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Some Special Techniques for Calculating Definite Integrals

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Notations

1. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively denote the sets of natural integers, relative integers, rationals, reals and complexes numbers.

2. $A[x]$ denotes the ring of polynomials whose coefficients in ring A.

3. $\deg(P(x))$: polynomial degree $P(x)$.

4. $\binom{n}{k}$ binomial coefficient, such that n and k two integer where $0 \leq k \leq n$.

5. $\binom{z}{k}$ generalized binomial coefficient, such that z is a complex number and k an integer.

6. H_n the harmonic number, defined as
$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

7. $H_{n,m}$ the harmonic number of order m, defined as
$$H_{n,m} = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{n^m}.$$

8. $[x^n](P(x))$ designates the cofficient of x^n in the polynomial $P(x)$.

9. B_n the Bernoulli number defined by the recurrence relation
$$B_0 = 1 \text{ and } B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k, \; n \geqslant 1.$$

10. ζ the Riemann zeta function, defined as :
$$\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \quad \text{where } \operatorname{Re}(s) > 1.$$

11. $\mathbf{Li}_2(x)$ the dilogarithm function defined by the power series
$$\mathbf{Li}_2(x) = \sum_{n=1}^{+\infty}\frac{1}{n^2}x^n \quad \text{for } |x| < 1.$$

12. $s(n,k)$ the Stirling numbers of the first kind, defined as
$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)\, x^k.$$

13. LHS is informal shorthand for the left-hand side of an equation.

14. RHS is informal shorthand for the right-hand side of an equation.

Abstract

The study of the calculus of indefinite integrals concerning logarithms, log-sin integrals or others is a crucial research topic being widely focused on. In this thesis we have studied a problem related to an interesting logarithm integral such

$$\int_0^1 \frac{\ln^n(1-t)}{t^m}dt \quad \text{where} \quad n,m \in \mathbb{N}^*.$$

We extend certain sums involving the harmonic numbers and binomial coefficients which are linked to certain logarithmic integral representations.

Keywords: Harmonic numbers, Binomial coefficients, Logarithm integral.

Résumé

L'étude du calcul des intégrales indéfinies concernant les logarithmes, les intégrales log-sin ou autres est un sujet de recherche crucial largement focalisé. Dans cette thèse, nous avons étudié un problème lié à une intégrale logarithmique intéressante telle que

$$\int_0^1 \frac{\ln^n(1-t)}{t^m} dt \quad \text{where} \quad n, m \in \mathbb{N}^*.$$

Mots-clés : Nombres harmoniques, Coefficients binomiaux, Logarithme integrale.

Contents

Introduction

Introduction

In mathematics, an integral is the continuous analog of a sum, which is used to calculate areas, volumes and their generalizations. Integration, the process of calculating an integral, is one of the two fundamental operations of calculus. Integration began as a method for solving problems in mathematics and physics, it is used today in a wide variety of scientific fields. This thesis is devoted to an in-depth study of the main problems seen in some journals.

Our work has three chapters.

The first chapter is devoted to generalities, we start by defining the binomial coefficients and the generalized binomail coefficients, we study the important and general properties of these binomial coefficients, then we define in a natural way the sequences of harmonic numbers H_n and their generalization harmonic numbers of order $H_{n,m}$.
At the end of this chapter due to the important role played the riemann zeta fonction of the calculation, we given an simple proof of Basel problem and some intersing values of $\zeta(n)$.

The second chapter constitutes the essential part of this thesis, it is devoted to an in -depth study on the study of certain properties and theorems concerning more specifically the sums involving binomial coefficients and sums involving harmonic numbers such

$$\sum_{k=1}^{n} \frac{H_k}{k}.$$

At the end of this chapter, we recall the Cauchy product of tow series and we present some various series such the development of

$$f(x) = \frac{\ln(1-x)}{1-x},$$

to use for solving important integrals in chapter three.

The last chapter of the theses introduces a collection of some problems seen in some new papers. Most of the problems appeared in some mathematical journals sush MAA .

we present various important integrals evaluated using series, combinatorial identities and harmonic numbers.

The main results of this chapter represent an interesting contribution in integral logarithms. They are obtained by using technical operations on binomial coefficients and harmonic numbers.

Firstly, we solve some problems concerning logarithmic integrals related to the harmonic series like

$$\int_0^1 \frac{\ln^m x}{1+x} dx \quad \text{and} \quad \int_0^1 \frac{\ln^m (1-x)}{x^m} dx$$

and we have also given some properties of the dilogarithm function $\mathbf{Li}_2(x)$.

Finally, in this chapter, many of these integrals invite the use of combinatorial mathematical techniques that involve elegant connections between integrals and infinite series.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we recall the definitions and some properties of the binomial coefficients, generalized binomial coefficients, the harmonic numbers, we also recall the riemann zeta functions and we give some values of rieman zeta like $\zeta(2), \zeta(3), \zeta(4)$.

1.2 Binomial coefficients

Binomial coefficients are involved in many areas of mathematics: binomial development in algebra, enumeration, series expansion, laws of probability, etc.

For any natural integer n and for any polynomial $P(x)$ of $A[x]$, where $A[x]$ denotes the ring of polynomials whose coefficients in ring A.
The notation $[x^n](P(x))$ designates the coefficient of x^n in the polynomial $P(x)$. So if

$$P(x) = \sum_{k\geq 0} a_k x^k,$$

we write

$$[x^n]P(x) = a_n.$$

With this notation, we define the binomial coefficient $\binom{n}{k}$, where n and k are two natural integers by

$$\binom{n}{k} = [x^n](1+x)^n = [x^n]\sum_{k\geq 0}\binom{n}{k}x^k.$$

For all natural numbers n and k, the binomial coefficients satisfy

1.
$$\binom{n}{k+1} = \binom{n-1}{k+1} + \binom{n-1}{k} \tag{1.1}$$

2.
$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \tag{1.2}$$

3.
$$\binom{n}{k} = \binom{n}{n-k}, \tag{1.3}$$

4.
$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k} \tag{1.4}$$

5.
$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k} \tag{1.5}$$

Remark 1.1 *Andreas von Ettingshausen introduced the notation* $\binom{n}{k}$ *in 1826, although the numbers were known centuries earlier as* $C(n,k)$, C_n^k, *and* $C_{n,k}$.

1.3 Generalized binomial coefficients

Mathematicians often generalize definitions, and the binomial coefficients are no exceptions. The typical way to generalize $\binom{n}{k}$ is to let n be an arbitrary complex number.

Let $\alpha \in \mathbb{C}$ et $k \in \mathbb{Z}$, We define the generalized binomial coefficient $\binom{\alpha}{k}$ as follows

$$\binom{\alpha}{k} = \prod_{j=0}^{k-1}(\alpha - j) = \frac{\alpha\,(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{for } k \geq 0.$$

and

$$\binom{\alpha}{k} = 0 \quad \text{for } k < 0.$$

The binomial coefficients intervene in the development in whole series of $(1+z)^\alpha$ for $|z| < 1$. we have

$$(1+z)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k} z^k \quad \text{for } |z| < 1$$

In the case where $\alpha \in \mathbb{Q}$, (rep $\mathbb{Z}$, $\mathbb{N}$), we have $\binom{\alpha}{k} \in \mathbb{Q}$, (rep $\mathbb{Z}$, $\mathbb{N}$) For $\alpha = n$ with $n \in \mathbb{N}$, we also find the definition of the binomial coefficient classic $\binom{n}{k}$.

We have the following properties

1. **Pascal's Identity for general binomial coefficients**

$$\binom{z}{k+1} = \binom{z-1}{k+1} + \binom{z-1}{k} \qquad k \in \mathbb{N} \text{ and } z \in \mathbb{C} \tag{1.6}$$

2. **Chair Identity**

$$\binom{z}{k} = \frac{z}{k}\binom{z-1}{k-1} \qquad \text{for} \quad z \in \mathbb{C} \text{ and } k \in \mathbb{N}^* \tag{1.7}$$

3. **Transformation**

$$\binom{-z}{k} = (-1)^k \binom{z+k-1}{k} \text{ for } z \in \mathbb{C} \text{ and } k \in \mathbb{N} \tag{1.8}$$

4. **Cancellation Identity**

$$\binom{z}{n}\binom{n}{k} = \binom{z-k}{n-k}\binom{z}{k} \qquad \text{for } z \in \mathbb{C} \text{ and } k, n \in \mathbb{N} \tag{1.9}$$

5. **Vandermonde formula**

$$\sum_{i+j=k} \binom{\alpha}{i}\binom{\beta}{j} = \binom{\alpha+\beta}{k} \text{ for } \alpha, \beta \in \mathbb{C} \text{ and } k \in \mathbb{N} \tag{1.10}$$

1.4 Harmonic numbers

Harmonic numbers have been studied since antiquity and are important in various branches of number theory. They are sometimes loosely termed harmonic series, are closely related to the Riemann zeta function, and appear in the expressions of various special functions.

In mathematics, the n-th harmonic number is the sum of the reciprocals of the first n natural numbers, defined as :

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \text{ for } n \in \mathbb{N}^*.$$

Recall that harmonic numbers of order m are given by

$$H_{0,m} = 0, \quad H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \text{ for } n \in \mathbb{N}^*.$$

with $H_{n,1} = H_n$ are the classical harmonic numbers.

An integral representation given by Euler is

Lemma 1.1 *Let $n \in \mathbb{N}$, the following identity holds*

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx \tag{1.11}$$

Proof. we have

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= \int_0^1 dx + \int_0^1 x dx + \int_0^1 x^2 dx + \cdots + \int_0^1 x^{n-1} dx \\ &= \int_0^1 \left(1 + x + x^2 + \cdots + x^{n-1}\right) dx, \end{aligned}$$

using the sum of a geometric sequence, the proof is complete. ■

1.5 Some values of Riemann zeta function

The Riemann zeta function or Euler–Riemann zeta function, denoted by the Greek letter ζ (zeta), is an analytic function defined, for any complex number s such that $\operatorname{Re}(s) > 1$, by the Riemann series

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Euler calculated (as part of his solution to the Basel problem) the value of the function ζ for even strictly positive integers; he deduced the formula

$$\zeta(2m) = \sum_{n=1}^{+\infty} \frac{1}{n^{2m}} = \frac{|B_{2m}| (2\pi)^{2m}}{2(2m)!}$$

where Bernoulli numbers $(B_n)_{n\in\mathbb{N}}$ are defined by the recurrence relation:

$$B_0 = 1 \text{ and } B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \ n \geqslant 1.$$

B_n is called the n-th Bernoulli number.

These values of $\zeta(2k)$ are therefore expressed using the even powers of π,for example

$$\zeta(2) = \frac{\pi^2}{6}; \ \zeta(4) = \frac{\pi^2}{90}; \ \zeta(6) = \frac{\pi^2}{945}; \cdots$$

1.5.1 Basel problem $\zeta(2)$

The Basel problem is an important problem in number theory that was first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734. The Basel problem so is named for the Swiss city in whose university two of the Bernoulli brothers successively served as professor of mathematics (Jakob, 1687 - 1705, and Johann, 1705 - 1748).
Coincidentally Euler was born in Basel.

The Basel problem asks whether the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$

has a closed form solution, that is, does it converge to a finite number and if it does, what number does it converge to?
The Basel problem resisted solution for some 84 years until the then 26 year old Euler finally solved it. Euler's surprising solution is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

This result has been proven by many mathematicians using different methods and here we have provided a simple proof presented in 2017 by A. Novac [21]

Let

$$I = \int_0^1 \int_0^\infty \frac{1}{1 + y^2 - x^2 y^2} dy dx$$

It follows that

$$\begin{aligned}
I &= \int_0^1 \int_0^\infty \frac{1}{1 + y^2 - x^2 y^2} dy dx = \int_0^1 \int_0^\infty \frac{1}{1 + \left(y\sqrt{1-x^2}\right)^2} dy dx \\
&= \int_0^1 \frac{1}{\sqrt{1-x^2}} \int_0^\infty \frac{\sqrt{1-x^2}}{1 + \left(y\sqrt{1-x^2}\right)^2} dy dx \\
&= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\arctan\left(y\sqrt{1-x^2}\right)\right]_0^\infty dx \\
&= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}} dx \\
&= \frac{\pi}{2} \left[\arcsin x\right]_0^1 = \frac{\pi^2}{4}.
\end{aligned}$$

then

$$I = \frac{\pi^2}{4} \tag{1.12}$$

and on the other nother hand by changing the order of integration and using the

$$\begin{aligned}
\int_0^1 \frac{1}{1-a^2x^2}dx &= \frac{1}{2}\int_0^1 \frac{1}{1+ax} + \frac{1}{1-ax}dx \\
&= \frac{1}{2a}\left[\ln\left(\frac{1+ax}{1-ax}\right)\right]_0^1 = \frac{1}{2a}\ln\left(\frac{1+a}{1-a}\right)
\end{aligned}$$

we have

$$\begin{aligned}
I &= \int_0^1\int_0^\infty \frac{1}{1+y^2-x^2y^2}dydx = \int_0^\infty \frac{1}{1+y^2}\int_0^1 \frac{1}{1-\left(\frac{y}{\sqrt{1+y^2}}\right)^2 x^2}dxdy \\
&= \frac{1}{2}\int_0^\infty \frac{\ln\left(y+\sqrt{1+y^2}\right)}{y\sqrt{1+y^2}}dy
\end{aligned}$$

With the substitution

$$y = \mathbf{sh}t$$

we find that:

$$\begin{aligned}
I &= \int_0^\infty \frac{\ln\left(\mathbf{sh}t+\mathbf{ch}t\right)}{\mathbf{sh}t\mathbf{ch}t}\mathbf{ch}tdt = \int_0^\infty \frac{\ln\left(\mathbf{sh}t+\mathbf{ch}t\right)}{\mathbf{sh}t}dt \\
&= 2\int_0^\infty \frac{t}{e^t-e^{-t}}dt \\
&= 2\int_0^\infty \frac{te^{-t}}{1-e^{-2t}}dt
\end{aligned}$$

using the developement series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ we obtain

$$I = 2\int_0^\infty t\sum_{n=0}^{\infty} e^{(-2n-1)t}dt = 2\sum_{n=0}^{\infty}\int_0^\infty te^{(-2n-1)t}dt$$

integration by parts

$$I = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

using the remark

$$\sum_{k=0}^{n} \frac{1}{(2k+1)^2} + \sum_{k=1}^{n} \frac{1}{(2k)^2} = \sum_{k=1}^{2n+1} \frac{1}{k^2}$$

we have

$$I = 2\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 2\left(\sum_{n=0}^{\infty} \frac{1}{n^2} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{n^2}\right) = \frac{3}{2}\sum_{n=0}^{\infty} \frac{1}{n^2}. \tag{1.13}$$

Eqs. (1.12) and (1.13) conclude the proof of Euler's formula.

1.5.2 Apéry's constant $\zeta(3)$

In mathematics, Apéry's constant is the sum of the reciprocals of the positive cubes. That is, it is defined as the number

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

where ζ is the Riemann zeta function. It has an approximate value

$$\zeta(3) = 1.202056903159594285399738161511449990764986292\ldots$$

$\zeta(3)$ was named Apéry's constant after the French mathematician Roger Apéry, who proved in 1978 that it is an irrational number.

Chapter 2

Sum involving harmonic numbers and binomial coefficients

Investigation of the harmonic numbers properties traces it is history back to Ancient Greece and has the fundamental importance to the several fields of mathematics. These numbers are closely related to Riemann zeta function and appear in many expressions of other special functions.

Sums involving numbers numbers and binomial coefficients have been of considerable interest throughout the 20th century. We extend certain sums involving the harmonic numbers and binomial coefficients which are related to certain logarithmic integral representations.

In order to prove the main problems of chapter 3 , we need some auxiliary results given by the following lemmas.

2.1 Sums involving binomial coefficients

Lemma 2.1 *Let n be positive integers, then*

$$\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{n}{k}=-H_n \tag{2.1}$$

and

$$\sum_{k=1}^{n}(-1)^k\frac{1}{k^2}\binom{n}{k}=\frac{H_n}{n}. \tag{2.2}$$

Proof. By induction on n.
Let

$$P(n): \sum_{k=1}^{n} (-1)^k \frac{1}{k}\binom{n}{k} = -H_n.$$

We have, LHS of $P(1)$ is

$$\sum_{k=1}^{1} (-1)^k \frac{1}{k}\binom{1}{k} = -1,$$

and RHS of $P(1)$ is

$$-H_1 = -1,$$

so $P(1)$ is true.
Now assume $P(n)$ is true, for some natural number $n \geq 1$, i.e. by recurence suppose that

$$\sum_{k=1}^{n} (-1)^k \frac{1}{k}\binom{n}{k} = -H_n$$

and deduce $P(n+1)$.
We have

$$\begin{aligned}
\sum_{k=1}^{n+1} (-1)^k \frac{1}{k}\binom{n+1}{k} &= \sum_{k=1}^{n+1} (-1)^k \frac{1}{k}\left[\binom{n}{k} + \binom{n}{k-1}\right] \\
&= \sum_{k=1}^{n+1} (-1)^k \frac{1}{k}\binom{n}{k} + \sum_{k=1}^{n+1} (-1)^k \frac{1}{k}\binom{n}{k-1} \\
&= \sum_{k=1}^{n} (-1)^k \frac{1}{k}\binom{n}{k} + \frac{1}{n+1}\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \\
&= -H_n - \frac{1}{n+1} \\
&= -H_{n+1}
\end{aligned}$$

Using the integral representation of harmonic number and the the substitution $x = 1-u$, we have

$$\begin{aligned}
H_n &= \int_0^1 \frac{1-x^n}{1-x} dx = \int_0^1 \frac{1-(1-u)^n}{u} du \\
&= -\int_0^1 \sum_{k=1}^{n} (-1)^k \binom{n}{k} u^{k-1} du = -\sum_{k=1}^{n} (-1)^k \binom{n}{k} \int_0^1 u^{k-1} du \\
&= -\sum_{k=1}^{n} (-1)^k \frac{1}{k}\binom{n}{k}.
\end{aligned}$$

■

Example 2.1 *Show that the above identity (2.2) holds.*

2.2 Sums involving harmonic numbers

. Recently, many other remarkable finite sum identities involving the harmonic numbers have been developed by many authors in different forms using a variety of methods.

In this section we intersting to calculate the following sums.

Lemma 2.2 *Let $n \geq 1$ be integers. The following identity holds*

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_n^2 + H_{n,2}}{2}. \tag{2.3}$$

Proof. We proceed as follow

$$\begin{aligned}
\sum_{k=0}^{n-1} H_k^2 + H_n^2 &= \sum_{k=0}^{n} H_k^2 = \sum_{k=1}^{n} H_k^2 = \sum_{k=0}^{n-1} H_{k+1}^2 \\
&= \sum_{k=0}^{n-1} \left(H_k + \frac{1}{k+1} \right)^2 \\
&= \sum_{k=0}^{n-1} \left(H_k^2 + 2\frac{H_k}{k+1} + \frac{1}{(k+1)^2} \right) \\
&= \sum_{k=0}^{n-1} H_k^2 + 2\sum_{k=0}^{n-1} \frac{H_k}{k+1} + \sum_{k=0}^{n-1} \frac{1}{(k+1)^2} \\
&= \sum_{k=0}^{n-1} H_k^2 + 2\sum_{k=0}^{n-1} \frac{H_{k+1} - \frac{1}{k+1}}{k+1} + \sum_{k=0}^{n-1} \frac{1}{(k+1)^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{n-1} H_k^2 + H_n^2 &= \sum_{k=0}^{n-1} H_k^2 + 2\sum_{k=0}^{n-1} \frac{H_{k+1}}{k+1} - \sum_{k=0}^{n-1} \frac{1}{(k+1)^2} \\
&= \sum_{k=0}^{n-1} H_k^2 + 2\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2} \\
&= \sum_{k=0}^{n-1} H_k^2 + 2\sum_{k=1}^{n} \frac{H_k}{k} - H_{n,2}
\end{aligned}$$

the proof is complete.

■

Lemma 2.3 *For n be integers we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k=-\frac{1}{n}. \tag{2.4}$$

Proof. We proceed as follow

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k k\binom{n}{k}H_k &= \sum_{k=1}^{n+1}(-1)^k k\binom{n}{k}H_k=\sum_{k=0}^{n}(-1)^{k+1}(k+1)\binom{n}{k+1}H_{k+1}\\
&= \sum_{k=0}^{n}(-1)^{k+1}(n-k)\binom{n}{k}\left(H_k+\frac{1}{k+1}\right)\\
&= \sum_{k=0}^{n}(-1)^{k+1}(n-k)\binom{n}{k}H_k+\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{n-k}{k+1}\\
&= \sum_{k=0}^{n}(-1)^k k\binom{n}{k}H_k-n\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k+\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{n-k}{k+1}
\end{aligned}$$

then

$$n\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k=\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{n-k}{k+1}$$

and since

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{n-k}{k+1} &= \sum_{k=0}^{n}(-1)^{k+1}\frac{n+1}{k+1}\binom{n}{k}-\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\\
&= \sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1}-\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\\
&= \sum_{k=1}^{n+1}(-1)^{k}\binom{n+1}{k}-\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\\
&= -1-0\\
&= -1
\end{aligned}$$

the proof is complete. ■

Corollay 2.1 *The following equality holds:*

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}H_{k+1}=-\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1}H_{k+1}=\frac{1}{(n+1)^2}.$$

Corollay 2.2 *By the Binomial inversion formula*

$$a_k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}b_k \Longleftrightarrow b_k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k$$

where $(a)_n$ *and* $(b)_n$ *are two complexe sequences, we have*

$$\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{n}{k}=-H_n\Longleftrightarrow\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k=-\frac{1}{n}.$$

2.3 Some applications of Cauchy product

In mathematics, more specifically in mathematical analysis, the Cauchy product is the convolution of two infinite series. It is named after the French mathematician Augustin-Louis Cauchy.

2.3.1 Cauchy product of two infinite series

Let $\sum\limits_{n\geq 0}a_n$, $\sum\limits_{n\geq 0}b_n$ be two infinite series with complex terms. The Cauchy product of these two infinite series is defined by

$$\left(\sum_{j\geq 0}a_j\right)\left(\sum_{j\geq 0}b_j\right)=\sum_{n\geq 0}c_n\ \text{ where } c_n=\sum_{i=0}^{n}a_ib_{n-i}.$$

2.3.2 Cauchy product of two power series

Let $\sum\limits_{n\geq 0}a_nx^n$, $\sum\limits_{n\geq 0}b_nx^n$ be two power series, with complex terms.
The Cauchy product formula of these two power series is defined by

$$\left(\sum_{j\geq 0}a_jx^i\right)\left(\sum_{j\geq 0}b_jx^j\right)=\sum_{n\geq 0}\left(\sum_{i=0}^{n}a_ib_{n-i}\right). \tag{2.5}$$

The generalat forumula is

$$\left(\sum_{j\geq r}a_jx^i\right)\left(\sum_{j\geq s}b_jx^j\right)=\sum_{n\geq r+s}\left(\sum_{i=r}^{n-s}a_ib_{n-i}\right)x^n \text{ or } =\sum_{n\geq r+s}\left(\sum_{j=s}^{n-r}a_{n-j}b_j\right)x^n. \tag{2.6}$$

Lemma 2.4 *we have*

$$\frac{\ln(1+x)}{1+x} = \sum_{n=1}^{\infty}(-1)^{n+1}H_nx^n \qquad |x|<1, \tag{2.7}$$

$$-\frac{\ln(1-x)}{1-x} = \sum_{n=1}^{\infty}H_nx^n \qquad |x|<1. \tag{2.8}$$

Proof. Stating from the known expansion of two series

$$\ln(1+x)=\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}x^n \text{ and } \frac{1}{1+x}=\sum_{n\geq 0}(-1)^n x^n,\quad |x|<1 \tag{2.9}$$

from the relation (2.6), we have

$$\begin{aligned}\frac{\ln(1+x)}{1+x} &= \left(\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}x^n\right)\left(\sum_{n\geq 0}(-1)^n x^n\right)=\sum_{n\geq 1}\left(\sum_{i=1}^{n-0}\frac{(-1)^{i+1}}{i}(-1)^{n-i}\right)x^n\\ &= \sum_{n\geq 1}(-1)^{n+1}\left(\sum_{i=1}^{n}\frac{1}{i}\right)x^n\\ &= \sum_{n\geq 1}(-1)^{n+1}H_n x^n.\end{aligned}$$

Stating from the known expansion of two series

$$\ln(1-x)=-\sum_{n\geq 1}\frac{1}{n}x^n \text{ and } \frac{1}{1-x}=\sum_{n\geq 0}x^n,\quad |x|<1$$

and from the relation (2.6), we have

$$\begin{aligned}-\frac{\ln(1-x)}{1-x} &= \left(\sum_{n\geq 1}\frac{1}{n}x^n\right)\left(\sum_{n\geq 0}x^n\right)=\sum_{n\geq 1}\left(\sum_{i=1}^{n-0}\frac{1}{i}\right)x^n\\ &= \sum_{n\geq 1}H_n x^n.\end{aligned}$$

■

Corollay 2.3 *We find by integration of relation (2.7)*

$$\frac{1}{2}\ln^2(1+x)=\sum_{n\geq 0}(-1)^{n+1}\frac{H_n}{n+1}x^{n+1}$$

then

$$\frac{\ln^2(1+x)}{x}=2\sum_{n\geq 0}(-1)^{n+1}\frac{H_n}{n+1}x^n.$$

Also we have

$$\frac{1}{2}\ln^2(1-x)=\sum_{n\geq 0}\frac{H_n}{n+1}x^{n+1}$$

then

$$\frac{\ln^2(1-x)}{x}=2\sum_{n\geq 0}\frac{H_n}{n+1}x^n$$

Chapter 3

A binomial formula for evaluating some Logarithmic integrals

3.1 Introduction

This chapter constitutes the essential part of this thesis, contains some techniques of integration which are not found in standard calculs and advanced calculus book. it can be considered as a map to explore many classical approaches to evaluate integrales.

3.2 values of $\int_0^1 x^m \ln(1-x)\,dx$ and $\int_0^1 x^m \ln^2(1-x)\,dx$

The follwing formula (3.1), which is quite old, is recorded in various tables of definite integrals. It appears as formula 865.5 in [10] , It is worth mentioning that the origin of such integrals dates back to the time of the English mathematician Joseph Wolstenholme (1829–1891), and the first proposed integral appeared in his book with mathematical problems.

Problem 3.1 *Let $n \geq 1$ be an integer. The following identity holds*

$$\int_0^1 x^{n-1} \ln(1-x)\,dx = -\frac{H_n}{n}. \tag{3.1}$$

Proof. .
First proof

Let

$$I_n = \int_0^1 x^{n-1} \ln(1-x)\, dx.$$

Using integration by parts

$$\begin{aligned} u &= x^{n-1} \longrightarrow u' = (n-1)\, x^{n-2} \\ v' &= \ln(1-x) \longrightarrow v = (x-1)\ln(1-x) - x \end{aligned}$$

then

$$\begin{aligned} I_n &= \left[x^{n-1}\left((x-1)\ln(1-x) - x\right)\right]_0^1 - (n-1)\int_0^1 x^{n-2}\left((x-1)\ln(1-x) - x\right) dx \\ &= \left[x^{n-1}\left((x-1)\ln(1-x)\right)\right]_0^1 - \left[x^n\right]_0^1 - (n-1)\left(I_n - I_{n-1}\right) + (n-1)\int_0^1 x^{n-1} dx \\ &= -(n-1)\left(I_n - I_{n-1}\right) - \frac{1}{n}, \end{aligned}$$

which yields the recurrence relation in k,

$$kI_k - (k-1)\, I_{k-1} = \frac{-1}{k}.$$

Giving values to k from $k = 2$ to n and using that $\int_0^1 \ln(1-x)\, dx = -1$, we obtain that

$$\begin{aligned} 2I_2 - I_1 &= \frac{-1}{2}. \\ 3I_3 - 2I_2 &= \frac{-1}{3} \\ &\vdots \\ &\vdots \\ (n-1)\, I_{n-1} - (n-2)\, I_{n-2} &= \frac{-1}{n-1} \\ nI_n - (n-1)\, I_{n-1} &= \frac{-1}{n} \end{aligned}$$

then

$$nI_n - I_1 = -\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

the proof of relation (3.1) is given.

Second proof

we have

$$\begin{aligned}
\int_0^1 x^{n-1}\ln(1-x)\,dx &= \int_0^1 (1-x)^{n-1}\ln x dx \\
&= \int_0^1 \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^k \ln x dx \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^1 x^k \ln x dx \\
&= \sum_{k=0}^{n-1} (-1)^{k+1} \frac{1}{(k+1)^2}\binom{n-1}{k} \\
&= \frac{1}{n}\sum_{k=0}^{n-1} (-1)^{k+1} \frac{1}{k+1}\binom{n}{k+1} \\
&= \frac{1}{n}\sum_{k=1}^{n} (-1)^{k} \frac{1}{k}\binom{n}{k}
\end{aligned}$$

using the formula (2.4) the proof of formula (3.1) is given. ■

the following problem is proved in 2016 by Vălean [27]

Problem 3.2 *Let $n \geq 1$ be an integer. The following identity holds*

$$\int_0^1 x^{n-1}\ln^2(1-x)\,dx = \frac{H_n^2 + H_{n,2}}{n}$$

Proof. Let

$$J_n = \int_0^1 x^{n-1}\ln^2(1-x)\,dx$$

using integration by parts as in above problem , we obtain the recurrence relation in k,

$$kJ_k - (k-1)\,J_{k-1} = 2\frac{H_k}{k}.$$

Giving values to k from $k = 2$ to n and using that $\int_0^1 \ln^2(1-x)\,dx = 2$, we obtain that

$$J_n = 2\sum_{k=1}^{n} \frac{H_k}{k}$$

using the relation (2.3), the proof is complete. ■

3.3 Values of $\int_0^1 \frac{\ln^m x}{1+x} dx$

Problem 3.3 *Let m be a positive integer. Then the following equality holds*

$$\int_0^1 \frac{\ln^m t}{1+t} dt = \frac{(-1)^{m+1} m! (1-2^m)}{2^m} \sum_{n=1}^{+\infty} \frac{1}{n^{m+1}}$$

Proof. We have

$$\int_0^1 \frac{\ln^m t}{1+t} dt = \int_0^1 \ln^m t \sum_{n=0}^{+\infty} (-1)^n t^n dt = \sum_{n=0}^{+\infty} (-1)^n \int_0^1 t^n \ln^m t dt$$

using the subtitition $t = e^{-x}$ and using

$$I_{n,k} = \int_0^{+\infty} t^n e^{-(m+1)t} dt$$

we have by part

$$\begin{aligned} u &= t^n \longrightarrow u' = nt^{n-1} \\ v' &= e^{-(m+1)t} \longrightarrow v = -\frac{1}{m+1} e^{-(m+1)t} \end{aligned}$$

hence, we have the identity

$$I_{n,m} = \frac{n}{k+1} I_{n-1,m} = \frac{n!}{(m+1)^{n+1}}$$

we immediately have

$$\begin{aligned} \int_0^1 t^n \ln^m t dt &= -\int_{+\infty}^0 e^{-nx} (-x)^m e^{-x} dx \\ &= (-1)^m \int_0^{+\infty} x^m e^{-(n+1)x} dx \\ &= \frac{(-1)^m m!}{(n+1)^{m+1}} \end{aligned}$$

then

$$\int_0^1 \frac{\ln^m t}{1+t} dt = (-1)^m m! \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)^{m+1}} = (-1)^{m+1} m! \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^{m+1}}$$

and by the remark

$$\sum_{n=1}^{+\infty}\frac{(-1)^n+1}{n^{\grave{u}+1}}=2\sum_{n=1}^{+\infty}\frac{1}{(2n)^{m+1}}=\frac{1}{2^m}\sum_{n=1}^{+\infty}\frac{1}{n^{m+1}} \tag{3.2}$$

we have

$$\int_0^1\frac{\ln^m t}{1+t}dt=(-1)^{m+1}m!\left(\frac{1}{2^m}\sum_{n=1}^{+\infty}\frac{1}{n^{m+1}}-\sum_{n=1}^{+\infty}\frac{1}{n^{m+1}}\right)=\frac{(-1)^{m+1}m!\,(1-2^m)}{2^m}\sum_{n=1}^{+\infty}\frac{1}{n^{m+1}}$$

■

Corollay 3.1 *As an application for some values of m we have*

$$\int_0^1\frac{\ln t}{1+t}dt = \frac{-1}{2}\sum_{n=1}^{+\infty}\frac{1}{n^2}=-\frac{1}{12}\pi^2 \tag{3.3}$$

$$\int_0^1\frac{\ln^2 t}{1+t}dt = \frac{3}{2}\sum_{n=1}^{+\infty}\frac{1}{n^3}=\frac{3}{2}\zeta(3) \tag{3.4}$$

$$\int_0^1\frac{\ln^3 t}{1+t}dt = \frac{-21}{4}\sum_{n=1}^{+\infty}\frac{1}{n^4}=\frac{-21}{4}\zeta(4) \tag{3.5}$$

3.4 Some values of $\int_0^1\frac{\ln^n(1-t)}{t^m}dt$

The College Mathematics Journal is an expository magazine it publishes well-written and captivating articles exploring new mathematics, or old mathematics in a new way. Most of its articles are accessible to upper-level undergraduate students.

In this section we have detailed the proof of the problem 1117 (College Mathematics Journal 2018), proposed by C. I. Vălean (Romania). Solution given by Khristo N. Boyadzhiev, Ohio Northern University Department of Mathematics and Statistics.

Problem 3.4 *Let n and m be a positive integer. Then the following equality holds*

$$\int_0^1\frac{\ln^n(1-x)}{x^m}dx=(-1)^n n!\sum_{k=1}^{+\infty}\frac{1}{k^{n+1}}\binom{m+k-2}{k-1}. \tag{3.6}$$

Proof. We have by the substitution $x=1-e^{-t}$ we write

$$\begin{aligned}
\int\limits_0^1 \frac{\ln^n(1-x)}{x^m}dx &= (-1)^n \int\limits_0^{+\infty} \frac{t^n e^{-t}}{(1-e^{-t})^m}dt = (-1)^n \int\limits_0^{+\infty} t^n e^{-t} \sum_{k=0}^{\infty} \binom{-m}{k} (-1)^k e^{-kt}dt \\
&= (-1)^n \sum_{k=0}^{\infty} (-1)^k \binom{-m}{k} \int\limits_0^{+\infty} t^n e^{-(k+1)t}dt \\
&= (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{n+1}} \binom{-m}{k} \\
&= (-1)^n n! \sum_{k=0}^{\infty} \frac{1}{(k+1)^{n+1}} \binom{m+k-1}{k}. \\
&= (-1)^n n! \sum_{k=1}^{+\infty} \frac{1}{k^{n+1}} \binom{m+k-2}{k-1}.
\end{aligned}$$

■

Corollay 3.2 *As an application for this problem we have*

$$\int\limits_0^1 \frac{\ln(1-x)}{x}dx = -\xi(2) \tag{3.7}$$

$$\int\limits_0^1 \frac{\ln^2(1-x)}{x}dx = 2\xi(3) \tag{3.8}$$

$$\int\limits_0^1 \left(\frac{\ln(1-x)}{x}\right)^2 dx = 2\xi(2) \tag{3.9}$$

$$\int\limits_0^1 \frac{\ln^3(1-x)}{x}dx = -6\xi(4) \tag{3.10}$$

Problem 3.5 *Let n be a positive integer. Then the following identity holds*

$$\int\limits_0^1 \frac{\ln^n(1-x)}{x^m}dx = n \sum_{k=1}^{n-1} (-1)^{k-1} s(n-1,k)\,\zeta(n+1-k).$$

where $s(n-1,k)$ is the Stirling numbers of the first kind defined as

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)\,x^k.$$

Proof. For the proof we shall transform both sides of this equation to one and the same expression.
Left hand side. Taking $n = m$ in relation (3.6), we get

$$\int_0^1 \left(\frac{\ln(1-x)}{x}\right)^n dx = (-1)^n n! \sum_{k=1}^{+\infty} \frac{1}{k^{n+1}} \binom{n+k-2}{k-1} = (-1)^n n! \sum_{k=0}^{+\infty} \frac{1}{(k+1)^{n+1}} \binom{n+k-1}{k}.$$

Right hand side

$$\begin{aligned} n\sum_{k=1}^{n-1} (-1)^{k-1} s(n-1,k)\,\zeta(n+1-k) &= n\sum_{k=1}^{n-1} (-1)^{k-1} s(n-1,k) \sum_{j=1}^{+\infty} \frac{j^k}{j^{n+1}} \\ &= n\sum_{j=1}^{+\infty} \frac{1}{j^{n+1}} \sum_{k=1}^{n-1} (-1)^{k-1} s(n-1,k)\, j^k \\ &= n\sum_{j=1}^{+\infty} \frac{(-1)^n}{j^{n+1}} \sum_{k=1}^{n-1} (-1)^{n+k-1} s(n-1,k)\, j^k. \end{aligned}$$

By using the definition of $s(n-1,k)$, we get

$$\begin{aligned} n\sum_{k=1}^{n-1} (-1)^{k-1} s(n-1,k)\,\zeta(n+1-k) &= (-1)^n n \sum_{j=1}^{+\infty} \frac{j(j+1)\cdots(j+n-2)}{j^{n+1}} \\ &= (-1)^n n \sum_{j=0}^{+\infty} \frac{(j+1)(j+2)\cdots(j+n-1)}{(j+1)^{n+1}} \\ &= (-1)^n n(n-1)! \sum_{j=0}^{+\infty} \frac{(j+n-1)!}{(n-1)!j!(j+1)^{n+1}} \\ &= (-1)^n n! \sum_{k=0}^{+\infty} \frac{1}{(k+1)^{n+1}} \binom{n+k-1}{k} \end{aligned}$$

which is exactly the expression above. The proof is completed. ∎

3.5 Some properties of dilogarithm function $\mathbf{Li}_2(x)$

The dilogarithm function is the function defined by the power series

$$\mathbf{Li}_2(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2} x^n \quad \text{for } |x| < 1$$

is a classical function of mathematical physics. Introduced by Leibniz in 1696 [[15], p. 351] and thoroughly discussed by Euler some seventy years later [[11], pp. 124–126], it has subsequently been well studied in the literature (for further historical details concerning the function see, for example, [18]). The canonical integral representation for the dilogarithm is

$$\mathbf{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} dt. \tag{3.11}$$

The first work on it seems to have been done by Landen and published in 1760. Independently Euler studied it, and they obtained results such as

$$\begin{aligned} \mathbf{Li}_2(x) + \mathbf{Li}_2(1-x) &= \frac{\pi^2}{6} - \ln x \ln(1-x) \quad \text{Euler's reflexion formula} \\ \mathbf{Li}_2(x) + \mathbf{Li}_2\left(\frac{x}{x-1}\right) &= -\frac{1}{2}\ln^2(1-x) \quad \text{Landen's identity} \end{aligned}$$

The first book on this function was by Spence [24] in 1809. Considering the importance of this fonction, where he aded

$$\begin{aligned} \mathbf{Li}_2(x) + \mathbf{Li}_2(-x) &= \frac{1}{2}\mathbf{Li}_2\left(x^2\right), \quad \text{duplication formula} \\ \mathbf{Li}_2(-x) + \mathbf{Li}_2\left(\frac{-1}{x}\right) &= -\frac{\pi^2}{6} - \frac{1}{2}\ln^2 x, \quad \text{inversion formula.} \end{aligned}$$

Firstly, we given simple proof of Euler's reflexion formula.

we have by the substitution $z = 1-t$ and integration by part in the following secondary integrale

$$\begin{aligned} \mathbf{Li}_2(x) + \mathbf{Li}_2(1-x) &= -\int_0^x \frac{\ln(1-t)}{t} dt - \int_0^{1-x} \frac{\ln(1-t)}{t} dt \\ &= -\int_0^x \frac{\ln(1-t)}{t} dt + \int_1^x \frac{\ln z}{1-z} dz \\ &= -\int_0^x \frac{\ln(1-t)}{t} dt - [\ln z \ln(1-z)]_1^x + \int_1^x \frac{\ln(1-z)}{z} dz \\ &= -\int_0^x \frac{\ln(1-t)}{t} dt - \ln x \ln(1-x) + \int_1^0 \frac{\ln(1-z)}{z} dz + \int_0^x \frac{\ln(1-z)}{z} dz \\ &= -\ln x \ln(1-x) - \int_0^1 \frac{\ln(1-z)}{z} dz \end{aligned}$$

by the identity (3.7) the proof is complete of Euler's reflexion formula is complete.

Substituting $x=\frac{1}{2}$ into Euler's reflexion formula leads to the special value

$$\mathbf{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.$$

The following theorem is proved In 2022 by S. M. Stewart "Some simple proofs of Lima's two-term dilogarithm identity.

Theorem 3.1 *For* $|x|\leq 1$ *the following dilogarithms holds*

$$\mathbf{Li}_2\left(\frac{1-x}{1+x}\right)-\mathbf{Li}_2\left(-\frac{1-x}{1+x}\right)=\frac{\pi^2}{4}+\mathbf{Li}_2\left(-x\right)-\mathbf{Li}_2\left(x\right)+\ln x\ln\frac{1+x}{1-x}$$

Proof. In view of (3.11) it is immediate that

$$\frac{d}{dx}\mathbf{Li}_2\left(x\right)=-\frac{\ln\left(1-x\right)}{x}.$$

Consider

$$\begin{aligned}\frac{d}{dx}\left[\mathbf{Li}_2\left(\frac{1-x}{1+x}\right)-\mathbf{Li}_2\left(-\frac{1-x}{1+x}\right)\right] &= \frac{2}{1-x^2}\ln\frac{2x}{x+1}-\frac{2}{1-x^2}\ln\frac{2}{x+1}\\ &= \frac{2}{1-x^2}\ln x.\end{aligned}$$

Integrating the above expression with respect to x gives

$$\begin{aligned}\mathbf{Li}_2\left(\frac{1-x}{1+x}\right)-\mathbf{Li}_2\left(-\frac{1-x}{1+x}\right) &= \int\frac{2}{1-x^2}\ln x dx+C\\ &= \int\frac{\ln x}{1-x}dx+\int\frac{\ln x}{1+x}dx+C\end{aligned}$$

after a partial fraction decomposition has been employed. Here C is an arbitrary constant of integration. Making the change of variable $x=1-t$, we see that the first above integral appearing is

$$\int\frac{\ln x}{1-x}dx=\mathbf{Li}_2\left(1-x\right)$$

integrating by parts followed by a change of variable of $x=-t$ leads to

$$\int\frac{\ln x}{1+x}dx=\ln x\ln\left(1+x\right)+\mathbf{Li}_2\left(-x\right)$$

then

$$\mathbf{Li}_2\left(\frac{1-x}{1+x}\right)-\mathbf{Li}_2\left(-\frac{1-x}{1+x}\right)=\mathbf{Li}_2\left(1-x\right)+\mathbf{Li}_2\left(-x\right)+\ln x\ln\left(1+x\right)+C$$

To find the constant C, we set $x=0$, we find then

$$C=-\mathbf{Li}_2\left(-1\right)=\frac{\pi^2}{12}.$$

■

3.6 Values of $\int\limits_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}dx$

In this section we have detailed the proof of the problem 11993 (American Mathematical Monthly, Vol.124, August-September 2017), this problem is proposed by C. I. Vălean (Romania). Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Universit'a di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Problem 3.6 *The following identity holds*

$$\int\limits_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}dx = -\frac{\pi^4}{280}$$

Proof. By letting $a = \ln(1-x)$ and $b = \ln(1+x)$ in the following identity

$$6ab^2 = (a+b)^3 + (a-b)^3 - 2a^3,$$

we get

$$I = \int\limits_0^1 \frac{\ln(1-x)\ln^2(1+x)}{x}dx = \frac{I_1 + I_2 - 2I_3}{6}$$

where

$$I_1 = \int\limits_0^1 \frac{\ln^3(1-x^2)}{x}dx,\ I_2 = \int\limits_0^1 \frac{\ln^3\frac{1-x}{1+x}}{x}dx \text{ and } I_3 = \int\limits_0^1 \frac{\ln^3(1-x)}{x}dx.$$

By the substitution $t = 1 - x^2$ and the relation (3.10)

$$\begin{aligned} I_1 &= \int\limits_0^1 \frac{\ln^3(1-x^2)}{x}dx = \frac{1}{2}\int\limits_0^1 \frac{\ln^3 t}{1-t}dt \\ &= \frac{1}{2}\int\limits_0^1 \frac{\ln^3(1-x)}{x}dt = -3\xi(4). \end{aligned}$$

and by the substitution ($t=\frac{1-x}{1+x}$, $dt=\frac{-2}{(1+x)^2}$, $x=\frac{1-t}{1+t}$) and the relations (3.5), (3.10)

$$\begin{aligned} I_2 &= \int_0^1 \frac{\ln^3 \frac{1-x}{1+x}}{x}dx = 2\int_0^1 \frac{\ln^3 t}{(1-t)(1+t)}dt \\ &= \int_0^1 \frac{\ln^3 t}{1-t}dt + \int_0^1 \frac{\ln^3 t}{1+t}dt = \int_0^1 \frac{\ln^3(1-t)}{t}dt + \int_0^1 \frac{\ln^3 t}{1+t}dt \\ &= -6\xi(4) - \frac{21}{4}\zeta(4) = -\frac{45}{4}\zeta(4) \end{aligned}$$

finaly

$$I = \frac{I_1 + I_2 - 2I_3}{6} = -\frac{3}{8}\xi(4) = -\frac{1}{240}\pi^4.$$

■

3.7 Some applications of Logarithmic Integrals

A variety of identities involving harmonic numbers and generalized harmonic numbers have been investigated since the distant past and involved in a wide range of diverse fields such as analysis of algorithms in computer science, various branches of number theory. Here we show how one can obtain certain infinite series involving harmonic numbers.

We have detailed the proof of the problem 11682 (American Mathematical Monthly, Vol.119, December 2012) Proposed by Ovidiu Furdui (Romania).

Problem 3.7 *The following identity holds*

$$\sum_{n=1}^{+\infty} \frac{H_n}{n^2} = 2\xi(3).$$

Proof. We proceed as follow

$$\sum_{n=1}^{+\infty} \frac{H_n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n}\frac{H_n}{n} = -\sum_{n=1}^{+\infty} \frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx$$

by the relations (3.1), (2.9) and (3.8) we have

$$\begin{aligned}
\sum_{n=1}^{+\infty}\frac{H_n}{n^2} &= \sum_{n=1}^{+\infty}\frac{1}{n}\frac{H_n}{n} = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx \\
&= -\int_0^1 \ln(1-x)\sum_{n=1}^{+\infty}\frac{1}{n}x^{n-1}dx \\
&= -\int_0^1 \frac{\ln(1-x)}{x}\sum_{n=1}^{+\infty}\frac{x^n}{n}dx \\
&= \int_0^1 \frac{\ln^2(1-x)}{x}dx \\
&= 2\xi(3).
\end{aligned}$$

■

Problem 3.8 *The following identity holds*

$$\sum_{n=0}^{+\infty}(-1)^n\left(\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{n+k}\right)^2 = \frac{\pi^2}{24}. \tag{3.12}$$

Proof. Let

$$S = \sum_{n=0}^{+\infty}(-1)^n\left(\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{n+k}\right)^2,$$

And starting again from the representations

$$\frac{1}{n+k} = \int_0^1 x^{n+k-1}dx$$

then

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{n+k} = \int_0^1 x^n\sum_{k=1}^{\infty}(-x)^{k-1}\,dx = \int_0^1 \frac{x^n}{1+x}dx$$

we continue this way

$$\begin{aligned} S &= .\sum_{n=0}^{+\infty}(-1)^n\left(\int_0^1 \frac{x^n}{1+x}dx\right)^2 = \sum_{n=0}^{+\infty}(-1)^n\left(\int_0^1 \frac{x^n}{1+x}dx\right)\left(\int_0^1 \frac{y^n}{1+y}dy\right) \\ &= \sum_{n=0}^{+\infty}(-1)^n\left(\int_0^1\int_0^1 \frac{(xy)^n}{(1+x)(1+y)}dxdy\right) \\ &= \int_0^1\int_0^1 \left\{\sum_{n=0}^{+\infty}(-xy)^n\right\}\frac{dxdy}{(1+x)(1+y)} \\ &= \int_0^1\int_0^1 \frac{dxdy}{(1+xy)(1+x)(1+y)} \end{aligned}$$

Here we set $u = \frac{y}{x}$ to get

$$S = \int_0^1 \left\{\int_0^x \frac{1}{(1+u)(u+x)}du\right\}\frac{1}{x+1}dx.$$

Using partial fractions

$$\frac{1}{(1+u)(u+x)} = \frac{1}{x-1}\left(\frac{1}{(u+1)} - \frac{1}{(u+x)}\right)$$

we can evaluate the inside integral. The result is

$$S = \int_0^1 \frac{1}{x^2-1}\ln\frac{2}{1+x}dx.$$

The substitution ($\frac{2}{1+x} = 1+t$, $x = \frac{1-t}{1+t}$, $dx = \frac{-2}{(1+t)^2}dt$) transforms this integral into a more transparent one

$$S = \frac{1}{2}\int_0^1 \frac{\ln(1+t)}{t}dt.$$

By the relation (3.2) and the folowing remark

$$S = \frac{1}{2}\int_0^1 \left\{\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}t^k\right\}\frac{1}{t}dt = \frac{1}{2}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}.$$

the proof is complete of this problem. ■

Remark 3.1 *The proof of the above problem (3.12) is given in 2022 by K. N. Boyadzhiev (see [8])*

3.8 Example and problem

In this section, we solved an example and left one problem unsolved, out of curiosity to look for the solution.

Example 3.1 *1-Prove the following equality holds*

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx = -\frac{5}{8}\zeta(3).$$

2- Deduce

$$\sum_{n=1}^{+\infty}(-1)^n \frac{H_n}{n^2} = -\frac{5}{8}\zeta(3).$$

Let

$$I = \int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx.$$

By letting $a = \ln(1-x)$ *and* $b = \ln(1+x)$ *in the following indication*

$$4ab = (a+b)^2 - (a-b)^2,$$

we get

$$I = \int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx = \frac{I_1 - I_2}{4}$$

where

$$I_1 = \int_0^1 \frac{\ln^2(1-x^2)}{x}dx \text{ and } I_2 = \int_0^1 \frac{\ln^2 \frac{1-x}{1+x}}{x}dx .$$

by the substitution $t = 1-x^2$ *and the relation (3.8)*

$$\begin{aligned} I_1 &= \int_0^1 \frac{\ln^2(1-x^2)}{x}dx = \frac{1}{2}\int_0^1 \frac{\ln^2 t}{1-t}dt \\ &= \frac{1}{2}\int_0^1 \frac{\ln^2(1-x)}{x}dt = \xi(3). \end{aligned}$$

and by the substitution ($t=\frac{1-x}{1+x}$, $dt=\frac{-2}{(1+x)^2}$, $x=\frac{1-t}{1+t}$) *and the relations (3.4), (3.8)*

$$\begin{aligned} I_2 &= \int_0^1 \frac{\ln^2\frac{1-x}{1+x}}{x}dx = 2\int_0^1 \frac{\ln^2 t}{(1-t)(1+t)}dt \\ &= \int_0^1 \frac{\ln^2 t}{1-t}dt + \int_0^1 \frac{\ln^2 t}{1+t}dt = \int_0^1 \frac{\ln^2(1-t)}{t}dt + \int_0^1 \frac{\ln^2 t}{1+t}dt \\ &= 2\xi(3) + \frac{3}{2}\zeta(3) = \frac{7}{2}\xi(3), \end{aligned}$$

then

$$I = \frac{I_1 - I_2}{4} = \frac{\xi(3) - \frac{7}{2}\xi(3)}{4} = -\frac{5}{8}\zeta(3).$$

2- By the relations (3.1), (2.9) and (3.8) we have

$$\begin{aligned} \sum_{n=1}^{+\infty}(-1)^n\frac{H_n}{n^2} &= \sum_{n=1}^{+\infty}\frac{(-1)^n}{n}\frac{H_n}{n} = -\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx \\ &= -\int_0^1 \ln(1-x)\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}x^{n-1}dx \\ &= -\int_0^1 \frac{\ln(1-x)}{x}\sum_{n=1}^{+\infty}\frac{(-x)^n}{n}dx \\ &= \int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx \\ &= -\frac{5}{8}\zeta(3). \end{aligned}$$

Problem 3.9 *1-Prove the following equality holds*

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{1+x}dx = \frac{1}{24}\left(8\ln^3 2 - \pi^2\ln 4 + 3\zeta(3)\right).$$

2- Deduce

$$\sum_{n=1}^{+\infty}(-1)^n\frac{H_n^2}{n} = \frac{1}{12}\left(\pi^2\ln 2 - 4\ln^3 2 - 9\zeta(3)\right).$$

Conclusion

The main results of last chapter represent an interesting contribution in integral logarithms. They are obtained by using technical operations on binomial coefficients and harmonic numbers.

Even today, the study of the calculation of indefinite integrals concerning logarithms, log-sin integrals or others is an important research topic have been widely studied in many papers.

Many questions arise, how to evaluate the integral or calculate the partial sum of the series such

$$\sum_{n=1}^{+\infty} \frac{H_{n,l}^{m}}{n^{s}}.$$

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