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Introduction

The theory of inequalities is an important subject of research where several mathematical situations call on these inequalities. On the other hand, integral inequalities have known a great development in several fields such as real analysis, numerical analysis and differential equations, etc. They represent a powerful and indispensable tool.

The Interest in the study of integral inequalities has steadily grown to an abundant literature has been developed on this subject and for full details see the work of Pachpatte [8], Burton, Pecaric[5], S.S Dragomir [18], where we can find a very good description of the historical evolution of inequalities.

Many researchers have generalized the classical and fractional operators by introducing parameters about ten years ago.

Recently many researchers have presented new dierential operators and fractional integrals and they have generalized by iteration procedure and introducing new strictly positive parameters.

In [6], Belarbi and Dahmani presented theorems related to Chebyshev's inequality for fractional integral operators of Riemann-Liouville ([16, 17, 19]).

In our memory we are interested to the Chebyshev type inequalities which have been applied almost to all kinds of functions and applicable in several fields such as in numerical integration and in nonlinear. The objective of this memory is to make a small synthesis concerning the Chebyshev-type integral inequalities for functions of single variables, and for functions of several variables.

This memory is divided as follows:
Chapter 1 is dedicated to a reminder of fractional integration [13] presents some preliminary notions, which will be used later in the demonstrations, and some definitions of functional analysis.
In the second chapter, we will present some integral inequalities of classical and fractional Chebyshev type for functions of one variable. While the last chapter will be entirely devoted to new integral Chebyshev-type inequalities for functions of several variables [20].

Chapter 1

Basic Notions

We mention fundamental notions and resuls of the functionnal analysis .

1.1 Functional Spaces

1.1.1 Spaces of Integrable Functions

Definition 1.1.1. *For $1 \leq p < \infty$ we denote by $L_p := L_p(0, \infty)$ the set of all Lebesgue measurable functions f such that*

$$\|f\|_p \left(\int_0^\infty |f(x)|^p dx\right)^{1/p} \leq \infty.$$

Definition 1.1.2. *We denote by $L^\infty(\Omega)$the space of essentially bounded functions on (Ω),*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \ \ measurable \ \ and \ \ \exists c \geq 0, \quad |f(x)| \leq c \text{ a.e on } \quad \Omega\}$$

such that

$$\|f\|_{L^\infty(\Omega)} = ess \sup_{t \in \Omega} |f(t)| = \inf \{M \geq 0 : |f(x)| \leq M \quad a.e \quad on \quad \Omega\}.$$

Theorem 1.1.1. *Let $\Omega = (a, b)$finite or infinite interval of $\mathbb{R}$.*
For $1 \leq p \leq +\infty$ the space $L^p(\Omega)$ is a Banach space.

1.1.2 Space of Continious Functions

Definition 1.1.3. *Let $\Omega = (a, b)(-\infty \leq a < b \leq +\infty)$ and $n \in \mathbb{N}$, we denote by $C^n(\Omega)$ the space of functions f such that $f^{(n)}$ is continuous on Ω .*

Theorem 1.1.2. *The space $C^n(\Omega)$ normed by*

$$\|f\|_{C^n} = \sum_{k=0}^{n} \|f^k\|_C = \sum_{k=1}^{n} \max_{x \in \Omega} |f^{(k)}|, \qquad n \in \mathbb{N}.$$

is Banach space.

Especially, if $n = 0$ $C^{(0)}(\Omega) \equiv C(\Omega)$ the space of continuous functions f on Ω normed by

$$\|f\|_c = \max_{x \in \Omega} |f(x)|$$

is Banach space.

1.1.3 Absolutely Continuous Functions

Definition 1.1.4. *Let $f : [a, b] \longrightarrow \mathbb{R}$ and $([a_k, b_k])_{k \in \mathbb{N}}$ a finite suite of under interval disjoint of $[a, b]$,we say that $f$is absolutely continuous on $[a, b]$, if for all real $\xi > 0$, it existe a $\delta(\xi) > 0$ such as*

$$\sum_{k=1}^{n} |b_k - a_k| < \delta(\xi)$$

then

$$\sum_{k=0}^{n} |f(b_k) - f(b_k)| < \xi$$

.

1.1.4 Weighted Continuous Functions

Definition 1.1.5. *A real valued function $f : [a, \infty) \longrightarrow \mathbb{R}$ is said to be in the space C_μ,$\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1 \in C[0, \infty)$.*

1.2 Probability Density Function

The function f defined On $[a, b]$ is called a probability density function if it is continuous, positive and

$$\int_a^b f(x) = 1.$$

Example 1.2.1. *The function* $f(x) = \dfrac{1}{\pi(1+x^2)}$ *defined on* $\mathbb{R}$ *is probability density function.*

1.3 Inquality of Hölder

Theorem 1.3.1. *Let* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *with* $1 < p < \infty$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *so*

$$\int_\Omega |(f.g)(x)|dx \leq \left(\int_\Omega |f(x)|^p dx\right)^{1/p} \left(\int_\Omega |g(x)|^q dx\right)^{1/q}.$$

Remark 1.3.1. *If* $p = 1$, $q = \infty$, *then*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}.\|g\|_{L^\infty(\Omega)}.$$

and reciprocally if $q = 1$, $p = \infty$, *then*

$$\|fg\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}.\|f\|_{L^\infty(\Omega)}.$$

1.4 Theorem of Fubini

Let $((a, b) \subset \mathbb{R}^n)$ and $((c, d) \subset \mathbb{R}^n)$ measurable sets and the function $f(x, y)$ is integrable on $(a, b) \times (c, d)$ so for all $x \in [a, b)$, $f(x, y)$ is integrable on (c, d) and :

$$\begin{aligned} \int_{(a,b)\times(c,d)} f(x,y)dxdy &= \int_a^b \left(\int_c^d f(x,y)dy\right) dx, \\ &= \int_c^d \left(\int_a^b f(x,y)dx\right) dy, \end{aligned} \quad (1.1)$$

i.e if $f(x, y)$ is a measurable on $(a, b) \times (c, d)$ and is finite one of the integrals

$$\int_{(a,b)} \left(\int_{(c,d)} |f(x,y)dydx\right) = \int_c^d \int_a^b f(x,y)dxdy.$$

1.4.1 Formula of Dirichler

It is particular case of theorem of Fubini , we have the following equality :

$$\int_a^b \int_a^x f(x,y)dydx) = \int_a^b \int_y^b f(x,y)dxdy.$$

Where one of two previous integrals at least is absolutely convergent.

1.5 Some Concepts in Fractional Calculus

1.5.1 Specific Functions

In this section we recall the functions Gamma and Bêta because they have an important role in the theory of fractional calculus and its applications.

Definition 1.5.1. *Let $z > 0, r, s > 0$. The gamma and the beta functions are defined as follows*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad z > 0.$$

$$B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}dt,$$

proposition 1.5.1. *For all $z \in \mathbb{C}$ with $Re(z) > 0$, we have*

$$\Gamma(z+1) = z\Gamma(z),$$

proposition 1.5.2. *For all $n \in \mathbb{N}$ we have*

$$\Gamma(n+1) = n!.$$

proposition 1.5.3. *Bêta function is related by Gamma function by the following relation*

$$B(z,w) = \frac{\Gamma(z).\Gamma(w)}{\Gamma(z+w)}.$$

for all $z > 0$; $w > 0$.

proposition 1.5.4. *For all $z > 0$; $w > 0$.*

$$\beta(z,w) = \beta(w,z).$$

1.6 The Fractional Integral Over an Interval[a,b]

Definition 1.6.1. *Let f a continuous function on [a,b], we consider the integrals:*

$$\begin{aligned} J^{(1)}f(x) &= \int_a^x f(t)dt \\ J^{(2)}f(x) &= \int_a^x \left(J^{(1)}f(\mu)\right) d\mu \\ &= \int_a^x \left(\int_a^\mu f(t)dt\right) d\mu \\ &= \int_a^x \left(\int_t^x d\mu\right) dt \\ &= \int_a^x (x-t)f(t)dt; \quad (n \in \mathbb{N}^*). \end{aligned} \tag{1.2}$$

More generally the n-th of the operator J can be written:

$$\begin{aligned} J^{(n)}f(x) &= \int_0^{x_1} dx_1 \int_0^{x_2} dx_2 \int_a^{x_{n-1}} f(x_n)dx_n \\ &= \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt. \end{aligned} \tag{1.3}$$

For all $n \in \mathbb{N}$.
This formula is called formula of Cauchy, and from the generalization of factorial by the Gamma function $\Gamma(n) = (n-1)$, Riemann releazed that the 2^{nd} member of (1.3) might have meaning even when n take a non integer value, it defined the fractional integral as follows.

Definition 1.6.2. *The Riemann-Liouville fractional integral operators of order $\alpha \geq 0$ of function $f(x) \in L_1[a,b], -\infty < a < b < +\infty$, are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \quad x > a.$$

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

For $a = 0$ we denote J_{a+}^α by J^α.

Theorem 1.6.1. *If $f \in L^1([a,b])$ then J_a^α f exist a.e $x \in [a,b]$ and further $J_a^\alpha f \in L^1([a,b])$.*

proposition 1.6.1. *Let $\alpha, \beta \in \mathbb{C}$, such that $Re(\alpha), Re(\beta) > 0$, for all function $f \in L^1([a,b])$ we have,*

$$J_a^\alpha \left(J_a^\beta f\right) = J_a^{\alpha+\beta} f = J_a^\beta \left(J_a^\alpha f\right).$$

For almost everywhere $x \in [a,b]$. If more over $f \in C([a,b])$, then this identity is true for all $x \in [a,b]$.

Example 1.6.1. *We consider the function $f(x) = (x-a)^m$ for $\alpha > 0$ and $m > -1$ so:*

$$J_a^\alpha (x-a)^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)}(x-a)^{\alpha+m},$$

in effect

$$J_a^\alpha (x-a)^m = \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1}(t-a)^m dt.$$

Using the change of variable $t = a + (x-a)\tau, 0 \leq \tau \leq 1$ so taking in account the Bêta function and proposition (1.5.3) we get

$$J_a^\alpha (x-a)^m = \frac{1}{\Gamma(\alpha)}(x-a)^{\alpha+m} \times \beta(m+1,\alpha)$$

$$J_a^\alpha (x-a)^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)}(x-a)^{\alpha+m}.$$

Chapter 2

Some Chebyshev type inqualities of a single variable

This chapter is divided into two sections:
The first section is devoted to some classical Chebyshev inequalities.
In second section we are intersted in the famous fractionnal inequalities of a single variable introduced by Chebyshev .
The Chebyshev type integral inqualities play an important role in all branches of mathematics, these inequalities apply to derivable ,absolutely continuous, lipchitzian, monotonic functions and to functions and to function with limited variation.

2.1 Classica chebychev's Functional

Definition 2.1.1. *Let $0 \leq a < b < \infty$ f and g be two integrable fuctions on [a,b] and*

$$T(f,g) := \int_a^b f(x)g(x)dx - \frac{1}{(b-a)} \left(\int_a^b f(x)dx \right) \left(\int_a^b g(x)dx \right). \qquad (2.1)$$

Definition 2.1.2. *For two measurable functions $f, g : [a, b] \longrightarrow \mathbb{R}$,define the functional, which is known in the literature as Chebychev's functional*

$$T(f,g;a,b) := \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx. \int_a^b g(x)dx. \qquad (2.2)$$

Cerone and Dragomir [3] have pointed out generalizations of the above results for integrals defined on two different intervals [a,b] and [c,d]. They defined the functional (generalised Chebyshev functional)

$$T(f,g;a,b,c,d) := M(f,g;a,b) + M(f,g;c,d) - M(f;a,b)M(g;c,d). \tag{2.3}$$

Where the integral mean is defined by

$$M(f;a,b) := \frac{1}{b-a}\int_a^b f(x)dx. \tag{2.4}$$

2.2 Some Chebyshev Type Inequalities of a Single Variable

2.2.1 Some Classical Chebyshev Inequalities

The following Chebyshev inequalities for synchronous functions (asynchronous) arae given by the following lemma:

Lemma 2.2.1 *If f and g are synchronous on $[a,b]$ i.e $(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geq 0$. for each $\tau,\rho \in [a,b]$, then*

$$T(f,g) \geq 0. \tag{2.5}$$

If f,g asynchronous on $[a,b]$, i.e $(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \leq 0$, For each $\tau,\rho \in [a,b]$. The constant $\frac{1}{b-a}$ is the best possible in inequality (2.5).
Where $T(f,g)$ difined by (2.1)
The following inequality is well known as the Gruss inequality [1].

Lemma 2.2.2 *Let $f,g : [a,b] \to \mathbb{R}$ two integrable functions on $[a,b]$and the function $f.g$ is integrable on $[a,b]$, then*

$$|T(f,g;a,b)| \leq \frac{1}{4}(M-m)(N-n), \tag{2.6}$$

provided that $m \leq f \leq M$ and $n \leq g \leq N$ a.e. on $[a,b]$, where m,M,n,N are real numbers. The constant $\frac{1}{4}$ in (2.6) is the best possible.

Another inequality of this type is due to Chebychev (see for example [2]. Namely, if f, g are absolutely continous on $[a, b]$ and $f', g' \in L_\infty[a, b]$ and $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|$, then

$$|T(f, g; a, b)| \leq \frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b - a)^2 \tag{2.7}$$

and the constant $\frac{1}{12}$ is the best possible.

Theorem 2.2.1. *Let f,g : $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be measurable on I, and the intervals $[a, b]$,$[c, d] \subset I$. Further, suppose that f and g are of Hölder type so that for $x \in [a, b], y \in [c, d]$*

$$|f(x) - f(y)| \leq H_1|x - y|^r and|g(x) - g(y)| \leq H_2|x - y|^8 \tag{2.8}$$

where $H_1, H_2 > 0$ and $r, s \in (0, 1]$ are fixed. The following inequality then holds,

$$(\theta+1)(\theta+2)|T(f, g; a, b, c, d)| \leq \frac{H_1H_2}{(b - a)(d - c)}[|b-c|^{\theta+2}-|b-d|^{\theta+2}+|d-a|^{\theta+2}-|c-a|^{\theta+2}], \tag{2.9}$$

where $\theta = r + s$ and $T(f, g; a, b, c, d)$ is as defined by (2.3) and (2.4).

2.2.2 Classical Chebyshev Type Inequalities For Riemann-Liouuville Operator

Theorem 2.2.2. *Let f and g be two synchronous functions on $(0, \infty)$, then*

$$\int_0^t (f.g)(x)dx \geq \frac{1}{t}\int_0^t f(x)dx \times \int_0^t g(x)dx; \qquad t > 0. \tag{2.10}$$

The inquality (2.10) is reversed if the functions are asynchronous on $(0, \infty)$.

proof Since the functions f and g are synchronous on $(0; \infty)$,then for all $\tau \geq 0, \rho \geq 0$ we have:

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0.$$

Hence

$$f(\tau)g(\tau)+f(\rho)g(\rho)\geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{2.11}$$

Integrating inequality (2.11) with respect to τ over $(0,x)$, we obtain:

$$\int_0^t (fg)(\tau)d\tau+f(\rho)g(\rho)\times t\geq \int_0^t f(\tau)g(\rho)d\tau+\int_0^t f(\rho)g(\tau)d\tau$$

then

$$\int_0^t (fg)(\tau)d\tau+tf(\rho)g(\rho)\geq g(\rho)\int_0^t f(\tau)d\tau+f(\rho)\int_0^t g(\tau)d\tau \tag{2.12}$$

□

we integrale the obtained inequality with respect to ρ over $(0,t)$:

$$t\int_0^t (fg)(\tau)d\tau+t\int_0^t f(\rho)g(\rho)d\rho\geq \int_0^t f(\tau)d\tau\times\int_0^t g(\rho)d\rho+\int_0^t g(\tau)d\tau+\int_0^t f(\rho)d\rho$$

this yields $\int_0^t (fg)(x)dx\geq \frac{1}{t}\int_0^t f(x)dx\times\int_0^t g(x)dx.$ □

Theorem 2.2.3. *Let f and g be two synchronous functions on* $(0,\infty)$, *then*

$$t\int_0^t (f.g)(x)dx\geq \int_0^t f(x)dx\times\int_0^t g(x)dx. \tag{2.13}$$

proof Multiplying both sides of (2.10) by t we get (2.13). □

Theorem 2.2.4. *Let* $\{f_i\}_{1\leq i\leq n}$ *be n positive increasing functions on* $(0,\infty)$, *then*

$$\int_0^x (\prod_{i=1}^n f_i)(t)dt\geq x^{(1-n)}\times\prod_{i=1}^n\int_0^x f_i(t)dt$$

for all $x\geq 0$.

Theorem 2.2.5. *Let f and g be two functions defined on* $(0,\infty)$, *such that f is increasing and g is differentiable and there exists a real number* $m=\inf_{x\geq 0} g'(x)$. *Then*

$$\int_0^x (f.g)(t)dt\geq \frac{1}{x}\int_0^x f(t)dt\times\int_0^x g(t)dt-\frac{m}{2}\int_0^x f(t)dt+m\int_0^x (xf)(t)dt$$

is valid for all $x>0$.

2.2.3 Classical Chebyshev Type Inequalities Using Pecaric Extention

Montgomery's identity is a very effective technique in establishing new generalizations of the Chebyshev type of integral inequalities. The following section discusses some extensions of this identity, then a generalization of some integral inequalities [5].

Lemma 2.2.3 *Let $f : [a, b] \to \mathbb{R}$ a differentiable function on $[a, b]$, we suppose that $f'(t)$ is integrable on $[a, b]$. Then the Montgomery's identity*

$$f(x) = \int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{2.14}$$

is satisfaite where the kernel of Peano is defined by

$$P(x,t) = \begin{cases} \dfrac{t-a}{b-a}, & a \leq t \leq x, \\ \dfrac{t-b}{b-a}, & x < t \leq b. \end{cases} \tag{2.15}$$

proof We use integrating by part. □

An extension of this identity was introduced by Pecaric [5]. This extension is given by the following Lemma:

Lemma 2.2.4 *Under the same assumptions of the Lemme(2.2.3) and if $w : [a, b] \to [0, +\infty[$ is a probability density function, then there is a generalization of Montgomery's identity*

$$f(x) = \int_a^b w(t)f(t)dt + \int_a^b P_w(x,t)f'(t)dt. \tag{2.16}$$

Where the kernel of Peano $P_w(x,t)$ is defined by:

$$P_w(x,t) = \begin{cases} W(t) & a \leq t \leq x \\ W(t) - 1 & x \leq t \leq b. \end{cases} \tag{2.17}$$

In 2007 Boukerrioua and Guezane Lakoud in their work [6] ,are established a new generalization of Montgomery's identity given below.

Lemma 2.2.5 *Let $f : [a,b] \to \mathbb{R}$ a differentiable function of derivative f' and Let $\varphi : [0,1] \to \mathbb{R}$ a differentiable function on $[0,1]$ with $\varphi(0) = 0, \varphi(1) \neq 0$ and φ' is integrable on $[0,1]$, then a generalization of Montgomery's identity is given by:*

$$f(x) = \frac{1}{\varphi(1)} \int_a^b w(t)\varphi'\left(\int_a^t w(s)ds\right) f(t)dt + \frac{1}{\varphi(1)} \int_a^b P_{w,\varphi}(x,t)f'(t)dt \tag{2.18}$$

where $P_{w,\varphi}$ is the kernel generalization defined below

$$P_{w,\varphi}(x,t) = \begin{cases} \varphi(W(t)), & a \leq t \leq x, \\ \varphi(W(t)) - \varphi(1), & x < t \leq b. \end{cases} \tag{2.19}$$

With

$$W(t) = \int_a^t w(s)ds.$$

Classical Chebyshev Type Inequalities Using Pecaric Extension

On 2014 Guezane-Lakoud and Aissaoui [7] the first Chebyshev integral inequality using Montgomrey's identity (2.14).

Corollary 1 *Let $f, g : [a,b] \to \mathbb{R}$ two differentiable functions on $[a,b]$,the derivatives f' and g' are integrable, and $f', g' \in L_\infty([a,b])$ So:*

$$|T(f,g)| \leq \frac{7}{60}(b-a)^2 \|f'\|_\infty \|g'\|_\infty \tag{2.20}$$

where $T(f,g)$ is defined in (2.2).

proof Let's define $F, G, \tilde{F}$, and $\breve{G}$ as follows

$$F = f(x) - \frac{1}{b-a}\int_a^b f(t)dt, \quad \tilde{F} = \int_a^b P(x,t)f'(t)dt$$
$$G = g(x) - \frac{1}{b-a}\int_b^a g(t)dt, \tilde{G} = \int_a^b P(x,t)g'(t)dt.$$

using Montgomery's identity (2.14), We have FG $= \tilde{F}\tilde{G}$ and:

$$\begin{aligned} F.G &= \left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right)\left(g(x) - \frac{1}{b-a}\int_a^b g(t)dt\right) \\ &= f(x)g(x) - f(x)\frac{1}{b-a}\int_a^b g(t)dt \\ &- g(x)\frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b f(t)dt\frac{1}{b-a}\int_a^b g(t)dt. \end{aligned} \tag{2.21}$$

Integrate (2.21) on $[a, b]$, then multiply the result by $\frac{1}{b-a}$,we find

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

$$\begin{aligned} T(f,g) &= \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \quad (2.22) \\ &= \frac{1}{b-a}\int_a^b \int_a^b p(x,t)f'(t)dt \int_a^b p(x,t)g'(t)dtdx \end{aligned}$$

therefore

$$\begin{aligned} |T(f,g)| &= \left|\frac{1}{b-a}\int_a^b \int_a^b p(x,t)f'(t)d \int_a^b p(x,t)g'(t)dtdx\right| \\ &\leqslant \frac{1}{b-a}\int_a^b \int_a^b |p(x,t)|\,|f'(t)|\,dt \int_a^b |p(x,t)|\,|g'(t)|\,dtdx \\ &\leqslant \frac{1}{b-a}\|f'\|_\infty \|g'\|_\infty \int_a^b \left(\int_a^b |p(x,t)|dt\right)^2 dx \quad (2.23) \end{aligned}$$

let's calculate

$$\begin{aligned} \int_a^b |p(x,t)|dt &= \int_a^x |p(x,t)|dt + \int_x^b |p(x,t)|dt \\ &= \frac{1}{2(b-a)}\left[(x-a)^2 + (x-b)^2\right], \end{aligned}$$

then

$$\left(\int_a^b |p(x,t)|dt\right)^2 = \frac{1}{4(b-a)^2}\left[(x-a)^2 + (x-b)^2\right]^2,$$

and

$$\begin{aligned} \int_a^b \left(\int_a^b |p(x,t)|dt\right)^2 dx &= \frac{1}{4(b-a)^2}\int_a^b \left[(x-a)^2 + (x-b)^2\right]^2 dx \\ &= \frac{1}{4(b-a)^2}\left[\int_a^b (x-a)^4 dx + \int_a^b (x-b)^4 dx + 2\int_a^b (x-a)^2(x-b)^2 dx\right] \\ &= \frac{1}{4(b-a)^2}\left[\frac{(b-a)^5}{5} + \frac{(b-a)^5}{5} + \frac{(b-a)^5}{15}\right] \\ &= \frac{7(b-a)^3}{60}, \quad (2.24) \end{aligned}$$

□

so replacing the result (2.24) in (2.23) gets (2.20).

$$|T(f,g)| \leq \frac{1}{b-a} \|f'\|_\infty \|g'\|_\infty \frac{7(b-a)^3}{60} = \frac{7}{60}(b-a)^2 \|f'\|_\infty \|g'\| \|_\infty .$$

Pachpatte, in his work [8], established a new generalization of Chebyhev-type inequality using the extension given by (2.14). The following theorem without proof shows this result. □

Theorem 2.2.6. *Let $f, g : [a,b] \to \mathbb{R}$ two differentiable functions on $[a,b]$ and f' and g' are integrable on $[a,b]$ and w is a probability density function then we have the following inequality:*

$$|T(w,f,g)| \leq \|f'\|_\infty \|g\prime\|_\infty \int_a^b w(x)H^2(x)dx, \tag{2.25}$$

where

$$H(x) = \int_a^b |p_w(x,t)|\, dt,$$

and

$$T(w,f,g) = \int_a^b w(x)f(x)g(x)dt - \int_a^b w(x)dx \int_a^b w(x)g(x)dx. \tag{2.26}$$

The following theorem presents a new generalization of the Chebyshev type integral inequality obtained by K. Boukerrioua and A. Guezane-Lakoud in [6]. Using the extension given by (2.18).

Theorem 2.2.7. *Let $f, g : [a,b] \to \mathbb{R}$ two differentiable functions on $[a,b]$, and f', g' are integrable on $[a,b]$. Let w and φ the two functions defined in lemma(2.2.3) so we have:*

$$|T(w,f,g,\varphi')| \leq \frac{1}{\varphi^2(1)} \|f'\|_\infty \|g'\|_\infty \|\varphi'\|_\infty \int_a^b w(x)H^2(x)dx \tag{2.27}$$

where

$$H(x) = \int_a^b |p_{w,\varphi}(x,t)|\, dt$$

$$\|\varphi'\|_\infty = \underset{t\in[0,1]}{ess} \sup |\varphi'(t)|,$$

and

$$\begin{aligned} T(w,f,g,\varphi') &= \int_a^b w(x)\varphi'\left(\int_a^t w(t)dt\right) f(x)g(x)dx \\ &- \frac{1}{\varphi(1)}\left[\int_a^b w(x)\varphi'\left(\int_a^t w(s)ds\right) f(x)dx\right]\left[\int_a^b w(x)\varphi'\left(\int_a^t w(s)ds\right) g(x)dx\right]. \end{aligned} \tag{2.28}$$

2.2.4 Fractional Chebyshev Type Inqualities for Generalized Riemann-Liouvill Operator

Theorem 2.2.8. *Let f and g be two synchronous functions on $(0, \infty)$. Then ,*

$$J^\alpha(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} J^\alpha f(t) J^\alpha g(t) \tag{2.29}$$

for all $t > 0, \alpha > 0$.

The inequality (2.29) is reversed if the functions are asynchronous on $(0, \infty)$.

Theorem 2.2.9. *Let f and g be two synchronous functions on $(0, \infty)$. Then*

$$\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta(fg)(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha(fg)(t) \geq J^\alpha f(t) J(\beta) g(t) + J^\beta f(t) J^\alpha g(t) \tag{2.30}$$

for all $t > 0, \alpha > 0, \beta > 0$.

The inequality (2.30) is reversed if the functions are asynchronous on $(0, \infty)$.

Theorem 2.2.10. *Let $p \geq 1$,$p^{'} \geq 1$ such that $\frac{1}{p}+\frac{1}{p^{'}} = 1$,if f and g are two functions in L^p and $L^{p^{'}}$, respectively.*

Then

$$J^\alpha(fg)(x) \leq (J^\alpha f^p(x))^{\frac{1}{p}} (J^\alpha g^{p^{'}}(x))^{\frac{1}{p^{'}}}$$

for all $x > 0, \alpha > 0$. The following theorems were proved in [10].

Theorem 2.2.11. *Let $\{f_i\}_{1 \leq i \leq n}$ be n positive increasing functions on $(0, \infty)$ then*

$$J^\alpha \left(\prod_{i=1}^{i=n} \right)(x) \geq (J^\alpha(1)(x))^{(1-n)} \prod_{i=1}^{i=n} J^\alpha f_i(x)$$

for all $x > 0, \alpha > 0$.

Theorem 2.2.12. *Let f and g be two functions defined on (0,∞), such that f is increasing and g is differentiable and there exists a real number $m := \inf_{x \geq 0} g'(x)$.*

Then

$$J^\alpha(fg)(x) \geq (J^{\alpha(1)})^{-1} J^\alpha g(x) - \frac{m}{\alpha+1} J^\alpha f(x) + m J^\alpha(xf)(x)$$

is valid for all $x > 0, \alpha > 0$.

Definition 2.2.1. *Let $\alpha > 0, \beta \geq 1, 1 \leq p < \infty$ and the integral operator $K_{u,v}^{\alpha,\beta}$ be defined as*

$$K_{u,v}^{a,b} f(x) = \frac{v(x)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [\ln(\frac{x}{t})]^{\beta-1} f(t) u(t) dt, \quad x > 0 \tag{2.31}$$

defined from L_p to L_p space, with locally integrable non-negative weight functions u and v.

We mention that for $\alpha > 0, \beta \geq 1$ necessary and sufficient conditions for the boundedness, see [9], and compactness,see [9], of the integral operator $K_{u,v}^{\alpha,\beta}$ from L_p to L_p spaces with $0 <$ p,q $< \infty$ were found for locally integrable non-negative weight functions u,v.

Remark 2.2.1. *If $v(x) = u(x) = 1, \beta = 1$, the operator $K_{1,1}^{\alpha,1}$ coincides with the classical Riemann-Liouville fractional integral operator.*

To simplify the calculations, we denote

$$K := K_{u,v}^{\alpha,\beta}, \quad k(x,t) := (x-t)^{\alpha-1} \ln^{\beta-1}\left(\frac{x}{t}\right) \neq 0.$$

Then the integral operator in inequality (2.31) becomes

$$Kf(x) = \frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x,t) f(t) u(t) dt, \quad x > 0$$

.

Theorem 2.2.13. *Let f,g be two sunchronous functions on $(0, \infty)$, u and v locally integrable non-negative weight functions.*

Then

$$K(fg)(x) \geq (K(1))^{-1} Kf(x) Kg(x), \tag{2.32}$$

where

$$K(1)(x) = \frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x,t) u(t) dt.$$

Inequality (2.32) is reversed if the functions are asynchronous on $(0, \infty)$.

proof Since the functions f and g are synchronous on $(0, \infty)$, then for all $\tau \geq 0, \rho \geq 0$ we have:

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0.$$

Hence,

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\tau)g(\rho). \tag{2.33}$$

Multiplying both sides of inequality (2.35) by $\frac{v(x)}{\Gamma(\alpha)} k(x, \tau)u(\tau), \tau \in (0, x)$, we get:

$$\frac{v(x)}{\Gamma(\alpha)} k(x, \tau)f(\tau)g(\tau)u(\tau) + \frac{v(x)}{\Gamma(\alpha)} k(x, \tau)f(\rho)g(\rho)u(r)$$
$$\geq \frac{v(x)}{\Gamma(\alpha)} k(x, \tau)f(\tau)g(\rho)u(\tau) + \frac{v(x)}{\Gamma(\alpha)} k(x, \tau)f(\rho)g(\tau)u(\tau). \tag{2.34}$$

Integration inequality (2.34) with respect to τ over $(0, x)$, we obtain

$$\frac{v(x)}{\Gamma(\rho)} \int_0^x k(x, \tau)f(\tau)g(\tau)u(\tau)d\tau + f(\rho)g(\rho)\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x, \tau)u(\tau)d\tau$$
$$\geq g(\rho)\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x, \tau)f(\tau)u(\tau)d\tau + f(\rho)\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x, \tau)f(\tau)u(\tau)d\tau$$

. This implies:

$$K(fg)(x) + f(\rho)g(\rho)K(1)(x) \geq g(\rho)K(f)(x) + f(\rho)g(\rho)K(1)(x) \tag{2.35}$$

□

Multiplying both sides of (2.35)by $\frac{v(x)}{\Gamma(\alpha)} k(x, \rho)u(\rho)$, we get

$$\frac{v(x)}{\Gamma(\alpha)} k(x, \rho)u(\rho)K(fg)(x) + \frac{v(x)}{\Gamma(\alpha)} k(x, \rho)u(\rho)f(\rho)g(\rho)K(1)(x)$$
$$\geq \frac{v(x)}{\Gamma(\alpha)} k(x, \rho)g(\rho)K(f)(x) + \frac{v(x)}{\Gamma(\alpha)} k(x, \rho)u(\rho)f(\rho)K(g)(x)$$

We integrate the obtained inequality with respect to ρ over $(0, x)$:

$$K(fg)(x)\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x, \rho)u(\rho)d\rho + K(1)(x)\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x, \rho)u(\rho)f(\rho)g(\rho)d(\rho)$$
$$\geq Kf(x)\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x, \rho)u(\rho)g(\rho)d\rho + Kg(x)\frac{v(x)}{\Gamma(\alpha)} \int_0^x k(x, \rho)f(\rho)u(\rho)d\rho.$$

Hence,

$$K(fg)(x)K(1)(x) + K(1)(x)K(fg)(x) \geq Kf(x)Kg(x) + Kg(x)Kf(x).$$

This yields:

$$K(fg)(x) \geq (K(1))^{-1} Kf(x) Kg(x).$$

If f and g are asynchronous, the proof is similar to that of synchronous case.
The proof is complete. □

Remark 2.2.2. *Theorem (2.2.13) applied with $v(x) = u(x) = 1, \beta = 1$ gives Theorem (2.2.8).*

Theorem 2.2.14. *Lef $\{f_i\}_{1 \leq i \leq n}$ be n positive increasing functions on $[0, \infty)$ and u and v be localement integrables non-negative weight functions. Then .*

$$K\left(\prod_{i=1}^{i=n} f_i\right)(x) \geq (K(1)(x))^{(1-n)} \prod_{i=1}^{i=n} Kf_i(x)$$

for all $x > 0$.

proof We prove this Theorem by induction. We suppose that

$$K\left(\prod_{i=1}^{i=n-1} f_i\right)(x) \geq (K(1)(x))^{2-n} \prod_{i=1}^{i=n-1} Kf_i(x). \tag{2.36}$$

□

Sience $\{f_i\}_{1 \leq i \leq n}$ are positive increasing function, then $\prod_{i=1}^{i=n-1} f_i$ is an increasing function.
Hence, we can apply Theorem (2.2.13) with $\prod_{i=1}^{i=n-1} f = g, f_n = f$,and we obtain

$$K\left(\prod_{i=1}^{i=n} f_i\right)(x) = K(fg)(x) \geq (K(1))^{-1} K\left(\prod_{i=1}^{i=n-1} f_i\right)(x) f_n(x).$$

Therefore, by (2.32), we get

$$K\left(\prod_{i=n}^{i=n} f_i\right)(x) \geq (K(1))^{-1} (K(1))^{2-n} \left(\prod_{i=n}^{i=n} f_i\right)(x) Kf_n(x)$$

and the proof is complete. □

Remark 2.2.3. *Theorem (2.2.14) $v(x) = u(x) = 1, \beta = 1$ gives Theorem (2.2.11). Considering $f_i = f, i = 1, 2, ..., n$,in Theorem (2.2.14), we get the following Corollary.*

Corollary 2 *Let f be an increasing positive function on $(0,\infty)$,u and v locally integrable non-negative weight function.*
Then

$$K(f^n)(x) \geq (K(1)(x))^{(1-n)}(Kf(x))^n$$

Now we consider the next two operators

$$K_1f(x) = \frac{v_1(x)}{\Gamma(\alpha_1)}\int_0^x (x-t)^{\alpha_1-1}\ln^{\beta_1-1}(\frac{x}{t})f(t)u_1(t)dt$$

$$K_2f(x) = \frac{v_2(x)}{\Gamma(\alpha_2)}\int_0^x (x-t)^{\alpha_2-1}\ln^{\beta_2-1}(\frac{x}{t})f(t)u_2(t)dt$$

Theorem 2.2.15. *Let f,g be two synochronous functions on $(0,\infty)$,*
$p,q:[a,b]\to(0,\infty)$ be integrable, u_i and v_i $i=1,2$, locally integrable non-negative weight functions. Then

$$K_2q(x)K_1(pfg)(x)+K_1p(x)K_2(qfg)(x) \geq K_2(qg)(x)K_1(pf)(x)+K_2(qf)(x)K_1(pg)(x). \tag{2.37}$$

For all $x>0$.
Inequality (2.37) is reversed if the functions are asynchronous on $(0,\infty)$.

proof We multiply both sides of inequality (2.33) by $\frac{v_1(x)}{\Gamma(\alpha)}K_1(x,\tau)u_1(\tau)p(\tau)$, $\tau\in(0,x)$,and integrating the resulting inequality with respect to τ over $(0,x)$, we find that

$$K_1(pfg)(x)+K_1(p)(x)f(\rho)g(\rho) \geq K_1(pf)(x)g(\rho)+K_1(pg)(x)f(\rho). \tag{2.38}$$

□

Again multiplying inequality (2.38) by $\frac{v_1(x)}{\Gamma(\alpha)}K_2(x,\rho)u_2(\rho)q(\rho)$ and integrating the resulting inequality with respect to ρ over $(0,x)$. This leads as to inequality (2.37). Letting $q(x)=p(x)$ in Theorem (2.2.15), we get the following Corollary. □

Corollary 3 *Let f,g be two synchronous functions on $[0,\infty)$, $P:[a,b]\longrightarrow(0,\infty)$, u_i be positive integrable weight functions and $v_i, i=1,2$, be positive functions. Then*

$$K_2p(x)K_1(pfg)(x)+K_1p(x)K_2(pfg)(x) \geq K_2(pg)(x)K_1(pf)(x)+K_2(pf)(x)K_1(pg)(x) \tag{2.39}$$

for all $x > 0$. *Inequality* (2.39)*is reversed if the functions are asynchronous on* $(0, \infty)$.
Theorem (2.2.15)*with* $K_1 = K_2 = K$ *and* $q(x) = p(x)$ *leads us to the following corolary.*

Corollary 4 *Let f,g be two synchronousn functions on* $(0, \infty)$, *u and v be locally ibtegrable non-negative weight functions. Then*

$$K(px)K(pfg)(x) \geq K(pf)(x)K(pg)(x) \tag{2.40}$$

for all $x > 0$.*Inequality* (2.40)*is reversed if the functions are asynchronous on* $(0, \infty)$.
Theorem (2.2.15) *with* $q(x) = p(x) = 1$ *gives the following corollary.*

Corollary 5 *Let f,g be two synchronousn functions on* $(0, \infty)$, u_i *and* $v_i, i = 1, 2$ *locally integrable non-negative weight functions. Then*

$$K_2(1)(x)K_1(fg)(x) + K_1(x)K_2(fg)(x) \geq K_2g(x)K_1f(x) + K_2f(x)K_1g(x) \tag{2.41}$$

for all $x > 0$.*Inequality* (2.41)*is reversed if the functions are asynchronous on* $[0, \infty[$.
If $f = g$ *in* (2.41), *we get the following corollary.*

Corollary 6 *Let* f, f^2 *be positive and integrable functions on* $(0, \infty)$, *and* u_i, u *and* $v_i, i = 1, 2$, *be locally integrable non-negative weight functions.Then*

$$K_2(1)K_1(f)^2(x) + K_1(1)K_2(f)^2(x) \geq K_2f(x)K_1f(x)$$

for all $x > 0$.

Corollary 7 *Let f be a positive and absolutely continunous functions on* $(0, \infty)$ *such that* $f' > 0$.*Let* u_i*and* $v_i, i = 1, 2$, *be locally integrable non-negative weight functions. Then*

$$\begin{aligned} K_2(1)(x)K_1(f^3)(x) + K_1(1)(x)K_2(f^3)(x) &\geq (K_1(1)(x))^{-1}K_2f(x)(K_1(px))^2 \\ &+ (K_2(1)(x))^{-1}K_1f(x)K_1f(x)(K_2f(x))^2 \end{aligned}$$

for all $x > 0$.

proof We observe that the conditions $f > 0, f^{'} > 0$ imply that the functions f and f^2 are synchronous on $(0, \infty)$.Hence, for all $\tau, \rho > 0$ we have

$$(f(\tau) - f(\rho))(f^2(\tau) - f^2(\rho)) \geq 0.$$

Therefore,

$$f^3(\tau)f^3(\rho) \geq f(\tau)f^2(\rho) + f(\rho)f^2(\tau).$$

Applying Theorem (2.2.13), we complete the proof. □

Remark 2.2.4. *By applying Corollary* (5) *with* $v_i(x) = u_i(x) = 1, \beta_i = 1, i = 1, 2,$ *we arrive at Theorem* (1.6.1)

In the following we shall make use a well known Hölder inequality for many functions.

Lemma 2.2.6 *Suppose that* $\frac{1}{p_1} + ... + \frac{1}{p_n} = 1$ *for* $P_i \geq 1 \quad i = 1, 2, ..., n$.*If* $f_i \in L_{p_i}$ *respectively, then* $\prod_{i=1}^{n} f_i \in L_1$*and*

$$\int_0^\infty \prod_{i=1}^{n} |f_i| dx \leq \prod_{i=1}^{n} \left(\int_0^\infty |f_i|^{p_i} dx \right)^{\frac{1}{p_i}} \tag{2.42}$$

Theorem 2.2.16. *Let* $P_i \geq 1$, $i = 1, 2, ..., n$ *such that*

$$\frac{1}{P_i} + ... + \frac{1}{P_n} = 1$$

If $f_i \in L_{p_i}$,u *and* v *locally integrable non-negative weight functions,then*

$$K \left(\prod_{i=1}^{i=n} f_i \right) (x) \leq \prod_{i=1}^{i=n} (K f_i^{p_i}(x))^{\frac{1}{p_i}}. \tag{2.43}$$

For all $x > 0$.

proof For $i = 1, 2, ..., n$ we consider the functions F_i, defined on $(0, x)$ as follows $F_i(t) = k(x, t)^{\frac{1}{p_i}} f_i(t)$. By applying Hölder's inequality, we obtain

$$\begin{aligned}
K(\prod_{i=1}^{i=n} f_i)(x) &= \frac{v(x)}{\Gamma(\alpha)}\int_0^x \prod_{i=1}^{i=n} f_i(t)k(x,t)u(t)dt \\
&= \frac{v(x)}{\Gamma(\alpha)}\int_0^x \prod_{i=1}^{i=n} F_i(t)u(t)dt \\
&\leq \prod_{i=1}^{i=n}\left(\frac{v(x)}{\Gamma(\alpha)}\int_0^x F_i^{p_i}(t)u(t)dt\right)^{\frac{1}{p_i}} \\
&= \prod_{i=1}^{i=n}\left(\frac{v(x)}{\Gamma(\alpha)}\int_0^x k(x,t)f_i^{p_i}(t)u(t)dt\right)^{\frac{1}{p_i}} \\
&= \prod_{i=1}^{i=n}(Kf_i^{p_i}(x))^{\frac{1}{p_i}}.
\end{aligned}$$

This proves inequality (2.43) and completes the proof. □

Remark 2.2.5. *Theorem (2.2.16) applied with* $v(x) = u(x) = 1, \beta = 1, n = 2$ *proves Theorem (2.2.10).*

Theorem 2.2.17. *Let f,g be two functions defined on* $(0,\infty)$*,* u *and* v *be locally integrable non-negative weight functions. If* f *is increasing.* g *is differentiable and there exists a real number* $m := \inf_{x\geq 0} g^{'}(x)$*,then*

$$K(fg)(x) \geq (K(1))^{-1}Kf(x)Kg(x) - m(K(-1))^{-1}Kf(x)K(id)(x) + mK(xf)(x)$$

holds for all $x > 0$*,where* $id(x) = x$*.*

proof We consider a function $h(x) = g(x) - mx$, where h is differentiable and increasing on $[0,\infty)$.

Then f and h are synchronous on $(0,\infty)$.

By applying Theorem (2.2.13) , we conclude that

$$K(f(x)(g - mx)) \geq (K(1))^{-1}Kf(x)K(g - mx).$$

Since K is linear, we have

$$K(f(x)(g - mx)) = K(fg)(x) - mK(xf)(x).$$

This yields:

$$K(fg)(x) \geq (K(1))^{-1}Kf(x)Kg(x) - m(K(1))^{-1}K(id)(x)Kf(x) + mK(xf)(x).$$

The proof is complete. □

Remark 2.2.6. *By applying Theorem (2.2.17) for $v(x) = u(x) = 1, \beta = 1$, we obtain Theorem (2.2.12). Theorem (2.2.13) applied to the decreasing functions $f(x)$ and $G(x) = g(x) - Mx$ for all $x > 0$, where $M := \sup_{x\geq 0} g^{'}(x)$, gives rise to the following Corollary.*

Corollary 8 *Let f g be two functions defined on $(0, \infty)$, u and v be locally integrable non-negative weight functions. If f is decreasing, g is differentiable and there exists a real number $M := \sup_{x\geq 0} g^{'}(x)$,then*

$$K(fg)(x) \geq (K(1))^{-1}Kf(x)Kg(x) - M(K(1))^{-1}Kf(x)K(id)(x) + MK(xf)(x)$$

is valid for all $x > 0$.
We observe that our results generalize Theorem (2.2.8), (2.2.9), (2.2.10), (2.2.11) and (2.2.12).

2.2.5 Fractional Chebyshev Type Inequalities Using The Pecaric Extention

Fractional Montgomery's Identities

In 2009, Annastassiou [11] established two fractional Montgomery identities. This was important work for the researchers, who stated:

Theorem 2.2.18. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, so Montgomery's identities for fractional integrals are given by:*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha} f(b) - J_a^{\alpha-1}\left(P_1(x,b)f(b)\right) + J_a^{\alpha}\left(P_1(x,b)f'(b)\right), \quad (2.44)$$

Where $P_1(x,t)$ the Peano fractional kernel is defined by:

$$P_1(x,t) = \begin{cases} \dfrac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), a \leq t \leq x \\ \dfrac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), x < t \leq b. \end{cases} \quad (2.45)$$

proof Based on (2.45) and taking into account the properties of the fractional calculation, we have

$$\begin{aligned}
\Gamma(\alpha)J_a^\alpha\left(P(x,b)f'(b)\right) &= \int_a^b (b-t)^{\alpha-1}P(x,t)f'(t)dt \\
&= \int_a^x (b-t)^{\alpha-1}\frac{t-a}{b-a}f'(t)dt + \int_x^b (b-t)^{\alpha-1}\frac{t-b}{b-a}f'(t)dt \\
&= \int_a^x (b-t)^{\alpha-1}\frac{t-a+b-b}{b-a}f'(t)dt \\
&- \frac{1}{b-a}\int_x^b (b-t)^{\alpha-1}(b-t)f'(t)dt \qquad (2.46)\\
&= \int_a^x \left(1+\frac{t-b}{b-a}\right)(b-t)^{\alpha-1}f'(t)dt - \frac{1}{b-a}\int_x^b (b-t)^\alpha f'(t)dt \\
&= \int_a^x (b-t)^{\alpha-1}f'(t)dt - \frac{1}{b-a}\int_x^b (b-t)^\alpha f'(t)dt - \frac{1}{b-a}\int_x^b (b-t)^\alpha f'(t)dt \\
&= \int_a^x (b-t)^{\alpha-1}f'(t)dt - \frac{1}{b-a}\int_a^b (b-t)^\alpha f'(t)dt.
\end{aligned}$$

We integrate (2.46) by part:

$$\begin{aligned}
\Gamma(\alpha)J_a^\alpha\left(P(x,b)f'(b)\right) &= (b-x)^{\alpha-1}f(x) - (b-a)^{\alpha-1}f(a) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}f(t)dt \\
&- \frac{1}{b-a}\left(\left[-(b-a)^\alpha f(a)\right] + \alpha\int_a^b (b-t)^{\alpha-1}f(t)dt\right) \\
&=(b-x)^{\alpha-1}f(x) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}f(t)dt - (b-a)^{\alpha-1}f(a) + (b-a)^{\alpha-1}f(a) \\
&- \frac{\alpha}{b-a}\int_a^b (b-t)^{\alpha-1}f(t)dt \\
&=(b-x)^{\alpha-1}f(x) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}f(t)dt - \frac{\alpha}{b-a}\int_a^b (b-t)^{\alpha-1}f(t)dt \\
&= (b-x)^{\alpha-1}f(x) - \frac{\alpha}{b-a}\Gamma(\alpha)J_a^\alpha f(b) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}f(t)dt
\end{aligned}$$

$$\Gamma(\alpha)J_a^\alpha\left(P(x,b)f'(b)\right) = (b-x)^{\alpha-1}f(x) - \frac{1}{b-a}\Gamma(\alpha)J_a^\alpha f(b) + \Gamma(\alpha)J_a^{\alpha-1}(P(x,b)f(b)). \qquad (2.47)$$

□

Finally, from (2.47) for $\alpha \geq 1$, we get:

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha f(b) - J_a^{\alpha-1}\left(P_1(x,b)f(b)\right) + J_a^\alpha\left(P_1(x,b)f'(b)\right)$$

Remark 2.2.7. *Replacing α with 1 in (2.44) we get the classicl Montgomery's identity (2.14).*

Theorem 2.2.19. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function, then a generalization of the fractional Montgomery identity is given by:*

$$f(x) = (b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha}(w(b)f(b)) - J_a^{\alpha-1}\left(Q_w(x,b)f(b)\right) + J_a^{\alpha}\left(Q_w(x,b)f'(b)\right). \tag{2.48}$$

Where $Q_w(x,t)$ is the fractional Peano kernel defined by:

$$Q_w(x,t) = \begin{cases} (b-x)^{1-\alpha}\Gamma(\alpha)W(t) & a \le t \le x \\ (b-x)^{1-\alpha}\Gamma(\alpha)(W(t)-1) & x \le t \le b. \end{cases} \tag{2.49}$$

Remark 2.2.8. *Replacing α with 1 in the (2.48) we get Montgomery's identity with weight (2.16).*

The thired fractional Montgomery indentity was established by A-Guezan.Lakoud-F.Aissaoui [7]on 2013, they replaced the fractional kernel $P_{W,\varphi}$ given by (2.19) , by a fractional kernel with weigh wich is compound by the function φ .

Their results were as follows.

Theorem 2.2.20. *Let $\varphi : [0, 1] \to \mathbb{R}$ a differentiable function such as $\varphi(0) = 0, \varphi(1) \neq 0$ and $\varphi' \in L^1[0, 1]$, then a generalization of the fractional Montgomery identity is given by:*

$$\begin{aligned} f(x) &= \frac{1}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha}\left(w(b)\varphi'(1)f(b)\right) \\ &- \frac{1}{\varphi(1)}J_a^{\alpha-1}\left(Q_{w,\varphi}(x,b)f(b)\right) + \frac{1}{\varphi(1)}J_a^{\alpha}\left(Q_{w,\varphi}(x,b)f'(b)\right). \end{aligned} \tag{2.50}$$

Where $Q_{w,\varphi}(x,t)$ is the weight fractional Peano kernel defined by:

$$Q_{w,\varphi}(x,t) = \begin{cases} (b-x)^{1-\alpha}\Gamma(\alpha)\varphi(W(t)) & a \le t \le x \\ (b-x)^{1-\alpha}\Gamma(\alpha)(\varphi(W(t)) - \varphi(1)) & x \le t \le b. \end{cases} \tag{2.51}$$

Remark 2.2.9. *Replacing a with 1 in the (2.50) we find Montgomery's identity with weight (2.18).*

Fractional Chebyshev Type Inequality

In this section we cite three integral inequalities of the fractional Chebyshev type. The first two inequalities are determined via the fractional Montgomery identities introduced by Annastassiou [11] and the third, is based on the fractional Montgomery identity established in 2013 by A. Guezane-Lakoud -F. Aissaoui [7] whose statements are given by the following theorems:

Theorem 2.2.21. *Let $f, g : [a, b] \to \mathbb{R}$ two differentiable functions, f', g' are integrable on $[a, b]$ then:*

$$|T_\alpha(f,g)| \leq \frac{(b-a)^{2\alpha}\left(2\alpha^2+11\alpha+8\right)}{(\alpha+1)^2(\alpha+2)(2\alpha+1)(2\alpha+3)}\left\|f'\right\|_\infty \left\|g'\right\|_\infty, \tag{2.52}$$

where $\alpha \geq 1$ and $T_\alpha(f,g)$ is defined by:

$$T_\alpha(f,g) = \frac{1}{b-a}\Gamma(2\alpha-1)J_a^{2\alpha-1}((fg)(b)) - \frac{\Gamma^2(\alpha)}{(b-a)^2}J_a^\alpha g(b) J_a^\alpha f(b) \tag{2.53}$$

proof Based on Montgomery fractional identity (2.44):

$$F = f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(f(b)) + J_a^{\alpha-1}\left(P_1(x,b)f(b)\right) = \breve{E} = J_a^\alpha\left(P_1(x,b)f'(b)\right)$$

$$G = g(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(g(b)) + J_a^{\alpha-1}\left(P_1(x,b)g(b)\right) = \breve{G} = J_a^\alpha\left(P_1(x,b)g'(b)\right),$$

since

$$F.G = \tilde{F}\cdot\tilde{G}$$

and

$$\begin{aligned}
F\cdot G &= \begin{pmatrix} f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(f(b)) \\ +J_a^{\alpha-1}\left(P_1(x,b)f(b)\right)\end{pmatrix}\begin{pmatrix} g(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(g(b)) \\ +J_a^{\alpha-1}\left(P_1(x,b)g(b)\right)\end{pmatrix} \\
&= f(x)g(x) - f(x)\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(g(b)) + f(x)J_a^{\alpha-1}\left(P_1(x,b)g(b)\right) \\
&- g(x)\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(f(b)) + \left(\frac{\Gamma(\alpha)}{b-a}\right)^2(b-x)^{2-2\alpha}J_a^\alpha(f(b))J_a^\alpha(g(b)) \\
&- \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(f(b))J_a^{\alpha-1}\left(P_1(x,b)g(b)\right) + g(x)J_a^{\alpha-1}\left(P_1(x,b)f(b)\right) \\
&- J_a^{\alpha-1}\left(P_1(x,b)f(b)\right)\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(g(b)) \\
&+ J_a^{\alpha-1}\left(P_1(x,b)f(b)\right)J_a^{\alpha-1}\left(P_1(x,b)g(b)\right).
\end{aligned} \tag{2.54}$$

We multiply (2.54) by $\frac{1}{b-a}(b-x)^{2\alpha-2}$ and integrate the result on $[a,b]$, taking into account the properties of the fractional calculation we find(2.52):

$$T_\alpha(f,g) = \frac{1}{b-a}\Gamma(2\alpha-1)J_a^{2\alpha-1}((fg)(b)) - \frac{\Gamma^2(\alpha)}{(b-a)^2}J_a^\alpha g(b)J_a^\alpha f(b)$$
$$= \frac{1}{b-a}\int_a^b (b-x)^{2\alpha-2}J_a^\alpha\left(P_1(x,b)f'(b)\right)J_a^\alpha\left(P_1(x,b)g'(b)\right)dx.$$

By a long calculation we find:

$$|T_\alpha(f,g)| = \frac{1}{(b-a)\Gamma^2(\alpha)} \mid \int_a^b (b-x)^{2\alpha-2}$$
$$\left[\int_a^b (b-t)^{\alpha-1}P_1(x,t)f'(t)dt\int_a^b (b-s)^{\alpha-1}P_1(x,s)g'(s)ds\right]dx \mid$$
$$\leqslant \frac{1}{(b-a)\Gamma^2(\alpha)}\int_a^b (b-x)^{2\alpha-2}$$
$$\left[\int_a^b (b-t)^{\alpha-1}\left|P_1(x,t)\right|\left|f'(t)\right|dt\int_a^b (b-s)^{\alpha-1}\left|P_1(x,s)\right|\left|g'(s)\right|ds\right]dx$$

$$\leqslant \frac{1}{(b-a)\Gamma^2(\alpha)}\left\|f'\right\|_\infty\left\|g'\right\|_\infty\int_a^b (b-x)^{2\alpha-2}\left(\int_a^b (b-t)^{\alpha-1}\left|P_1(x,t)\right|dt\right)^2 dx \quad (2.55)$$

ifintegrat par parts we get:

$$\int_a^b (b-t)^{\alpha-1}\left|P_1(x,t)\right|dt = \int_a^x (b-t)^{\alpha-1}\left|P_1(x,t)\right|dt + \int_x^b (b-t)^{\alpha-1}\left|P_1(x,t)\right|dt$$
$$= \frac{\Gamma(\alpha)(b-x)^{1-\alpha}}{b-a}\left[\int_a^x (b-t)^{\alpha-1}(t-a)dt + \int_x^b (b-t)^\alpha dt\right]$$
$$= \frac{\Gamma(\alpha)(b-x)^{1-\alpha}}{b-a}\left[(a-x)\frac{(b-x)^\alpha}{\alpha} - \frac{(b-x)^{\alpha+1}}{\alpha(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\alpha(\alpha+1)} + \frac{(b-x)^{\alpha+1}}{\alpha+1}\right]$$
$$= \frac{\Gamma(\alpha)(b-x)^{1-\alpha}}{b-a}\left[(a(\alpha+1)+b(\alpha-1)-2\alpha x)\frac{(b-x)^\alpha}{\alpha(\alpha+1)} + \frac{(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\right], \quad (2.56)$$

□

replacing (2.56) in (2.55) we get

$$|T_\alpha(f,g)| \leqslant \frac{1}{(b-a)^3}\left\|f'\right\|_\infty\left\|g'\right\|_\infty\left[\int_a^b \frac{(b-x)^{2\alpha}}{\alpha^2(\alpha+1)^2}(a(\alpha+1)+b(\alpha-1)-2\alpha x)^2 dx\right.$$
$$\left. + \int_a^b \frac{(b-a)^{2\alpha+2}}{\alpha^2(\alpha+1)^2}dx + 2\int_a^b (a(\alpha+1)+b(\alpha-1)-2\alpha x)\frac{(b-x)^\alpha}{\alpha(\alpha+1)}\frac{(b-a)^{\alpha+1}}{\alpha(\alpha+1)}dx\right]$$
$$= \frac{1}{(b-a)^3}\left\|f'\right\|_\infty\left\|g'\right\|_\infty\frac{(b-a)^{2\alpha+3}}{\alpha^2(\alpha+1)^2}\left[\frac{(\alpha-1)^2}{2\alpha+1} - \frac{4\alpha(\alpha-1)}{(2\alpha+1)(2\alpha+2)}\right.$$
$$\left. + \frac{8\alpha^2}{(2\alpha+1)(2\alpha+2)(2\alpha+3)} + 1 + \frac{2(\alpha-1)}{\alpha+1} - \frac{4\alpha}{(\alpha+1)(\alpha+2)}\right]$$
$$\leqslant \frac{(b-a)^{2\alpha}(2\alpha)^2+11\alpha+8}{(\alpha+1)^2(\alpha+2)(2\alpha+1)(2\alpha+3)}\|f'|_\infty\|g'\|_\infty.$$

Remark 2.2.10. *Replaces α with 1 in Chebyshev's fractional inequality (2.52), it is reduced to the classical Chebyshev inequality (2.20).*
To simplify the notation, and for two given functions f and $g : [a,b] \to \mathbb{R}$,we note

$$\begin{aligned} T_\alpha(f,g,w) &= \Gamma(2\alpha-1)J_a^{2\alpha-1}(w(b)f(b)g(b)) \\ &- \Gamma^2(\alpha)J_a^\alpha(w(b)g(b))J_a^\alpha(w(b)f(b)). \end{aligned} \tag{2.57}$$

The following theorem presents the second fractional Chebyshev type inequality with a weight kernel.

Theorem 2.2.22. *Let $f, g : [a,b] \to \mathbb{R}$ two differentiable functions on $[a,b]$, f' and g' are integrable on $[a,b]$. Then the fractional Chebyshev type inequality with a weight kermel is given by:*

$$|T_\alpha(f,g,w)| \leq \frac{1}{\Gamma^2(\alpha)} \|f'\|_\infty \|g'\|_\infty \int_a^b (b-x)^{2\alpha-2} w(x) + H_\alpha^2(x)dx \tag{2.58}$$

where $\alpha \geq 1$ and

$$H_\alpha(x) = \int_a^b (b-t)^{\alpha-1} |Q_w(x,t)| \, dt$$

with $Q_w(x,t)$ defined by (2.48).

Remark 2.2.11. *If the value of α is changed to 1 , in the Chebyshev fractional inequality (2.58), we get (2.4).*
The following theorem represents the third Chebyshev type fractional inequality with a weight kernel.
To simplify the notation, for two given functions f and $g : [a,b] \to \mathbb{R}$, we denote

$$\begin{aligned} T_\alpha(f,g,\varphi,w) = \Gamma(2\alpha-1)J_a^{2\alpha-1}\left(w(b)\varphi'(1)f(b)g(b)\right) \\ - \frac{\Gamma^2(\alpha)}{\varphi(1)} J_a^\alpha\left(w(b)\varphi'(1)g(b)\right) J_a^\alpha\left(w(b)\varphi'(1)f(b)\right). \end{aligned}$$

Theorem 2.2.23. *Let $f, g : [a,b] \to \mathbb{R}$ two differentiable functions on $[a,b] f'$ and g' are integrable on $[a,b]$. Then the Chebyshev type fractional inequality with weight kernel is given by:*

$$|T_\alpha(f,g,\varphi,w)| \leq \frac{1}{\Gamma^2(\alpha)\varphi^2(1)} \|f'\|_\infty \|g'\|_\infty \int_a^b (b-x)^{2\alpha-2} w(x) H_\alpha^2(x)dx \tag{2.59}$$

where $\alpha \geq 1$ *and*

$$H_\alpha(x) = \int_a^b (b-t)^{\alpha-1} \left| q_{w,\varphi}(x,t) \right| dt$$

with $Q_{w,\varphi}(x,t)$ *defined in* (2.51) *formula.*

Remark 2.2.12. *If the value of* α *is changed to 1 , in the fractional inequality of Chebyshev* (2.59)*, we get the inequality of Chebychev* (2.27)*.*

Chapter 3

A generalisation of chebyshev's inequalities for functions of serveral variables

3.1 Chebyshev type inequalities of two variables using the Montgomery identities:

In the last years, many articles have been devoted to the generalization of Chebyshev type inequalities. In this section we discuss our results for some Chebyhev-type integral inequalities for double integrals and [20, 9] multivariable functions, based on a new version of the Montgomery identity with two variables given in the following lemma:

Lemma 3.1.1 *Let $f : I = [a,b]\times[c,d] \to \mathbb{R}$ is differentiable, the derivative $\dfrac{\partial^2 f(t,s)}{\partial t \partial s}$ is integrable on I, so Montgomery's identity is given by:*

$$\begin{aligned} f(x,y) &= \frac{1}{(b-a)}\int_a^b f(t,y)dt + \frac{1}{(d-c)}\int_c^d f(x,s)ds \\ &- \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \\ &+ \int_a^b\int_c^d p(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \end{aligned} \tag{3.1}$$

where the $P(x,t)$ *kernel is defined in* (2.13) *and* $Q(y,s)$ *is such that*

$$Q(y,s) = \begin{cases} \dfrac{s-c}{d-c}, & c \le s \le y \\ \dfrac{s-d}{d-c}, & y \le s \le d \end{cases} \tag{3.2}$$

proof Let's calculate the following double integral

$$\begin{aligned} \int_a^b \int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \\ &= \int_a^x \int_c^y \left(\frac{t-a}{b-a}\right)\left(\frac{s-c}{d-c}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \\ &+ \int_a^x \int_y^d \left(\frac{t-a}{b-a}\right)\left(\frac{s-d}{d-c}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \\ &+ \int_x^b \int_c^y \left(\frac{t-b}{b-a}\right)\left(\frac{s-c}{d-c}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \\ &+ \int_x^b \int_y^d \left(\frac{t-b}{b-a}\right)\left(\frac{s-c}{d-c}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

We integrate A_i, $i \in 1,2,3,4$ by part and using the fubini theorem, we get

$$\begin{aligned} A_1 &= \int_a^x \int_c^y \left(\frac{t-a}{b-a}\right)\left(\frac{s-c}{d-c}\right)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \\ &= \frac{1}{(b-a)(d-c)} \int_a^x (t-a) \int_c^y (s-c)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \\ &= \frac{1}{(b-a)(d-c)} \int_a^x (t-a) \left[(y-c)\frac{\partial f(t,y)}{\partial t} - \int_c^y \frac{\partial f(t,s)}{\partial t} ds\right] dt \\ &= \frac{1}{(b-a)(d-c)} \int_a^x (t-a)(y-c)\frac{\partial f(t,y)}{\partial t} dt - \int_a^x (t-a)\left(\int_c^y \frac{\partial f(t,s)}{dt} ds\right) dt \\ A_1 &= \frac{(x-a)(y-c)}{(b-a)(d-c)} f(x,y) - \frac{(y-c)}{(b-a)(d-c)} \int_a^x f(t,y)dt \\ &- \frac{(x-a)}{(b-a)(d-c)} \int_c^y f(x,s)ds + \frac{1}{(b-a)(d-c)} \int_a^x \int_c^y f(t,s)dsdt \end{aligned}$$

In the same way, one obtains

$$\begin{aligned} A_2 &= \frac{(x-a)(d-y)}{(b-a)(d-c)}f(x,y) - \frac{(d-y)}{(b-a)(d-c)}\int_a^x f(t,y)dt \\ &- \frac{(x-a)}{(b-a)(d-c)}\int_y^d f(x,s)ds + \frac{1}{(b-a)(d-c)}\int_a^x\int_y^d f(t,s)dsdt \\ A_3 &= \frac{(y-c)(b-x)}{(b-a)(d-c)}f(x,y) - \frac{(y-c)}{(b-a)(d-c)}\int_x^b f(t,y)dt \\ &- \frac{(b-x)}{(b-a)(d-c)}\int_c^y f(x,s)ds + \frac{1}{(b-a)(d-c)}\int_x^b\int_c^y f(t,s)dsdt \end{aligned}$$

and

$$\begin{aligned} A_4 &= \frac{(d-y)(b-x)}{(b-a)(d-c)}f(x,y) - \frac{(d-y)}{(b-a)(d-c)}\int_x^b f(t,y)dt \\ &- \frac{(b-x)}{(b-a)(d-c)}\int_y^d f(x,s)ds + \frac{1}{(b-a)(d-c)}\int_x^b\int_y^d f(t,s)dsdt \end{aligned}$$

calculate $A_1+A_2+A_3+A_4$, term to term we get

$$\begin{aligned} &A_1+A_2+A_3+A_4 \\ &\qquad = \int_a^b\int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ &\qquad = f(x,y) - \frac{1}{(b-a)}\int_a^b f(t,y)dt - \frac{1}{(d-c)}\int_c^d f(x,s)ds \\ &\qquad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \end{aligned}$$

then

$$\begin{aligned} f(x,y) &= \frac{1}{(b-a)}\int_a^b f(t,y)dt + \frac{1}{(d-c)}\int_c^d f(x,s)ds \\ &- \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \\ &+ \int_a^b\int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \end{aligned}$$

hence (3.1), which is the end of the demonstration.

Montgomery's Identity With Two Variables and a Weight Kernel

We now give a new extension of the Montgomery identity with weight for the functions two variables similar to (2.14), for which we define two density probability functions.

Let $w : [a, b] \to [0, +\infty\,[\,]$ a density probability function, and $W(t) = \int_a^t w(x)dx$ for $a \le t \le b$, with $W(a) = 0$ and $W(b) = 1$, and $\varphi : [c, d] \to [0, +\infty[]$ a density probability function, and $\Psi(s) = \int_c^s \varphi y)dy$ for $c \le s \le d$, with $\Psi(c) = 0$ and $W(d) = 1$.

Theorem 3.1.1. *Assume that the partial derivatives* $\frac{\partial f(s,t)}{\partial s}, \frac{\partial f(s,t)}{\partial t}$ *and* $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ *exist and are continuous on* I *then:*

$$f(x, y) = \int_a^b w(t)f(t, y)dt + \int_c^d \psi(s)f(x, s)ds - \int_a^b w(t) \int_c^d \psi(s)f(t, s)dsdt + \int_a^b \int_c^d P_w(x, t)Q_\psi(y, s)\frac{\partial^2 f(t, s)}{\partial t \partial s}dsdt. \quad (3.3)$$

Where $P_w(x, t)$ *and* $Q_\varphi(y, s)$ *two Peano kernels defined by:*

$$P_w(x, t) = \begin{cases} W(t), & a \le t \le x \\ W(t) - 1, & x < t \le b. \end{cases} \quad (3.4)$$

$$Q_\varphi(y, s) = \begin{cases} \Psi(s), & c \le s \le y \\ \Psi(s) - 1, & y < s \le d. \end{cases}$$

We give a new extension of the Montgomery identity with weight for two-variable functions similar to (2.16).

Theorem 3.1.2. *Let* $f : I \to \mathbb{R}$ *a function that is twice differentiable and its second derivative* $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ *is integrable on* I*.*

$$\begin{aligned} f(x, y) &= \frac{1}{\varphi(1)} \int_a^b w(t)\varphi'(W(t))f(t, y)dt \\ &+ \frac{1}{\varphi(1)} \int_c^d \psi(s)\varphi'(\Psi(s))f(x, s)ds \\ &- \frac{1}{\varphi^2(1)} \int_a^b w(t)\varphi'(W(t)) \int_c^d \psi(s)\varphi'(\Psi(s))f(t, s)dsdt \\ &+ \frac{1}{\varphi^2(1)} \int_a^b \int_c^d P_{w,\varphi}(x, t)Q_{\psi,\varphi}(y, s)\frac{\partial^2 f(t, s)}{\partial t \partial s}dsdt. \end{aligned}$$

Where $P_{w,\varphi}$ *defined in* (2.6) *and* $Q_{\psi,\varphi}$ *the Peano kernel defined by:*

$$Q_{\psi,\varphi}(y,s) = \begin{cases} \varphi(\Psi(s)), c \leq s \leq y \\ \varphi(\Psi(s)) - \varphi(1), y \leq s \leq d. \end{cases} \tag{3.5}$$

Chebyshev Type Inequalities Using Montgomery's Identity for Double Integral:

The first Chebyshev-type inequality for the functions of two variables is now presented based on the identity of Montgomery (3.1).

Theorem 3.1.3. *Let* $f, g : I \to \mathbb{R}$ *two differentiatable functions such as second derivatives* $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ *and* $\frac{\partial^2 g(s,t)}{\partial s \partial t}$ *are integratable on* I*, then*

$$|T(f,g)| \leq \frac{49}{3600}(b-a)^2(d-c)^2 \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty, \tag{3.6}$$

where the $T(f,g)$ *operator is defined by:*

$$\begin{aligned} T(f,g) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dxdy \\ &- \frac{1}{(b-a)^2(d-c)} \int_a^b \int_c^d g(x,y) \int_a^b f(t,y)dtdxdy \\ &- \frac{1}{(b-a)(d-c)^2} \int_a^b \int_c^d g(x,y) \int_c^d f(x,s)dsdxdy \\ &+ \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_c^d f(x,s)dsdx \int_c^d \int_a^b g(t,y)dtdy. \end{aligned} \tag{3.7}$$

proof Let F, G, $\tilde{F}$ and $\tilde{G}$ by the quantities defined by:

$$\begin{aligned} F =& f(x,y) - \frac{1}{(b-a)} \int_a^b f(t,y)dt - \frac{1}{(d-c)} \int_c^d f(x,s)ds \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \end{aligned}$$

$$G = g(x,y) - \frac{1}{(b-a)} \int_a^b g(t,y)dt - \frac{1}{(d-c)} \int_c^d g(x,s)ds$$
$$+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(t,s)dsdt$$
$$\tilde{F} = \int_a^b \int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s}dsdt$$

and

$$\tilde{G} = \int_a^b \int_c^d P(x,t)Q(y,s)\frac{\partial^2 g(t,s)}{\partial t \partial s} dsdt \tag{3.8}$$

According to Montgomery's identity (3.1) we have:

$$FG = \tilde{F}\tilde{G}$$

By multiplying FG by $\dfrac{1}{(b-a)(d-c)}$ and integrating on I, we get

$$\begin{aligned} T(f,g) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(\int_a^b \int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \right) \\ &\times \left(\int_a^b \int_c^d P(x,t)Q(y,s)\frac{\partial^2 g(t,s)}{\partial t \partial s} dsdt \right). \end{aligned} \tag{3.9}$$

What implies that

$$\begin{aligned} |T(f,g)| = &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(\int_a^b \int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \right) \right. \\ &\times \left(\int_a^b \int_c^d P(x,t)Q(y,s)\frac{\partial^2 g(t,s)}{\partial t \partial s} dsdt. \right) \\ \leq &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(\int_a^b \int_c^d |P(x,t)Q(y,s)| \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| dsdt \right) \\ &\times \left(\int_a^b \int_c^d |P(x,t)Q(y,s)| \frac{\partial^2 g(t,s)}{\partial t \partial s} dsdt \right) dxdy \\ \leq &\frac{1}{(b-a)(d-c)} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \\ &\times \int_a^b \int_c^d \left(\int_a^b \int_c^d |P(x,t)Q(y,s)| dsdt \right)^2 dxdy \end{aligned} \tag{3.10}$$

Let's calculate

$$\begin{aligned} \int_a^b \int_c^d |P(x,t)Q(y,s)| dsdt &= \int_a^b |P(x,t)| \left(\int_c^d |Q(y,s)| ds \right) dt \\ &= \int_a^b |P(x,t)| \left(\int_c^y \frac{s-c}{d-c} ds + \int_y^d \frac{d-s}{d-c} ds \right) dt \\ &= \frac{1}{(d-c)} \int_a^b |P(x,t)| \left(\left[\frac{(s-c)^2}{2} \right]_c^y + \left[-\frac{(d-s)^2}{2} \right]_y^d \right) dt \\ &= \frac{[(y-c)^2 + (d-y)^2]}{2(b-a)(d-c)} \left(\int_a^x (t-a)dt + \int_x^b (b-t)dt \right) \\ &= \frac{[(x-a)^2 + (b-x)^2]\,[(y-c)^2 + (d-y)^2]}{4(b-a)(d-c)} \end{aligned} \tag{3.11}$$

□

replacing (3.11) in (3.10) we get

$$\begin{aligned}
|T(f,g)| &\le \frac{1}{16(b-a)^3(d-c)^3}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty \\
&\times \int_a^b\int_c^d \left(\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]\right)^2 dxdy \\
&= \frac{1}{16(b-a)^3(d-c)^3}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty \\
&\times \int_a^b \left[(x-a)^2+(b-x)^2\right]^2 dx \int_c^d \left[(y-c)^2+(d-y)^2\right]^2 dy \\
&= \frac{1}{16(b-a)^3(d-c)^3}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\| \|_\infty\| \frac{\partial^2 g(t,s)}{\partial t\partial s}\|_\infty \\
&\times \left(\frac{(b-a)^5}{5}+\frac{(b-a)^5}{5}+\frac{(b-a)^5}{15}\right)\left(\frac{(d-c)^5}{5}+\frac{(d-c)^5}{5}+\frac{(d-c)^5}{15}\right) \\
&= \frac{1}{16(b-a)^3(d-c)^3}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty \\
&\times \frac{(b-a)^5}{15}\times\frac{(d-c)^5}{15} \\
&= \frac{49}{3600}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty .
\end{aligned}$$

Wich completes the proof. □

Chebyshev Type Inequalities With Weigh kermel for Double Integral

Motivated by identity (3.4) we give an integral inequality of Chebyshev type with weight kernels. Let the T operator defined by:

$$\begin{aligned}
T(w,\psi,f,g) &= \int_a^b\int_c^d w(x)\psi(y)f(x,y)g(x,y)dxdy \qquad (3.12)\\
&- \int_a^b\int_c^d w(x)\psi(y)g(x,y)\left(\int_a^b w(t)f(t,y)dt\right)dxdy \\
&- \int_a^b\int_c^d w(x)\psi(y)g(x,y)\left(\int_c^d \psi(s)f(x,s)ds\right)dxdy \\
&+ \left(\int_a^b w(x)\left(\int_c^d \psi(s)f(x,s)ds\right)dx\right) \\
&\times \left(\int_c^d \psi(y)\left(\int_a^b w(t)g(t,y)dt\right)dy\right).
\end{aligned}$$

Theorem 3.1.4. *Let $f, g : I \to \mathbb{R}$ such as the partial derivatives $\dfrac{\partial f(s,t)}{\partial s}, \dfrac{\partial f(s,t)}{\partial t}$ and $\dfrac{\partial^2 f(s,t)}{\partial s \partial t}$ exist and are continuous on I. So:*

$$|T(w,\psi,f,g)| \leq \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \int_a^b \int_c^d w(x)\psi(y) H^2(x,y) dx dy \tag{3.13}$$

where

$$H(x,y) = \int_a^b \int_c^d |P_w(x,t) Q_\psi(y,s)| \, ds dt$$

Theorem 3.1.5. *Let $f, g : I \to \mathbb{R}$ two differentiable functions such as their second derivatives $\dfrac{\partial^2 g(s,t)}{\partial s \partial t}, \dfrac{\partial^2 g(s,t)}{\partial s \partial t}$ are integrable on I.*
Then the inequality is given by:

$$\begin{aligned} |T(w,\psi,f,g,\varphi')| &\leq \frac{1}{\varphi^4(1)} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \\ &\times \|\varphi'\|_\infty^2 \int_a^b \int_c^d w(x)\psi(y) H^2(x,y) dx dy \end{aligned} \tag{3.14}$$

where

$$H(x,y) = \int_a^b \int_c^d |P_{w,\varphi}(x,t) Q_{\psi,\varphi}(y,s)| \, ds dt$$

and the operator

$$\begin{aligned} &T(w,\psi,f,g,\varphi') = \\ &\int_a^b \int_c^d w(x)\varphi'(W(t))\psi(y)\varphi'(\Psi(y)) f(x,y) g(x,y) dx dy \\ &-\frac{1}{\varphi(1)} \int_a^b \int_c^d \Big[w(x)\varphi'(W(x))\psi(y)\varphi'(\Psi(y)) g(x,y) \\ &\qquad \int_a^b w(t)\varphi'(W(t)) f(t,y) dt \Big] dx dy \\ &-\frac{1}{\varphi(1)} \int_a^b \int_c^d \Big[w(x)\varphi'(W(x))\psi(y)\varphi'(\Psi(y)) g(x,y) \\ &\qquad \int_c^d \psi(s)\varphi'(\Psi(s)) f(x,s) ds \Big] dx dy \\ &+\frac{1}{\varphi^2(1)} \int_a^b w(x)\varphi'(W(x)) \left(\int_c^d \psi(s)\varphi'(\Psi(s)) f(x,s) ds \right) dx \\ &\qquad \int_c^d \psi(y)\varphi'(\Psi(y)) \left(\int_a^b w(t)\varphi'(W(t)) g(t,y) dt \right) dy. \end{aligned} \tag{3.15}$$

3.2 Bounds For The Chebyshev Functional For Function of Hölder Type

Cerone and Dragomir [3] obtained a variety of bounds using a generalisation of Korkine's identity.

If we consider the Chebyshev functional:

$$\begin{aligned} D_n(f,g) &:= \frac{1}{\prod_{k=1}^{n}(b_k-a_k)}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)g(x_1,...,x_n)dx_1...dx_n \\ &-\frac{1}{\prod_{k=1}^{n}(b_k-a_k)^2}\int_{a_1}^{b_1}...\int_{a_n}^{b_n} f(x_1,...,x_n)dx_1...dx_n \\ &\times\int_{a_1}^{b_1}...\int_{a_n}^{b_n} g(x_1,...,x_n)dx_1...dx_n \\ &= \frac{1}{v([\bar{a},\bar{b}])}\int_{\bar{a}}^{\bar{b}} f(\bar{x})g(\bar{x})d\bar{x} - \frac{1}{v([\bar{a},\bar{b}])}\int_{\bar{a}}^{\bar{b}} f(\bar{x})d\bar{x}.\int_{\bar{a}}^{\bar{b}} g(\bar{x})d\bar{x}, \end{aligned}$$

where $v([\bar{a},\bar{b}]) := \prod_{k=1}^{n}(b_k-a_k)$,then we can state the following generalisation of Chebyshev's inequality.

Theorem 3.2.1. *Let $f,g:[\bar{a},\bar{b}]\longrightarrow\mathbb{R}$ be of Hölder type.That is,*

$$|f(\bar{x})-f(\bar{y})| \le \sum_{i=1}^{n} L_i|x_i-y_i|^{p_i}, \bar{x},\bar{y}\in[\bar{a},\bar{y}], \tag{3.16}$$

$$|g(\bar{x})-g(\bar{y})| \le \sum_{i=1}^{n} H_i|x_i-y_i|^{q^i}, \bar{x},\bar{y}\in[\bar{a},\bar{y}], \tag{3.17}$$

where $L_i, H_i > 0$and $p_i, q_i \in (0,1]$ are fixed for $i = 1,2,...,n$ Then we have the inequality:

$$|D_n(f,g) \le \sum_{i=1}^{n} L_iH_i\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)} + 2\sum_{i\neq j;i,j=1}^{n} L_iH_j\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)} \tag{3.18}$$

and the inequality is sharp. We have

$$|f(\bar{x})-f(\bar{y}) \le \sum_{i=1}^{n} L_i|x_i-y_i|^{p_i} \tag{3.19}$$

and

$$|g(\bar{x})-g(\bar{y}) \le \sum_{i=1}^{n} H_i|x_i-y_i|^{q_i} \tag{3.20}$$

If we multiply (3.19)*and* (3.20), *we may get*

$$|(f(\bar{x})-f(\bar{y}))(g(\bar{x})-g(\bar{x}))|$$
$$\leq \sum_{i,j=1}^{n} L_i H_j |x_i-y_i|^{p_i}|x_j-y_j|^{q_j}$$
$$=\sum_{i=1}^{n} L_i H_j |x_i-y_i|^{p_i+q_i} + \sum_{i\neq j;i,j=1}^{n} L_i H_j |x_i-y_i|^{p_i}|x_j-y_j|^{q_j}$$

If we integrate over $\bar{x},\bar{y}\in[\bar{a},\bar{b}]=\prod_{i=1}^{n}[a_i,b_i]:=[a_1,b_1]\times[a_n,b_n]$ *we get from Korkine's identity*

$$|D_n(f,g)|\leq \frac{1}{2[\prod_{i=1}^{n}(b_i-a_i)]^2}\int_{\bar{a}}^{\bar{b}}\int_{\bar{a}}^{\bar{b}}|(f(\bar{x})-f(\bar{y}))(g(\bar{x})-g(\bar{y}))|d\bar{x}d\bar{y} \tag{3.21}$$
$$\leq \frac{1}{2[\prod_{i=1}^{n}(b_i-a_i)]^2}$$
$$\left[\sum_{i=1}^{n} L_i H_i \int_{\bar{a}}^{\bar{b}}\int_{\bar{a}}^{\bar{b}}|x_i-y_i|^{p_i+q_i}d\bar{x}d\bar{y}+\sum_{i\neq j;i,j=1} L_i H_i |x_i-y_i^{p_i}|x_j-y_j|^{q_j}d\bar{x}d\bar{y}\right].$$

Now, we have that

$$A_i:=\int_{\bar{a}}^{\bar{b}}\int_{\bar{a}}^{\bar{b}}|x_i-y_i|^{p_i+q_i}d\bar{x}d\bar{y}=\prod_{k\neq ik=1}^{n}(b_k-a_k)^2\int_{a_i}^{b_i}\int_{a_i}^{b_i}|x_i-y_i|^{p_i+q_i}dx_i dy_i \tag{3.22}$$

and as

$$\int_{c}^{d}\int_{c}^{d}|x-y|^r dxdy = 2\frac{(d-c)^{r+2}}{(r+1)(r+2)'} \tag{3.23}$$

then we get

$$A_{ij}:=\prod_{k\neq i;k=1}^{n}(b_k-a_k)^2.\frac{2(b_i-a_i)^{p_i}+q_i-2}{(p_i+q_i+1)(p_i+q_i+2)}$$
$$=2\prod_{k\neq i;k=1}^{n}(b_k-a_k)^2.\frac{(b_i-a_i)^{p_i}+q_i}{(p_i+q_i+1)(p_i+q_i+2)}.$$

Also,

$$A_{ij}:=\int_{\bar{a}}^{\bar{b}}\int_{\bar{a}}^{\bar{b}}|x_i-y_i|^{p_i}|x_j-y_j|^{p_j}d\bar{x}d\bar{y}$$
$$=\prod_{k\neq i,j;k=1}^{n}(b_k-a_k)^2\int_{a_i}^{b_i}\int_{a_i}^{b_i}|x_i-y_i|^{p_i}|dx_i dy_i\int_{a_j}^{b_j}\int_{a_j}^{b_j}|x_j-y_j|^{p_j}|dx_j dy_j$$
$$=\prod_{k\neq i,j;k=1}^{n}(b_k-a_k)^2.\frac{2(b_i-a_i)^{p_i+2}}{(p_i+1)(p_i+2)}.\frac{2(b_j-a_j)^{q_j+2}}{(q_j+1)(q_j+2)}$$

$$= 4\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)}$$

where we have utilised (3.22). *Further,by* (3.21)*,we have:*

$$\begin{aligned}
&|D_n(f,g)| \\
&\le \frac{1}{2[\prod_{i=1}^{n}(b_i-a_i)]^2}\left[\sum_{i=1}^{n}L_iH_i - 2\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)}\right. \\
&+ 4\sum_{i\neq j;i,j=1}^{n}L_iH_j\prod_{k=1}^{n}(b_k-a_k)^2\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)} \\
&= \sum_{i=1}^{n}L_iH_i\frac{(b_i-a_i)^{p_i+q_i}}{(p_i+q_i+1)(p_i+q_i+2)} \\
&+ 2\sum_{i=1}^{n}L_iH_i\frac{(b_i-a_i)^{p_i}(b_j-a_j)^{q_j}}{(p_i+1)(p_i+2)(q_j+1)(q_j+2)}
\end{aligned}$$

and the result (3.21) *is thus verified.*

The sharpness follows from the sharpness of the Chebychev functional for n=1 (see for example [18]).

Corollary 9 *Let f,g :*$[\bar{a},\bar{b}] \longrightarrow \mathbb{R}$ *be Lipschitzian with constants* $L_i, H_i > 0$. *That is,*

$$|f(\bar{x}) - f(\bar{y})| \le \sum_{i=1}^{n}L_i|x_i-y_i|, \bar{x},\bar{y}\in[\bar{a},\bar{b}]$$

and

$$|g(\bar{x}) - f(\bar{y})| \le \sum_{i=1}^{n}H_i|x_i-y_i|, \bar{x},\bar{y}\in[\bar{a},\bar{b}]$$

Then the inequality

$$|D_n(f,g)| \le \frac{1}{12}\sum_{i=1}^{n}L_iH_i(b_i-a_i)^2 + \frac{1}{18}\sum_{i\neq j;i,j=1}^{n}L_iH_i(b_i-a_i)(b_j-a_j) \quad (3.24)$$

holds and is sharp .

proof Taking $p_i = q_i = 1$ for $i = 1, 2, ..., n$ in (3.21)readiy produces (3.24) □

Remark 3.2.1. *Result (3.24)is presented in [18], however,the coefficients of the sums are ineterchanged .Further, it is apparent that for $x, y \in [a, b]$ and f ansolutely continuous,then*

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq \sup_{z\in[a,b]} |f'(z)| = L,$$

demonstrating that a function ssatisfying a Lipschitzian condition is a weaker condition than one whose dervative belongs to L_∞.

Remark 3.2.2. *if $n = 1$,then for $f, g \in [a_1, b_1] \longrightarrow \mathbb{R}$*

$$|D_1(f,g)| = |T(f,g,a,b) \leq L_1H_1\frac{(b_1-a_1)_1^p+q_1+1}{(p_1+q_1+1)(p_1+q_1+2)} \tag{3.25}$$

with

$$|f(x_1)-f(y_1)| \leq L_1|x_1-y_1|^{p_1}, |g(x_1)-g(y_1)| \leq H_1|x_1-y_1|^{q_1}$$

$x_1, y_1 \in [a_1, b_1]$ and $p_1, q_1 \in (0, 1]$.
If $n = 2$ then for $f, g \in [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R}$

$$\begin{aligned}|D_2(f,g)| = &\left|\frac{1}{(b_1-a_2)(b_2-a2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)g(x_1,x_2)dx_1dx_2\right.\\
&-\frac{1}{(b_1-a_2)^2(b_2-a2)^2}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)dx_1dx_2\\
&\times\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(x_1,x_2)dx_1dx_2\\
&\leq L_1H_1\frac{(b_1-a_1)^{p_1+q_1}}{(p_1+q_1+1)(p_1+q_1+2)} + L_2H_2\frac{(b_2-a_2)^{p_2+q_2}}{(p_2+q_2+1)(p_2+q_2+2)}\\
&+2L_1H_2\frac{(b_1-a_1)^{p_1}(b_2-a_2)^{q_2}}{(p_1+1)(p_1+2)(q_2+1)(q_2+2)}\\
&+2L_2H_1\frac{(b_2-a_2)^{p_2}(b_1-a_1)^{q_1}}{(p_2+1)(p_2+2)(q_1+1)(q_1+2)}\end{aligned}$$

with

$$|f(x_i)-f(y_i)| \leq L_i|x_i-y_i|^{p_i}, x_i, y_i \in [a_i, b_i], \qquad i = 1, 2,$$

and

$$|g(x_i)-g(y_i)| \leq H_i|x_i-y_i|^{q_i}, x_i, y_i \in [a_i, b_i], \qquad i = 1, 2,$$

thus (3.2.2) recaplures the result of hanna, Dragomir and cerone [13][11].

Conclusion

The objective of this work is to study several classical and fractional Chebyshev type inequalities for functions with one variable and for functions with several variables , using the fractional integral in the Riemann-Liouville sense, Montgomery's identities, Korkin's identity.

Bibliography

[1] **S**emigroups of Linear Operators and Applications to Partial Differential Equatuions

[2] S.S. DRAGOMIR et I FEDOTOV, Une inégalité de type Grüss pour l'intégrale de Riemann-Stieltjes et les applications de moyens spéciaux, Tamkang J. of Math., 29(4) (1998), 286-292 [9] AM. FINK, Un traité sur l'inégalité de Grüss, soumis

[3] **G. A. Anastassiou** , Fractional Differentiation Inequalities , Springer , Dordrecht , 2009 .

[4] **P.Agarwal**, J.Tariboon , and S.K.Ntouyas , Some generalized Riemann-Liouville k-fractional integral inequalities , J.of Ineq .Appl .2016 , Paper No .122,13 pp . A

[5] **Pecari·c J. E** and T. Pejkovicí, On an Integral Inequality, J. Inequal. Pure Appl. Math. 5 (2) (2004), Art. 47.

[6] **Boukerrioua K**, Guezane-Lakoud A. On Generalization of Cebyöev Type Inequalities. JIPAM. 2007 ;8 :Art 55, 4pp.

[7] **A. Guezane-Lakoud and F**. Aissaoui New Results for Cebyöev type Inequalities In- tegral Transforms and Special Functions, Vol. 25, No. 9, 711-720.

[8] **B. G. Pachpatte**, Inequalities for Di§erential and Integral Equation, Academic Press, New York, 1998.

[9] **S.M. Farsani**. On the boundedness and compactness of a certain integral operator // Banach J. Math. Anal. 7:2, 86–102 (2013).

[10] **S. Belarbi** and Z. Dahmani. On some new fractional integral inequality // J. Ineq. Pure Appl.Math. 10:3, 1–5 (2009).

[11] **G. Anastassiou** , M. R. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, Montgomery identities for fractional integrals and related fractional inequalities, J. Ineq. Pure Appl. Math. 10 (4) (2009) Art. 97.

[12] **S.S. DRAGOMIR**, Inégalité de Grüss dans les espaces internes des produits. L'Australian Math Soc. Gazette, 26 (1999), No. 2, 65-70.

[13] **G. HANNA, S.S.** DRAGOMIR et P. CERONE, Une formule de type Taylor pour les applications de deux variables définies sur un rectangle dans le plan, soumises (12) D.S. MITRINOVIĆ, JE PEČARIĆ et AM FINK, Classical and New Inequalities in Analyse, Kluwer Academic Publishers, Dordrecht, 1993

[14] **Kilbas**, A. Srivastava, Hari M. and Trujillo, Juan J., Theory and Applications of Fractional Di§erential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V, Amsterdam, 2006.

[15] **Boukerrioua K**, Guezane-Lakoud A. On Generalization of Cebyöev Type Inequalities. JIPAM. 2007 ;8 :Art 55, 4pp.

[16] **S.S. DRAGOMIR** et I FEDOTOV, Une inégalité de type Grüss pour l'intégrale de Riemann-Stieltjes et les applications de moyens spéciaux, Tamkang J. of Math., 29(4) (1998), 286-292

[17] **BUDIMIR, P.** CERONE et J. PEČARIĆ, Inégalités liées à la Fonctionnelle de Tchebychev impliquant des intégrales sur différents intervalles, soumis.

[18] **AM. FINK,** Un traité sur l'inégalité de Grüss, soumis

[19] **A. Guezane-Lakoud** and F. Aissaoui New Results for Cebyöev type Inequalities In- tegral Transforms and Special Functions, Vol. 25, No. 9, 711-720.

[20] **A. Guezane** Lakoud and F. Aissaoui, New double integrals weighted Cebyöev type inequalities. Journal of Mathematics and System Science 1(2011)1-6.