-a)^{\alpha})-\Gamma(\alpha+1)(I_{b^-}^{\alpha}f(x)+)I_{a^+}^{\alpha}| \leq M\frac{\alpha}{\alpha+1}((b-x)^{\alpha+1}+(x-a)^{\alpha+1}), x \in [a,b]$$

Corollary 25 *I $\beta = \alpha = 1$, then we lead to the Ostrowski inequality(2.2) .*

Corollary 26 *If $\beta = \alpha$ in theorem(23), then we lead to the lead to the following inequality*

$$|((b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a))-\Gamma(\alpha+1)(I_{x^+}^{\alpha}f(b))+I_{x^-}^{\alpha}| \leq \frac{M\alpha}{\alpha+1}((b-x)^{\alpha+1}+(x-a)^{\alpha+1}), x \in [a,b] \quad (4.35)$$

where $\alpha > 0$.

Remark 27 *Following the steps of the proof of theorem(21) line by line with $\alpha = \beta = 1$ an alternative proof of the Ostrowski inequality is followed see [27].*

Remark 28 *if m is replaced with M in theorem(22), then with some rerrangements one can get theorem(21).*

Remark 29 *A more general from of theorem (23) like theorem (22) for a differentiable function which is bounded below as well as bounded above holds.*

4.2 Fractional Inequalities of Ostrowski Type For Convex Functions

Theorem 30 *[28] Letf : $I \subset [0,\infty[. \to \mathbb{R}$, be a differentiable function on I^o (the interior of I)such that $f' \in L^1[a,b]$, where $a, b \in I$, $a < b$. If $|f'|^{\frac{p}{p-1}}$ is convex on*

$[a,b]$, then the following inequality

$$|f(x) - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{1}{(b-a)(p+1)^{\frac{1}{p}}}$$
$$\left[(b-x)^2\left(\frac{|f'(x)^q| + |f'(b)|^q}{2}\right)^{\frac{1}{q}} + (x-a)^2\left(\frac{|f'(x)^q| + |f'(a)|^q}{2}\right)^{\frac{1}{q}}\right] \quad (4.36)$$

holds.

Lemma 4.2.1 *[28] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o where $a, b \in I$,with $a < b$.If $f' \in L^1[a,b]$,then for all $x \in [a,b]$ and $\alpha > 0$ we have:*

$$\left[\frac{(x-a)^\alpha + (b-x)^\alpha}{(b-a)^{\alpha+1}}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}[I_{x^+}^\alpha f(b) + I_{x^-}^\alpha f(a)] = \int_0^1 m(t)f'(ta+(1-t)b)dt$$

For each $t \in [a,b]$, where

$$p(t) = \begin{cases} -t^\alpha, & t \in \left[0, \dfrac{b-x}{b-a}\right] \\ (1-t)^\alpha, & t \in \left[\dfrac{b-x}{b-a}, 1\right] \end{cases}$$

For all $x \in [a,b]$.

Proof 13 *By integration by parts ,we can obtain*

$$\begin{aligned} I &= \int_0^1 m(t)f'(ta + (1-t))dt \\ &= \int_0^{\frac{b-x}{b-a}} (-t^\alpha)f'(ta + (1-t)b)dt \\ &+ \int_{\frac{b-x}{b-a}}^1 (1-t)^\alpha f'(ta + (1-t)b)dt \\ &= \left(\frac{b-x}{b-a}\right)^\alpha \frac{f(x)}{b-a} - \frac{\alpha}{b-a}\int_{\frac{b-x}{b-a}}^1 (1-t)^{\alpha-1} f(ta + (1-t)b)dt. \end{aligned}$$

Using the change of the varriable $u = ta + (1-t)b$ for $t \in [0,1]$ which gives

$$\begin{aligned} I &= \frac{(b-x)^\alpha}{(b-a)^{\alpha+1}} f(x) - \frac{\alpha}{(b-a)^{\alpha+1}}\int_x^b (b-u)^{\alpha-1} f(u)du \\ &+ \frac{(x-a)^\alpha}{(b-a)^{\alpha+1}} f(x) - \frac{\alpha}{(b-a)^{\alpha+1}}\int_a^x (u-a)^{\alpha-1} f(u)du \\ &= \left[\frac{(x-a)^\alpha + (b-x)^\alpha}{(b-a)^{\alpha+1}}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[I_{x^+}^\alpha f(b) + I_{x^-}^\alpha f(a)\right]. \end{aligned}$$

this complet the proof.

Theorem 31 *[28] Let $f : [a,b] \to \mathbb{R}$, be differentiable function on (a,b) with $a < b$ such that $f' \in L^1[a,b]$.If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals with $\alpha > 0$,holds:*

$$\left|\left[\frac{(x-a)^\alpha + (b-x)^\alpha}{(b-a)^{\alpha+1}}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}[I_{x+}^\alpha f(b) + I_{x-}^\alpha f(a)]\right|$$
$$\frac{1}{\alpha+2}\left\{\left(\frac{(b-x)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha+1}}\left[\frac{1}{\alpha+1} + \frac{b-x}{b-a}\right]\right)|f'(a)|\right\}$$
$$+\frac{1}{\alpha+2}\left\{\left(\frac{(x-a)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha+1}}\left[\frac{1}{\alpha+1} + \frac{x-a}{b-a}\right]\right)|f'(b)|\right\} \tag{4.37}$$

Proof 14 *from lemma(4.2.1)and since $|f'|$ is convex on $[a,b]$,we have*

$$\left|\left[\frac{(x-a)^\alpha + (b-x)^\alpha}{(b-a)^{\alpha+1}}\right] f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[I_{x+}^\alpha f(b) + I_{x-}^\alpha f(a)\right]\right|$$
$$\leq \int_0^{\frac{b-x}{b-a}} t^\alpha[t|f'(a)| + (1-t)|f'(b)|]dt + \int_{\frac{b-x}{b-a}}^1 |(1-t)^\alpha||f'(ta+(1-t)b)|dt$$
$$\leq \int_0^{\frac{b-x}{b-a}} t^\alpha[t|f'(a)| + (1-t)|f'(b)|]dt + \int_{\frac{b-x}{b-a}}^1 (1-t)^\alpha[t|f'(a)| + (1-t)|f'(b)|]$$
$$= \left[\frac{1}{\alpha+2}\frac{(b-x)^{\alpha+2} - (x-a)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{1}{\alpha+1}\frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha+1}}\right]|f'(a)|$$
$$+ \left[\frac{1}{\alpha+2}\frac{(x-a)^{\alpha+2} - (b-x)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{1}{\alpha+1}\frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha+1}}\right]|f'(b)|$$
$$= \frac{1}{\alpha+2}\left(\frac{(b-x)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha+1}}\left[\frac{1}{\alpha+1} + \frac{b-x}{b-a}\right]\right)|f'(a)|$$
$$+\frac{1}{\alpha+2}\left(\frac{(x-a)^{\alpha+2}}{(b-a)^{\alpha+2}} + \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha+1}}\left[\frac{1}{\alpha+1} + \frac{x-a}{b-a}\right]\right)|f'(b)|.$$

Which complet the proof.

Corollary 32 *[28] If we take $x = \dfrac{a+b}{2}$ in theorem(31)*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I_{(\frac{a+b}{2})+}^\alpha f(b) + I_{(\frac{a+b}{2})-}^\alpha f(a)\right]\right| \leq \frac{b-a}{2(\alpha+1)}\left(\frac{|f'(a)| + |f'(b)|}{2}\right). \tag{4.38}$$

Theorem 33 *[28] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$ such that $f' \in L^1[a,b]$, if $|f'|^q$ is convex on $[a,b], q > 1$ and $x \in [a,b]$, then the* *following inequality for fractional integrals*

$$\left|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[I_{x^+}^{\alpha}f(b)+I_{x^-}^{\alpha}f(a)\right]\right|$$

$$\leq \frac{1}{(b-a)^{\alpha+1}(\alpha p+1)^{\frac{1}{p}}}\left[(b-x)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+(x-a)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right] \tag{4.39}$$

holds where $\frac{1}{p}+\frac{1}{q}=1, \alpha>0.$

Proof 15 *From Lemma(4.2.1) and using the well known Hôlder inequality,we have:*

$$\left|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[I_{x^+}^{\alpha}f(b)+I_{x^-}^{\alpha}f(a)\right]\right|\leq\int_0^{\frac{b-x}{b-a}}|-t|^{\alpha}|f'(ta+(1-t)b)|dt$$

$$+\int_{\frac{b-x}{b-a}}^{1}|(1-t)^{\alpha}||f'(ta+(1-t)b)|dt$$

$$\leq\left(\int_0^{\frac{b-x}{b-a}}t^{\alpha p}\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^{1}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

Since $|f'|$ *is convex,by Hermite-Hadamard inequality we have:*

$$\int_0^{\frac{b-x}{b-a}}|f'(ta+(1-t)b)|^q dt\leq\frac{b-x}{b-a}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right);$$

$$\int_{\frac{b-x}{b-a}}^{1}|f'(ta+(1-t)b)|^q dt\leq\frac{x-a}{b-a}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)$$

and by simple computation

$$\int_0^{\frac{b-x}{b-a}}t^{\alpha p}dt=\frac{1}{\alpha p+1}\left(\frac{b-x}{b-a}\right)^{\alpha p+1}$$

$$\int_{\frac{b-x}{b-a}}^{1}(1-t)^{\alpha p}dt=\frac{1}{\alpha p+1}\left(\frac{x-a}{b-a}\right)^{\alpha p+1}$$

There fore:

$$\left|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{(b-a)^{\alpha+1}}\right]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[I_{x^+}^{\alpha}f(b)+I_{x^-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{1}{(b-a)^{\alpha+1}}(\alpha p+1)^{\frac{1}{p}}\left[(b-x)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+(x-a)^{\alpha+1}\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$, *hence using the formula* $\Gamma(\alpha+1)=\alpha\Gamma(\alpha), (\alpha>0)$ *the proof is complete*

Remark 34 *[28] In theorem,if we choose* $\alpha=1$,*then we obtain inequality4.36*

Corollary 35 *[28] If we take* $x=\frac{a+b}{2}$ *in (4.39),we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)}+f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)}+f(a)\right]\right|$$

$$\frac{b-a}{4(\alpha p+1)^{\frac{1}{p}}}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)^{q}\right|+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)^{q}\right|+|f'(a)|^{q}}{2}\right)^{\frac{1}{q}}\right]$$

Conclusion

In this work we introduced Ostrowski type inequality and Ostrowski's inequality via functions whose derivatives are s-convex in the second sense then we discussed about fractional inequality via Rieman Liouville for convex functions.Also are given some applications to means of Ostrovski's inequality.

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